

# *Design of Stable Mechanical Structures*

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# Structural design problems

## Mathematical program with equilibrium constraints:

$$\min_{\rho, \mathbf{u}} F(\rho, \mathbf{u})$$

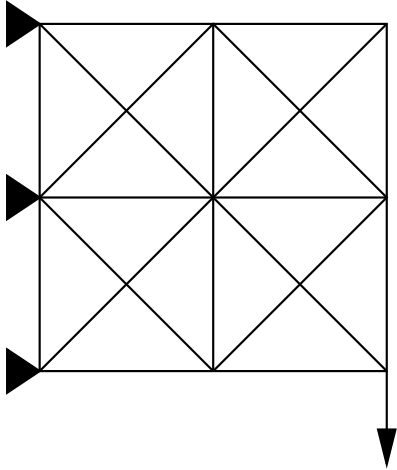
s.t.

$$\rho \in U_{\text{ad}}$$

$$\mathbf{u} \text{ solves } \mathcal{E}(\rho, \mathbf{u})$$

$F(\rho, \mathbf{u})$	...	cost functional (weight, stiffness, peak stress...)
$\rho$	...	design variable (thickness, material properties, shape...)
$\mathbf{u}$	...	state variable (displacements, stresses)
$U_{\text{ad}}$	...	admissible designs

# Truss design problem



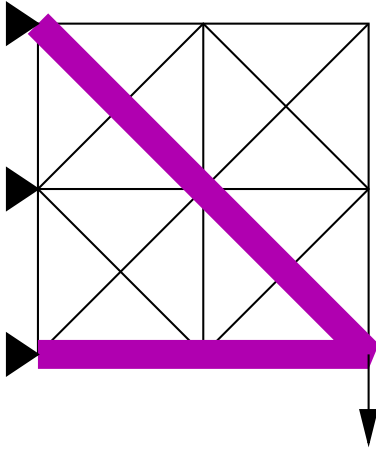
$$\mathbf{K}(\boldsymbol{\rho}) = \sum_{i=1}^m \rho_i \mathbf{K}_i,$$

$\rho_i \dots$  bar volumes

$u_i \dots$  displacements

$$\mathbf{K}_i = \frac{E_i}{l_i^2} \boldsymbol{\gamma}_i \boldsymbol{\gamma}_i^T$$

# Truss design problem



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$\rho_i \dots$  bar volumes  
 $u_i \dots$  displacements

$$\min_{\rho \in \mathbb{R}^m, u \in \mathbb{R}^n} \sum_{i=1}^m \rho_i$$

$$\text{s.t. } K(\rho)u = f, \quad f^T u \leq C, \quad \rho_i \geq 0, \quad i = 1, 2, \dots, m$$

# Structural design problems

## WEIGHT versus STIFFNESS:

•  $W$  weight  $\sum \rho_i$

•  $C$  stiffness (compliance)  $f^T u$

Equilibrium constraint:  $u$  solves  $\mathcal{E}(\rho, u) \longrightarrow \sum (\rho_i K_i) u = f$

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$\min C$

s.t.

$$W \leq \widehat{W}$$

equilibrium

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S. Timoshenko:

*Experience showed that structures like bridges or aircrafts may fail in some cases not on account of high stresses but owing to insufficient elastic stability.*

# Self-vibrations of a mechanical system

Free vibrations (motions in absence of external loads) of a mechanical system

$$M\ddot{x}(t) = -Kx(t) \quad (\star)$$

$K \succeq 0$ ... stiffness matrix,  $M \succ 0$ ... mass matrix

$$M = \sum \rho_i \beta_i \beta_i^T, \quad \beta_i = \sqrt{\mu} \gamma_i$$

The solutions to  $(\star)$  are of the form

$$x(t) = \sum_{j=1}^n [a_j \cos(\omega_j t) + b_j \sin(\omega_j t)] e_j$$

where  $a_j, b_j$  are free parameters,  $e_j$  are the eigenvectors of

$$(\lambda_j M - K)e_j = 0$$

and  $\omega_j = \sqrt{\lambda_j}$ .

