

COURSE ON LMI OPTIMIZATION  
WITH APPLICATIONS IN CONTROL  
PART II.3

**LMI<sub>s</sub> IN SYSTEMS CONTROL**  
**ROBUSTNESS ANALYSIS**  
**POLYNOMIAL METHODS**

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Vertical and diagonal Planes (1913-14)  
František Kupka (1871-1957)

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## Polynomial methods

Based on the algebra of **polynomials** and **polynomial matrices**, typically involve

- linear Diophantine equations
- quadratic spectral factorization

Pioneered in central Europe during the 70s mainly by Vladimír Kučera from the former Czechoslovak Academy of Sciences

Network funded by the European commission



[www.utia.cas.cz/europoly](http://www.utia.cas.cz/europoly)

Polynomial matrices also occur in Jan Willems' behavioral approach to systems theory

**Alternative** to state-space methods developed during the 60s most notably by Rudolf Kalman in the USA, rather based on

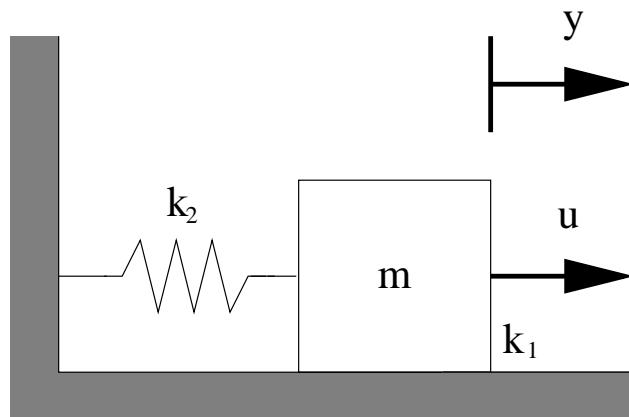
- linear Lyapunov equations
- quadratic Riccati equations

## Ratio of polynomials

A scalar transfer function can be viewed as the ratio of two **polynomials**

### Example

Consider the mechanical system



- $y$  displacement
- $k_1$  viscous friction coeff
- $m$  mass
- $u$  external force
- $k_2$  spring constant

Neglecting static and Coloumb frictions, we obtain the linear transfer function

$$G(s) = \frac{y(s)}{u(s)} = \frac{1}{ms^2 + k_1s + k_2}$$

## Ratio of polynomial matrices

Similarly, a MIMO transfer function can be viewed as the ratio of **polynomial matrices**

$$G(s) = N_R(s)D_R^{-1}(s) = D_L^{-1}(s)N_L(s)$$

the so-called matrix fraction description (MFD)

**Lightly damped structures** such as oil derricks, regional power models, earthquakes models, mechanical multi-body systems, damped gyroscopic systems are most naturally represented by second order polynomial MFDs

$$(D_0 + D_1s + D_2s^2)y(s) = N_0u(s)$$

### Example

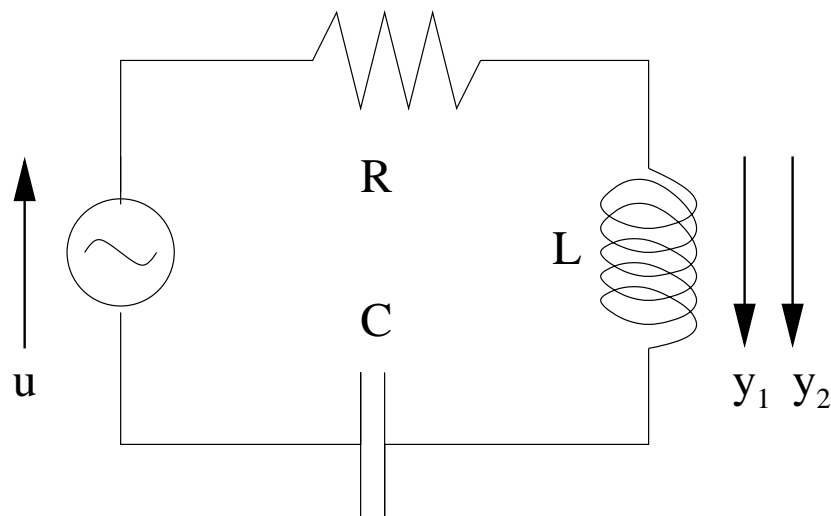
The (simplified) oscillations of a **wing in an air stream** is captured by properties of the quadratic polynomial matrix [Lancaster 1966]

$$D(s) = \begin{bmatrix} 121 & 18.9 & 15.9 \\ 0 & 2.7 & 0.145 \\ 11.9 & 3.64 & 15.5 \end{bmatrix} + \begin{bmatrix} 7.66 & 2.45 & 2.1 \\ 0.23 & 1.04 & 0.223 \\ 0.6 & 0.756 & 0.658 \end{bmatrix} s + \begin{bmatrix} 17.6 & 1.28 & 2.89 \\ 1.28 & 0.824 & 0.413 \\ 2.89 & 0.413 & 0.725 \end{bmatrix} s^2$$