# Structural design with free vibration control

## Three quantities to control:

- $W$  weight  $\sum \rho_i$
- $C$  stiffness (compliance)  $f^T u$
- $\lambda$  min. eigenfrequency  $K(\rho)u = \lambda M(\rho)u$

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$$\begin{array}{l} \max \lambda \\ \text{s.t.} \\ \\ W \leq \widehat{W} \\ C \leq \widehat{C} \\ \text{equilibrium} \end{array}$$

# Structural design with stability control

## Three quantities to control:

- $W$  weight  $\sum \rho_i$
- $C$  stiffness (compliance)  $f^T u$
- $\lambda$  critical buckling force  $K(\rho)u = \lambda G(\rho, u)u$

$$\begin{array}{l} \min C \\ \text{s.t.} \\ W \leq \widehat{W} \\ \lambda \geq 1 \\ \text{equilibrium} \end{array}$$

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# Structural design with stability control

Lowest (positive) eigenvalue of

$$K(\rho)u = \lambda G(\rho, u)u$$

(critical force) should be bigger than 1 (than  $\hat{\lambda}$  for vibration constraint)

$$\min_{\rho, u} W(\rho)$$

s.t.

$$K(\rho)u = f$$

$$f^T u \leq \hat{C}$$

$$\rho_i \geq 0, \quad i = 1, \dots, m$$

$$\lambda \geq 1$$

# Structural design with stability control

## Two standard tricks:

$$K(\boldsymbol{\rho}) \succ \mathbf{0}, \quad \mathbf{u} = K(\boldsymbol{\rho})^{-1} \mathbf{f}$$

$$\mathbf{f}^T K(\boldsymbol{\rho})^{-1} \mathbf{f} \leq \hat{C} \iff \begin{pmatrix} \hat{C} & \mathbf{f}^T \\ \mathbf{f} & K(\boldsymbol{\rho}) \end{pmatrix} \succeq \mathbf{0}$$

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$$\left. \begin{array}{l} K(\boldsymbol{\rho})\mathbf{u} = \lambda G(\boldsymbol{\rho}, \mathbf{u})\mathbf{u} \\ \lambda \geq 1 \end{array} \right\} \iff K(\boldsymbol{\rho}) - G(\boldsymbol{\rho}, \mathbf{u}) \succeq \mathbf{0}$$

$$\iff K(\boldsymbol{\rho}) - \tilde{G}(\boldsymbol{\rho}) \succeq \mathbf{0}$$

$$\tilde{G}(\boldsymbol{\rho}) = G(\boldsymbol{\rho}, K(\boldsymbol{\rho})^{-1} \mathbf{f})$$

# Structural design with stability control

Formulated as SDP problem:

$$\min_{\rho} W(\rho)$$

subject to

$$K(\rho) - \tilde{G}(\rho) \succeq 0$$

$$\begin{pmatrix} c & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0$$

$$\rho_i \geq 0, \quad i = 1, \dots, m$$

where

$$K(\rho) = \sum \rho_i K_i, \quad \tilde{G}(\rho) = \sum \tilde{G}_i$$

# Structural design with free vibration control

Formulated as SDP problem:

$$\min_{\rho} W(\rho)$$

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$$K(\rho) - \hat{\lambda}M(\rho) \succeq 0$$

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$$\rho_i \geq 0, \quad i = 1, \dots, m$$

where

$$K(\rho) = \sum \rho_i K_i, \quad M(\rho) = \sum \rho_i M_i$$

# Solving vibration problem as GEVP

Another option (vibration problems):

Solve the maximum eigenvalue problem formulated as GEVP:

$\lambda$  min. eigenfrequency of  $K(\rho)u = \lambda M(\rho)u$

$$\begin{array}{l} \max \lambda \\ \text{s.t.} \\ W \leq \widehat{W} \\ C \leq \widehat{C} \\ \text{equilibrium} \end{array}$$

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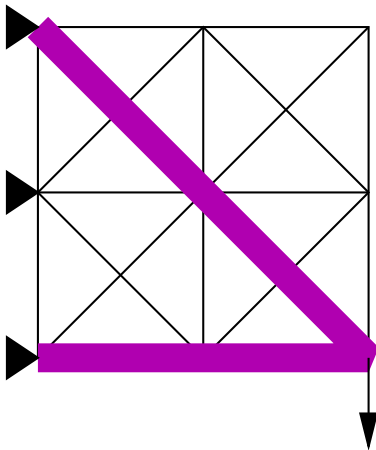
$$\begin{array}{l} \max_{\rho, \lambda} \lambda \\ \text{s.t.} \\ K(\rho) - \lambda M(\rho) \succeq 0 \\ \sum \rho_i \leq \widehat{W} \\ \rho_i \geq 0, \quad i = 1, \dots, m \\ \begin{pmatrix} \widehat{C} & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0 \end{array}$$

(quasiconvex) SDP problem with BMI constraints — solve by PENBMI

# Truss design problem

$$\min_{\rho \in \mathbb{R}^m, u \in \mathbb{R}^n} f^T u$$

$$\text{s.t. } K(\rho)u = f, \quad \sum_{i=1}^m \rho_i = V, \quad \rho_i \geq 0, \quad i = 1, 2, \dots, m$$



$$K(\rho) = \sum_{i=1}^m \rho_i K_i, \quad K_i = \frac{E_i}{l_i^2} \gamma_i \gamma_i^T$$

$\rho_i$  ... bar volumes

$u_i$  ... displacements

# Truss design with stability constraint

$$\min_{\rho} \sum \rho_i \quad (\text{minimize weight})$$

subject to

$$K(\rho) + \tilde{G}(\rho) \succeq 0$$

$$\begin{pmatrix} c & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0$$

$$\rho_i \geq 0, \quad i = 1, \dots, m$$

with

$$K(\rho) = \sum \rho_i K_i \quad K_i = \frac{E_i}{\ell_i^2} \gamma_i \gamma_i^T$$

$$\tilde{G}(\rho) = \sum \tilde{G}_i(\rho) \quad \tilde{G}_i(\rho) = \frac{E \rho_i}{\ell_i^3} (\gamma_i^T K(\rho)^{-1} f) (\delta_i \delta_i^T).$$

# Truss design with free vibration constraint

$$\min_{\rho} \sum \rho_i \quad (\text{minimize weight})$$

subject to

$$K(\rho) + \hat{\lambda}M(\rho) \succeq 0$$

$$\begin{pmatrix} c & f^T \\ f & K(\rho) \end{pmatrix} \succeq 0$$

$$\rho_i \geq 0, \quad i = 1, \dots, m$$

with

$$K(\rho) = \sum \rho_i K_i \quad K_i = \frac{E_i}{\ell_i^2} \gamma_i \gamma_i^T$$

$$M(\rho) = \sum \rho_i M_i \quad M_i = \text{diag}(\delta_i \delta_i^T)$$

# Free Material Optimization

Aim:

Given an amount of material, boundary conditions and external load  $f$ , find the material (distribution) so that the body is as stiff as possible under  $f$ .

The design variables are the **material properties at each point** of the structure.