## First-order polynomial MFD

### Example

### RCL network



- $y_1$  voltage through inductor
- $y_2$  current through inductor
- $u$  voltage

Applying Kirchoff's laws and Laplace transform we get

$$\begin{bmatrix} 1 & -Ls \\ Cs & 1 + RCs \end{bmatrix} \begin{bmatrix} y_1(s) \\ y_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ Cs \end{bmatrix} u(s)$$

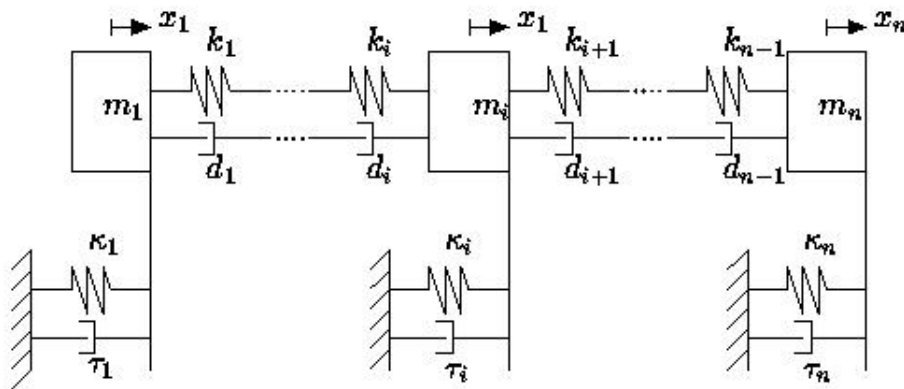
and thus the first-order left system MFD

$$G(s) = \begin{bmatrix} 1 & -Ls \\ Cs & 1 + RCs \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ Cs \end{bmatrix}.$$

## Second-order polynomial MFD

### Example

### mass-spring system



Vibration of system governed by 2nd-order differential equation  $M\ddot{x} + C\dot{x} + Kx = 0$  where e.g.  $n = 250$ ,  $m_i = 1$ ,  $\kappa_i = 5$ ,  $\tau_i = 10$  except  $\kappa_1 = \kappa_n = 10$  and  $\tau_1 = \tau_n = 20$

Quadratic matrix polynomial

$$D(s) = Ms^2 + Cs + K$$

with

$$M = I$$

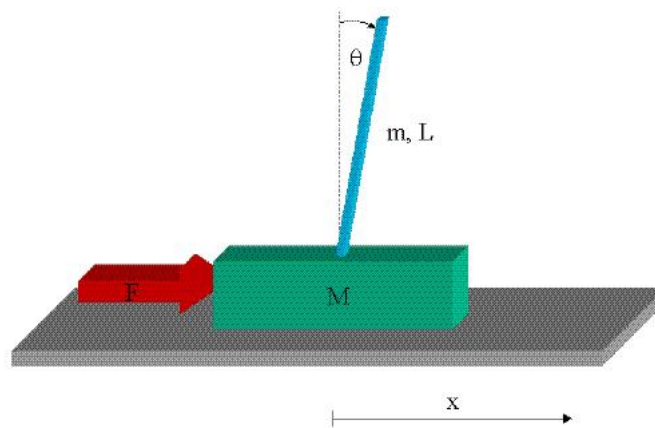
$$C = \text{tridiag}(-10, 30, -10)$$

$$K = \text{tridiag}(-5, 15, -5).$$

## Another second-order polynomial MFD

### Example

Inverted pendulum on a cart



Linearization around the upper vertical position yields the left polynomial MFD

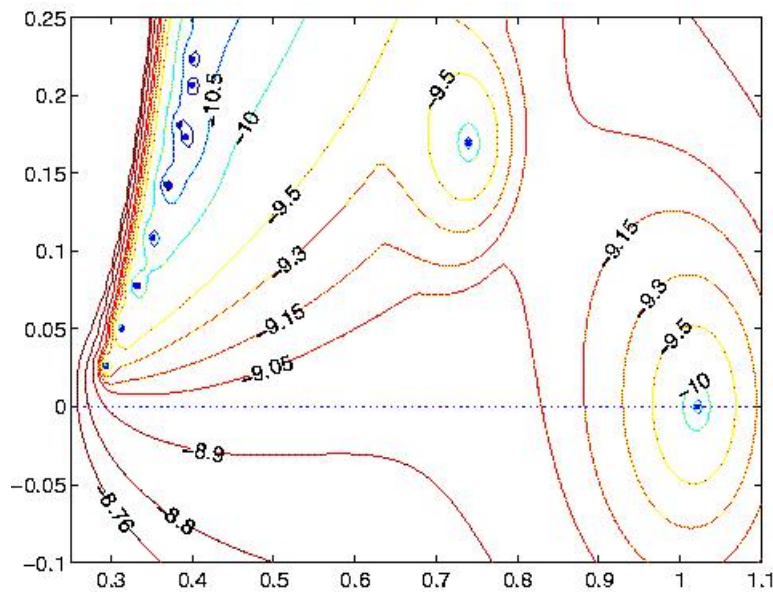
$$\begin{bmatrix} (M + m)s^2 + bs & lms^2 \\ lms^2 & (J + l^2m)s^2 + ks - lmg \end{bmatrix} \begin{bmatrix} x(s) \\ \phi(s) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} f(s)$$