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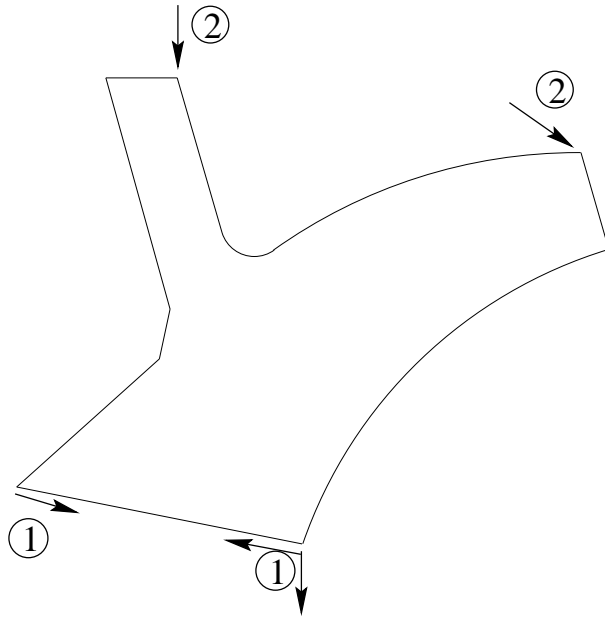
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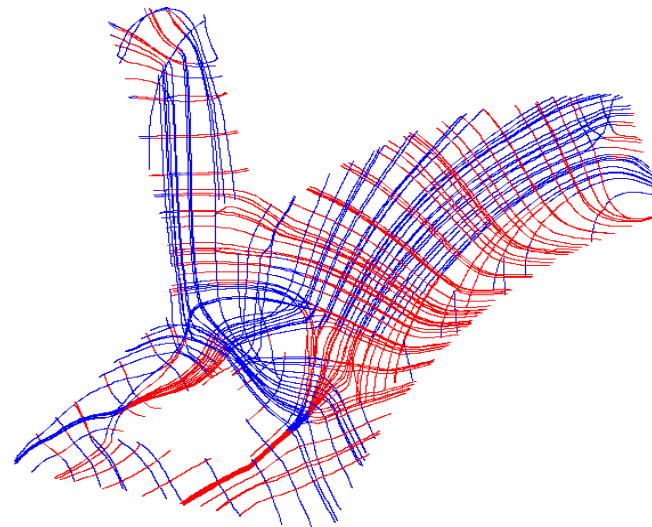
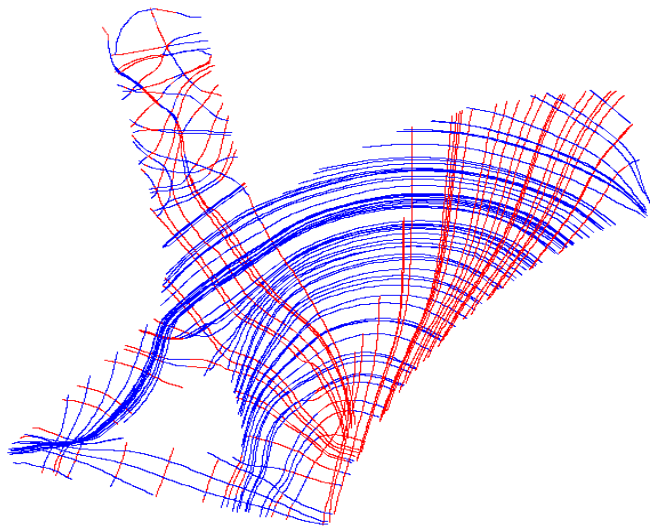
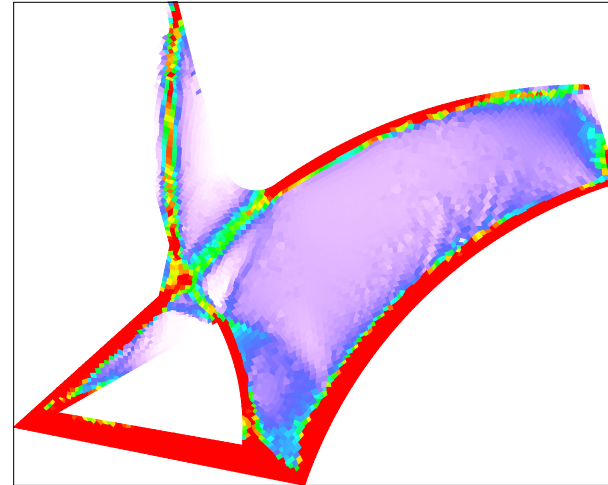
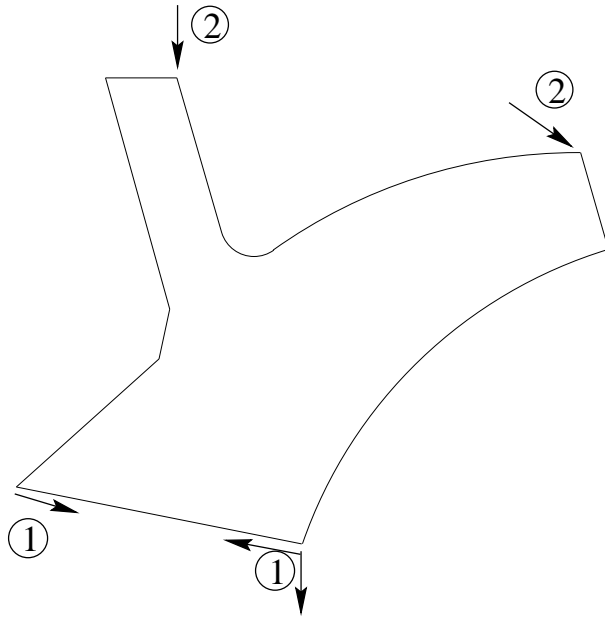
$$\inf_{\substack{\rho \geq 0 \\ \int \rho dx \leq 1}} \sup_{u \in U} -\frac{1}{2} \int_{\Omega} \rho \langle e(u), e(u) \rangle dx + \int_{\Gamma_2} f \cdot u dx$$

$$\inf_{\alpha \in \mathbb{R}, u \in U} \left\{ \alpha - f^T u \mid \alpha \geq \frac{m}{2} u^T A_i u \quad \text{for } i = 1, \dots, m \right\}$$

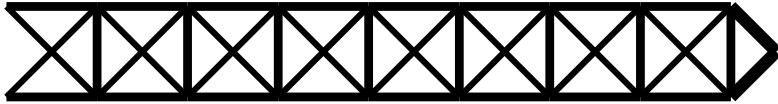
# FMO, example



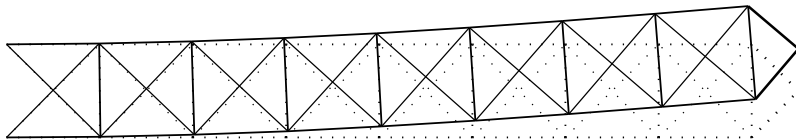
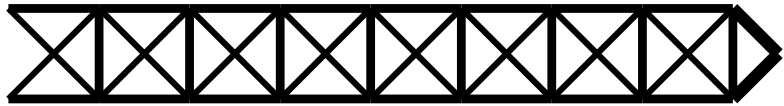
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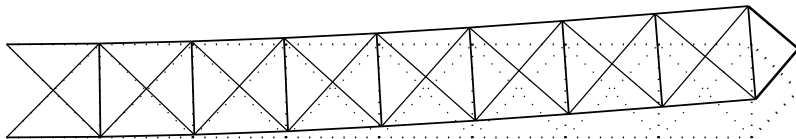
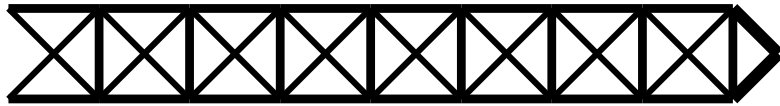
# *Truss design w. stability, vibration constraint*