With  $J = mL^2/12$ ,  $l = L/2$  and  $g = 9.8$ ,  $M = 2$ ,  $m = 0.35$ ,  $l = 0.7$ ,  $b = 4$ ,  $k = 1$ , we obtain the denominator polynomial matrix

$$D(s) = \begin{bmatrix} 5s + 3s^2 & 0.35s^2 \\ 0.35s^2 & -3.4 + s + 0.16s^2 \end{bmatrix}$$

## More examples of polynomial MFDs

Higher degree polynomial matrices can also be found in [aero-acoustics](#) (3rd degree) or in the study of the spatial stability of the Orr-Sommerfeld equation for plane Poiseuille flow in [fluid mechanics](#) (4rd degree)



Pseudospectra of Orr-Sommerfeld equation

For more info see Nick Higham's homepage at [www.ma.man.ac.uk/~higham](http://www.ma.man.ac.uk/~higham)

## Stability analysis for polynomials

Well established theory - LMIs are of no use here !

Given a continuous-time polynomial

$$p(s) = p_0 + p_1s + \dots + p_{n-1}s^{n-1} + p_ns^n$$

with  $p_n > 0$  we define its  $n \times n$  **Hurwitz matrix**

$$H(p) = \begin{bmatrix} p_{n-1} & p_{n-3} & & 0 & 0 \\ p_n & p_{n-2} & & \vdots & \vdots \\ 0 & p_{n-1} & \dots & 0 & 0 \\ 0 & p_n & & p_0 & 0 \\ \vdots & \vdots & & p_1 & 0 \\ 0 & 0 & & p_2 & p_0 \end{bmatrix}$$

**Hurwitz stability criterion:** Polynomial  $p(s)$  is stable iff all principal minors of  $H(p)$  are  $> 0$



Adolf Hurwitz  
(Hanover 1859 - Zürich 1919)

## Robust stability analysis for polynomials

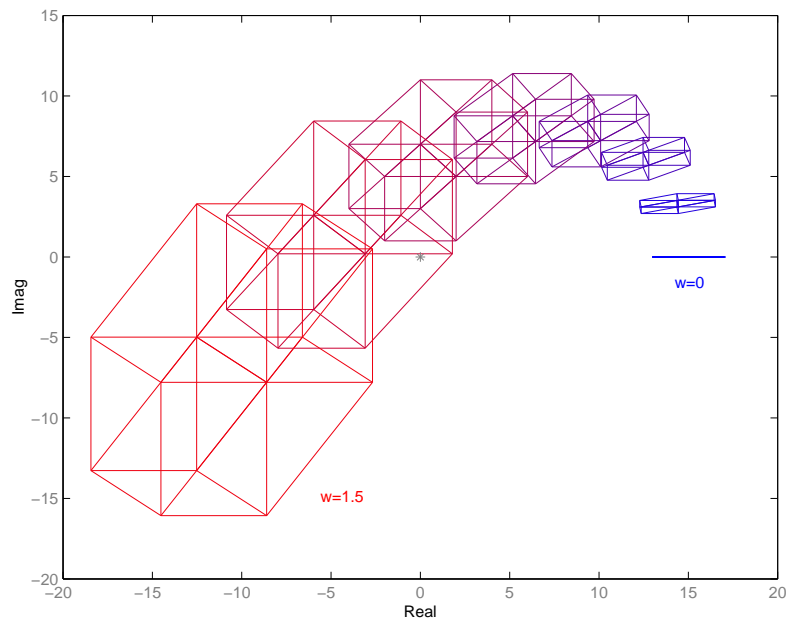
Analyzing **stability robustness** of polynomials is a little bit more interesting..

Here too **computational complexity** depends on the **uncertainty model**

In increasing order of complexity, we will distinguish between

- single parameter uncertainty  $q \in [q_{\min}, q_{\max}]$
- interval uncertainty  $q_i \in [q_{i\min}, q_{i\max}]$
- polytopic uncertainty  $\lambda_1 q_1 + \dots + \lambda_N q_N$
- multilinear uncertainty  $q_0 + q_1 \cdot q_2 \cdot q_3$

LMIs will not show up very soon..  
..just **basic linear algebra**



Graphic stability analysis of an interval plant controlled with a first-order compensator

## Single parameter uncertainty and eigenvalue criterion

Consider the uncertain polynomial

$$p(s, q) = p_0(s) + qp_1(s)$$

where

- $p_0(s)$  nominally stable with positive coeffs
- $p_1(s)$  such that  $\deg p_1(s) < \deg p_0(s)$

The **largest** stability interval

$$q \in ]q_{\min}, q_{\max}[$$

such that  $p(s, q)$  is **robustly stable** is given by

$$\begin{aligned} q_{\max} &= 1/\lambda_{\max}^+(-H_0^{-1}H_1) \\ q_{\min} &= 1/\lambda_{\min}^-(-H_0^{-1}H_1) \end{aligned}$$

where  $\lambda_{\max}^+$  is the max positive real eigenvalue  
 $\lambda_{\min}^-$  is the min negative real eigenvalue  
 $H_i$  is the Hurwitz matrix of  $p_i(s)$

## Higher powers of a single parameter

Now consider the continuous-time polynomial

$$p(s, q) = p_0(s) + qp_1(s) + q^2p_2(s) + \cdots + q^m p_m(s)$$

with  $p_0(s)$  stable and  $\deg p_0(s) > \deg p_i(s)$

Using the zeros (roots of determinant) of the [polynomial Hurwitz matrix](#)

$$H(p) = H(p_0) + qH(p_1) + q^2H(p_2) + \cdots + q^m H(p_m)$$

we can show that

$$\begin{aligned} q_{\min} &= 1/\lambda_{\min}^-(M) \\ q_{\max} &= 1/\lambda_{\max}^+(M) \end{aligned}$$

where

$$M = \begin{bmatrix} 0 & & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & & I & 0 \\ 0 & & 0 & I \\ -H_0^{-1}H_m & \cdots & -H_0^{-1}H_2 & -H_0^{-1}H_1 \end{bmatrix}$$

is a [block companion matrix](#)

## MIMO systems

Uncertain multivariable systems are modeled by uncertain **polynomial matrices**

$$P(s, q) = P_0(s) + qP_1(s) + q^2P_2(s) + \dots + q^m P_m(s)$$

where  $p_0(s) = \det P_0(s)$  is a stable polynomial

We can apply the **scalar** procedure to the **determinant** polynomial

$$\det P(s, q) = p_0(s) + qp_1(s) + q^2p_2(s) + \dots + q^r p_r(s)$$

### Example

MIMO design on the plant with left MFD

$$\begin{aligned} A^{-1}(s, q)B(s, q) &= \begin{bmatrix} s^2 & q \\ q^2 + 1 & s \end{bmatrix}^{-1} \begin{bmatrix} s + 1 & 0 \\ q & 1 \end{bmatrix} \\ &= \frac{\begin{bmatrix} s^2 + s - q^2 & -q \\ qs^2 - (q^2 + 1)s - (q^2 + 1) & s^2 \end{bmatrix}}{s^3 - q^2 - q} \end{aligned}$$

with uncertain parameter  $q \in [0, 1]$

## MIMO systems: example

Using some design method, we obtain a controller with right MFD

$$Y(s)X^{-1}(s) = \begin{bmatrix} 94 - 51s & -18 + 17s \\ -55 & 100 \end{bmatrix} \begin{bmatrix} 55 + s & -17 \\ -1 & 18 + s \end{bmatrix}$$

Closed-loop system with characteristic denominator polynomial matrix

$$\begin{aligned} D(s, q) &= A(s, q)X(s) + B(s, q)Y(s) \\ &= D_0(s) + qD_1(s) + q^2D_2(s) \end{aligned}$$

Nominal system poles: roots of  $\det D_0(s)$

Applying the [eigenvalue criterion](#) on  $\det D(s, q)$  yields the stability interval

$$q \in ] - 0.93, 1.17[ \supset [0, 1]$$

so the closed-loop system is **robustly stable**

## Independent uncertainty

So far we have studied polynomials affected by a **single** uncertain parameter

$$p(s, q) = (6 + q) + (4 + q)s + (2 + q)s^2$$

However in practice **several** parameters can be uncertain, such as in

$$p(s, q) = (6 + q_0) + (4 + q_1)s + (2 + q_2)s^2$$

**Independent** uncertainty structure: each component  $q_i$  enters into only one coefficient

**Interval** uncertainty: independent structure and uncertain parameter vector  $q$  belongs to a given **box**, i.e.  $q_i \in [q_i^-, q_i^+]$