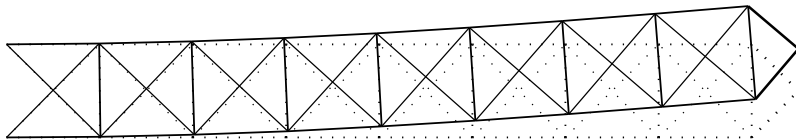
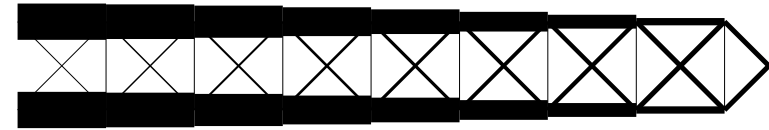
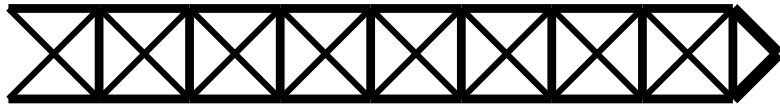
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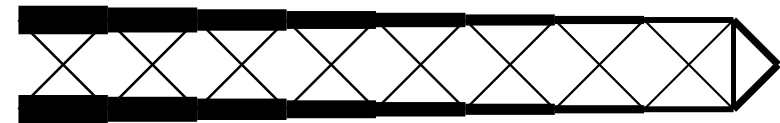
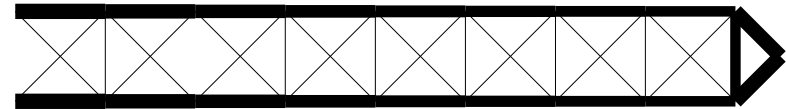
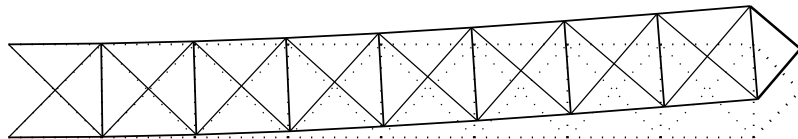
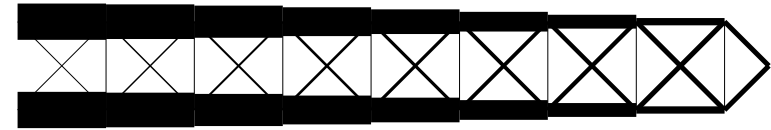
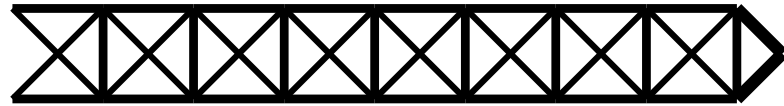