### Example

Uncertain polynomial

$$(6 + q_0) + (4 + q_1)s + (2 + q_2)s^2, \quad |q_i| \leq 1$$

has interval uncertainty, also denoted as

$$[5, 7] + [3, 5]s + [1, 3]s^2$$

Some coefficients can be **fixed**, e.g.

$$6 + [3, 5]s + 2s^2$$

## Kharitonov's polynomials

Associated with the interval polynomial

$$p(s, q) = \sum_{i=0}^n [q_i^-, q_i^+] s^i$$

are **four Kharitonov's polynomials**

$$\begin{aligned} p^{--}(s) &= q_0^- + q_1^- s + q_2^+ s^2 + q_3^+ s^3 + q_4^- s^4 + q_5^- s^5 + \dots \\ p^{-+}(s) &= q_0^- + q_1^+ s + q_2^+ s^2 + q_3^- s^3 + q_4^- s^4 + q_5^+ s^5 + \dots \\ p^{+-}(s) &= q_0^+ + q_1^- s + q_2^- s^2 + q_3^+ s^3 + q_4^+ s^4 + q_5^- s^5 + \dots \\ p^{++}(s) &= q_0^+ + q_1^+ s + q_2^- s^2 + q_3^- s^3 + q_4^+ s^4 + q_5^+ s^5 + \dots \end{aligned}$$

where we assume  $q_n^- > 0$  and  $q_n^+ > 0$

### Example

Interval polynomial

$$p(s, q) = [1, 2] + [3, 4]s + [5, 6]s^2 + [7, 8]s^3$$

Kharitonov's polynomials

$$\begin{aligned} p^{--}(s) &= 1 + 3s + 6s^2 + 8s^3 \\ p^{-+}(s) &= 1 + 4s + 6s^2 + 7s^3 \\ p^{+-}(s) &= 2 + 3s + 5s^2 + 8s^3 \\ p^{++}(s) &= 2 + 4s + 5s^2 + 7s^3 \end{aligned}$$

## Kharitonov's theorem

In 1978 the Russian researcher Vladimir Kharitonov proved the following fundamental result

A continuous-time interval polynomial is robustly stable iff its four Kharitonov polynomials are stable

Instead of checking stability of an **infinite** number of polynomials we just have to check stability of **four** polynomials, which can be done using the classical Hurwitz criterion



Peter and Paul fortress in St Petersburg

## Affine uncertainty

Sadly, Kharitonov's theorem is valid only

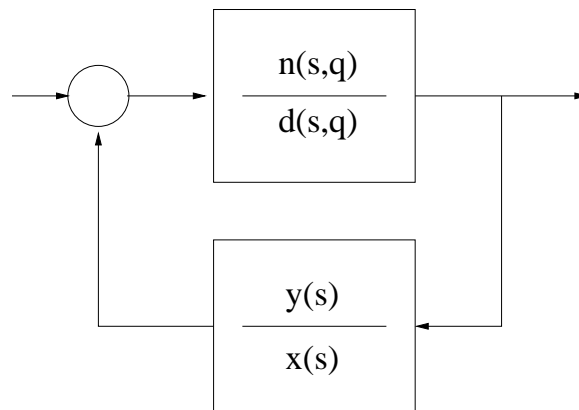
- for continuous-time polynomials
- for independent interval uncertainty

so that we have to use **more general tools** in practice

When coefficients of an uncertain polynomial  $p(s, q)$  or a rational function  $n(s, q)/d(s, q)$  depend affinely on parameter  $q$ , such as in

$$a^T q + b$$

we speak about **affine uncertainty**



The above feedback interconnection

$$\frac{n(s, q)x(s)}{d(s, q)x(s) + n(s, q)y(s)}$$

**preserves** the affine uncertainty structure of the plant

## Polytopes of polynomials

A family of polynomials  $p(s, q)$ ,  $q \in Q$  is said to be a **polytope of polynomials** if

- $p(s, q)$  has an **affine** uncertainty structure
- $Q$  is a **polytope**

There is a natural isomorphism between a polytope of polynomials and its **set of coefficients**

### Example

$p(s, q) = (2q_1 - q_2 + 5) + (4q_1 + 3q_2 + 2)s + s^2$ ,  $|q_i| \leq 1$   
Uncertainty polytope has 4 generating vertices

$$\begin{array}{ll} q^1 = [-1, -1] & q^2 = [-1, 1] \\ q^3 = [1, -1] & q^4 = [1, 1] \end{array}$$

Uncertain polynomial family has 4 generating vertices

$$\begin{array}{ll} p(s, q^1) = 4 - 5s + s^2 & p(s, q^2) = 2 + s + s^2 \\ p(s, q^3) = 8 + 3s + s^2 & p(s, q^4) = 6 + 9s + s^2 \end{array}$$