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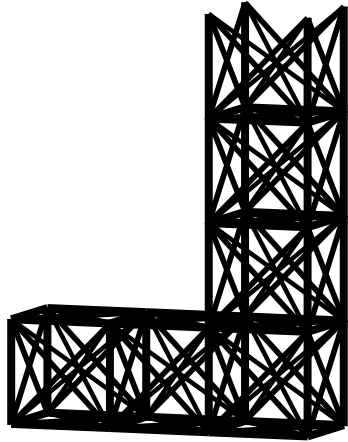
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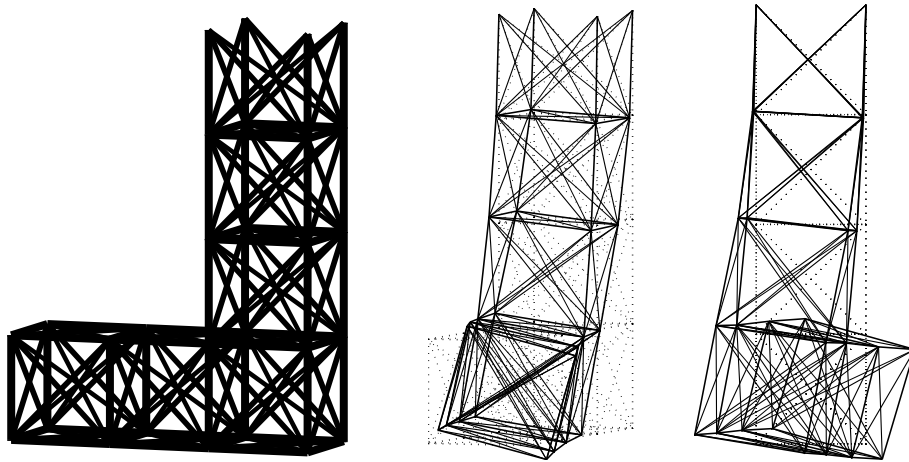
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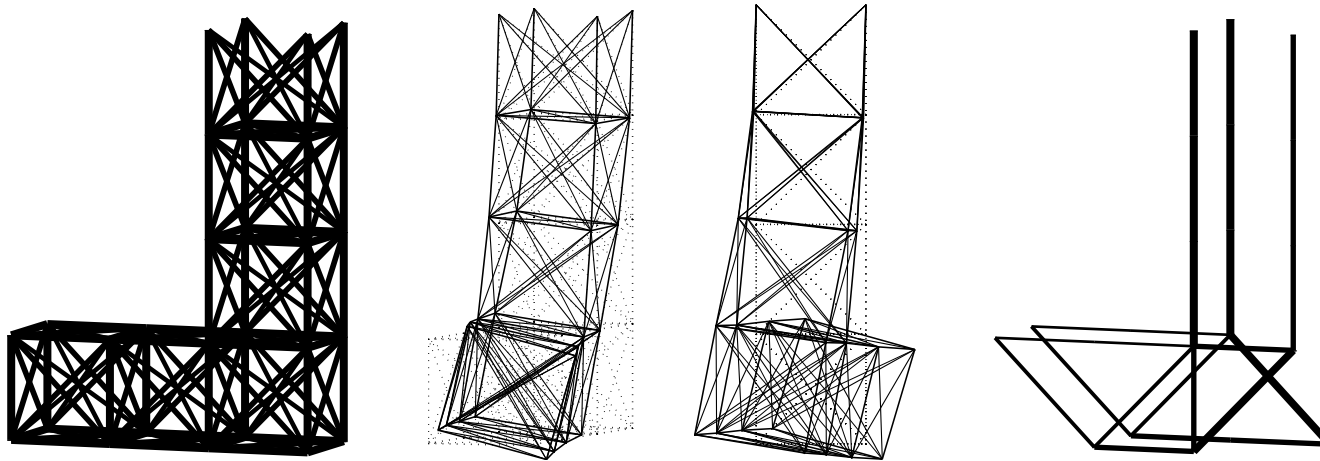
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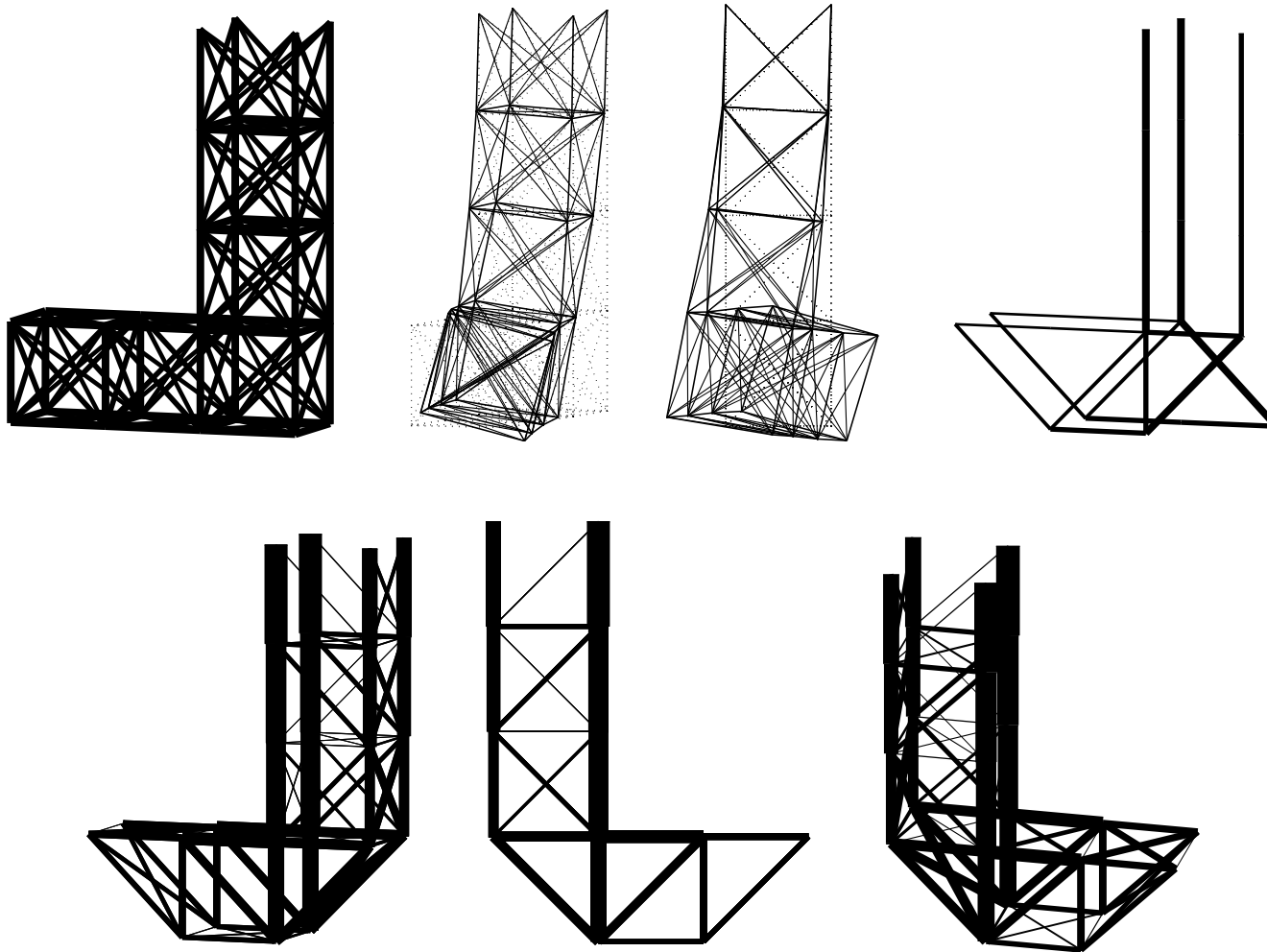
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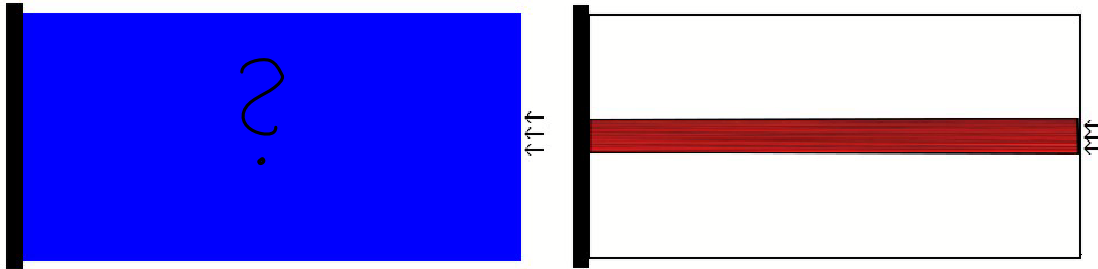