Any polynomial in the family can be written as

$$p(s, q) = \sum_{i=1}^4 \lambda_i p(s, q^i), \quad \sum_{i=1}^4 \lambda_i = 1, \quad \lambda_i \geq 0$$

## Interval polynomials

Interval polynomials are a **special case** of polytopic polynomials

$$p(s, q) = \sum_{i=0}^n [q_i^-, q_i^+] s^i$$

with at most  $2^{n+1}$  generating vertices

$$p(s, q^k) = \sum_{i=0}^n q_i^k s^i, \quad q_i^k = \begin{cases} q_i^- \\ \text{or} \\ q_i^+ \end{cases} \quad 1 \leq k \leq 2^{n+1}$$

### Example

The interval polynomial

$$p(s, q) = [5, 6] + [3, 4]s + 5s^2 + [7, 8]s^3 + s^4$$

can be generated by the  $2^3 = 8$  vertex polynomials

$$\begin{aligned} p(s, q^1) &= 5 + 3s + 5s^2 + 7s^3 + s^4 \\ p(s, q^2) &= 6 + 3s + 5s^2 + 7s^3 + s^4 \\ p(s, q^3) &= 5 + 4s + 5s^2 + 7s^3 + s^4 \\ p(s, q^4) &= 6 + 4s + 5s^2 + 7s^3 + s^4 \\ p(s, q^5) &= 5 + 3s + 5s^2 + 8s^3 + s^4 \\ p(s, q^6) &= 6 + 3s + 5s^2 + 8s^3 + s^4 \\ p(s, q^7) &= 5 + 4s + 5s^2 + 8s^3 + s^4 \\ p(s, q^8) &= 6 + 4s + 5s^2 + 8s^3 + s^4 \end{aligned}$$

## The edge theorem

Let  $p(s, q)$ ,  $q \in Q$  be a polynomial with invariant degree over polytopic set  $Q$

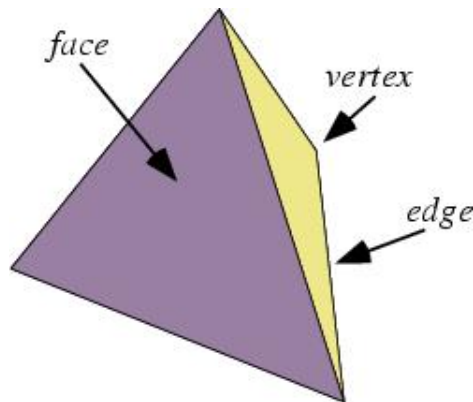
Polynomial  $p(s, q)$  is **robustly stable** over the whole uncertainty polytope  $Q$  iff  $p(s, q)$  is stable **along the edges** of  $Q$

In other words, it is **enough** to check robust stability of the **single parameter** polynomial

$$\lambda p(s, q^{i1}) + (1 - \lambda)p(s, q^{i2}), \quad \lambda \in [0, 1]$$

for **each pair** of vertices  $q^{i1}$  and  $q^{i2}$  of  $Q$

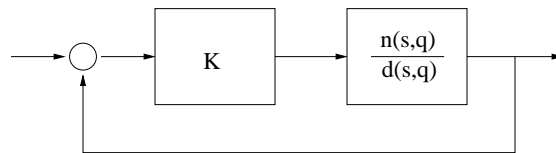
This can be done with the **eigenvalue criterion**



## Interval feedback system

### Example

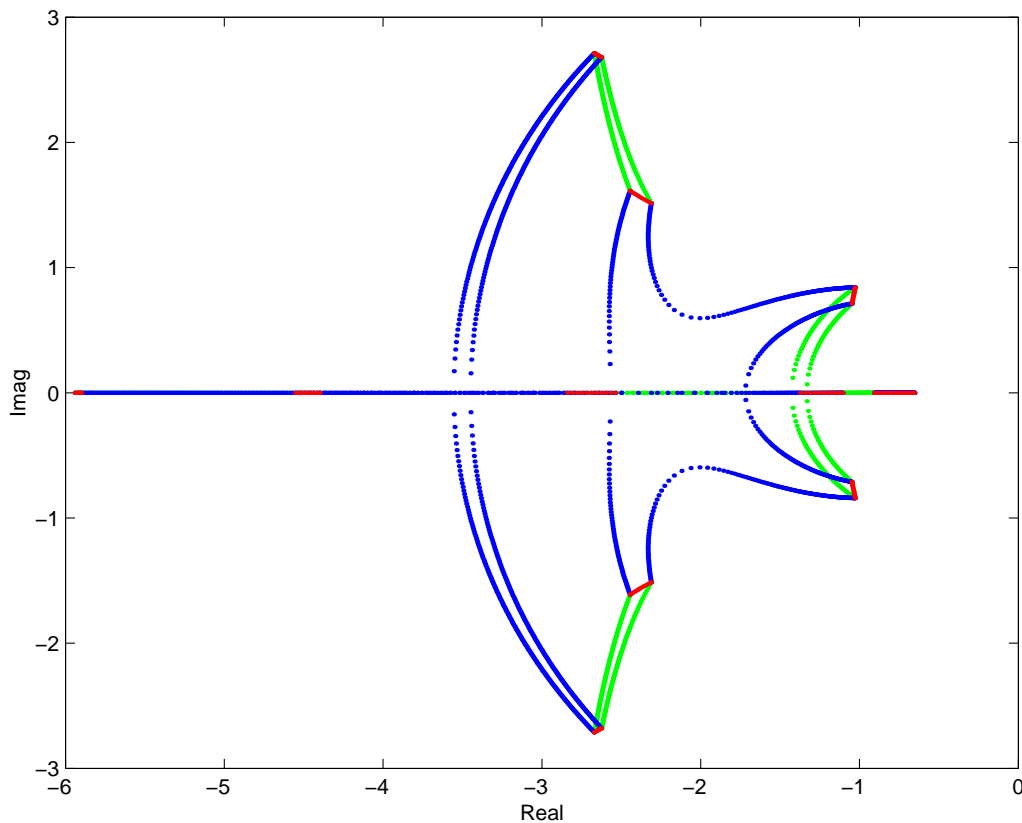
We consider the interval control system



with  $n(s, q) = [6, 8]s^2 + [9.5, 10.5]$ ,  $d(s, q) = s(s^2 + [14, 18])$  and characteristic polynomial

$$K[9.5, 10.5] + [14, 18]s + K[6, 8]s^2 + s^3$$

For  $K = 1$  we draw the 12 edges of its root set



The closed-loop system is **robustly stable**

## More about uncertainty structure

In typical applications, uncertainty structure is **more complicated** than interval or affine

Usually, uncertainty enters highly **non-linearly** in the closed-loop characteristic polynomial

We distinguish between

- **multilinear** uncertainty, when each uncertain parameter  $q_i$  is linear when other parameters  $q_j, i \neq j$  are fixed
- **polynomial** uncertainty, when coefficients are multivariable polynomials in parameters  $q_i$

We can define the following **hierarchy** on the uncertainty structures

interval  $\subset$  affine  $\subset$  multilinear  $\subset$  polynomial

## Examples of uncertainty structures

### Examples

The uncertain polynomial

$$(5q_1 - q_2 + 5) + (4q_1 + q_2 + q_3)s + s^2$$

has **affine** uncertainty structure

The uncertain polynomial

$$(5q_1 - q_2 + 5) + (4q_1q_3 - 6q_1q_3 + q_3)s + s^2$$

has **multilinear** uncertainty structure

The uncertain polynomial

$$(5q_1 - q_2 + 5) + (4q_1 - 6q_1 - q_3^2)s + s^2$$

has **polynomial** (here quadratic) uncertainty structure

The uncertain polynomial

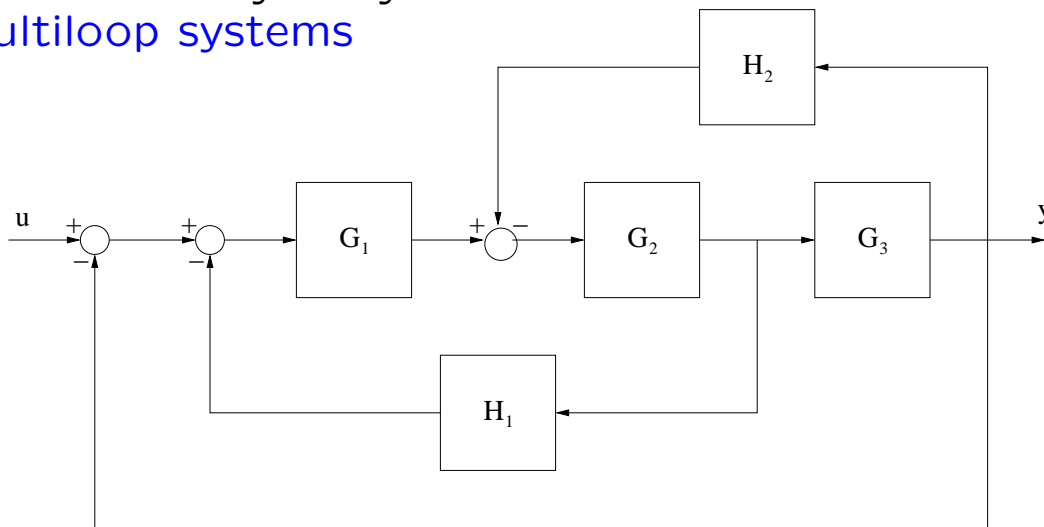
$$(5q_1 - q_2 + 5) + (4q_1 - 6q_1q_3^2 + q_3)s + s^2$$

has **polynomial** uncertainty structure

## Multilinear uncertainty

We will focus on **multilinear uncertainty** because it arises in a wide variety of system models such as:

- multiloop systems



Closed-loop transfer function

$$\frac{y}{u} = \frac{G_1 G_2 G_3}{1 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3}$$

- state-space models with **rank-one uncertainty**

$$\dot{x} = A(q)x, \quad A(q) = \sum_{i=1}^n q_i A_i, \quad \text{rank } A_i = 1$$

and characteristic polynomial

$$p(s, q) = \det(sI - A(q))$$

- polynomial MFDs with **MIMO interval** uncertainty

$$G(s) = A^{-1}(s, q)B(s, q), \quad C(s) = Y(s)X^{-1}(s)$$

and closed-loop characteristic polynomial

$$p(s, q) = \det(A(s, q)X(s) + B(s, q)Y(s))$$

## Robust stability analysis for multilinear and polynomial uncertainty

Unfortunately, there is **no systematic** computational tractable necessary and sufficient robust stability condition

On the one hand, sufficient condition through polynomial value sets, the zero exclusion condition and the **mapping theorem**

On the other hand, brute-force method: intensive parameter **gridding**, NP-hard in general

No easy trade-off between computational complexity and conservatism

## Polynomial stability analysis: summary

Checking robust stability can be

- **easy** (polynomial-time algorithms) or more
- **difficult** (NP-hard problem)

depending namely on the uncertainty model

We focused on polytopic uncertainty:

- **Interval scalar polynomials**

Kharitonov's theorem (ct only)

- **Polytope of scalar polynomials**

(affine polynomial families)

Edge theorem

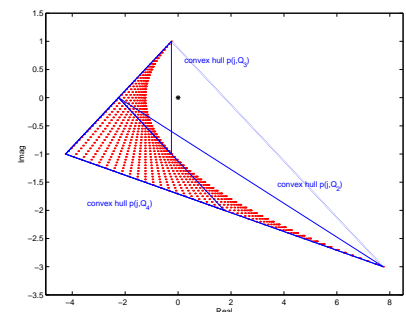
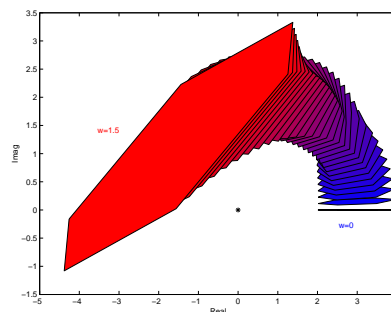
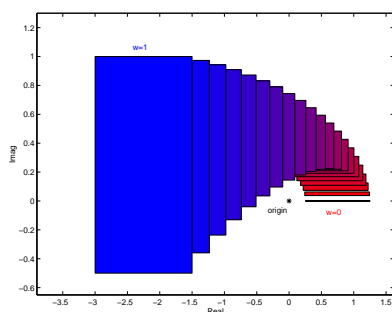
- **Interval matrix polynomials**

(multiaffine polynomial families)

Mapping theorem

- **Polytopes of matrix polynomials**

(polynomic polynomial families)



## Lessons from robust analysis: lack of extreme point results

Ensuring robust stability of the parametrized polynomial

$$p(s, q) = p_0(s) + qp_1(s) \\ q \in [q_{\min}, q_{\max}]$$

amounts to ensuring robust stability of the whole segment of polynomials

$$\lambda p(s, q_{\min}) + (1 - \lambda)p(s, q_{\max}) \\ \lambda = \frac{q_{\max} - q}{q_{\max} - q_{\min}} \in [0, 1]$$

A natural question arises: does stability of two vertices imply stability of the segment ?

Unfortunately, the answer is **no**

Example

First vertex:  $0.57 + 6s + s^2 + 10s^3$  **stable**

Second vertex:  $1.57 + 8s + 2s^2 + 10s^3$  **stable**

But middle of segment:

$1.07 + 7s + 1.50s^2 + 10s^3$  **unstable**

## Lessons from robust analysis: lack of edge results

In the same way there is **lack of vertex results** for **affine** uncertainty, there is a **lack of edge results** for **multilinear** uncertainty

Example

Consider the uncertain polynomial

$$\begin{aligned} p(s, q) = & (4.032q_1q_2 + 3.773q_1 + 1.985q_2 + 1.853) \\ & + (1.06q_1q_2 + 4.841q_1 + 1.561q_2 + 3.164)s \\ & + (q_1q_2 + 2.06q_1 + 1.561q_2 + 2.871)s^2 \\