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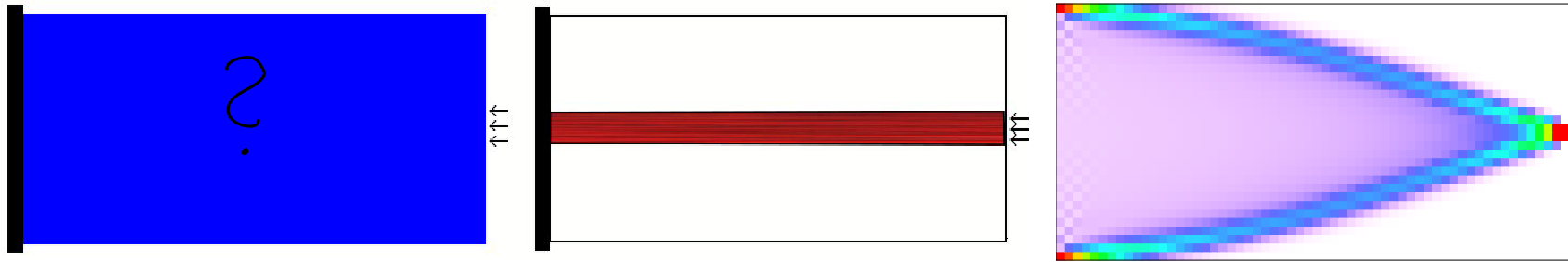
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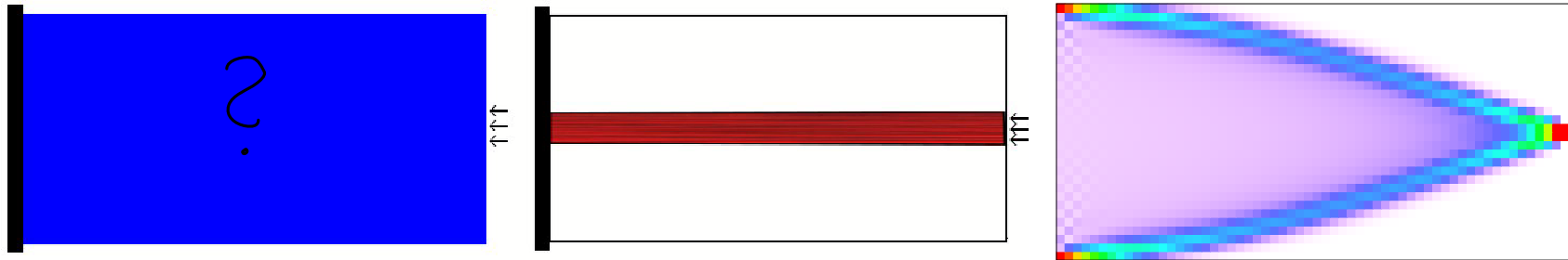
# Examples, FMO w. vibration constraint



# Examples, FMO w. vibration constraint



# Examples, FMO w. vibration constraint



FMO with vibration constraint: BMI formulation

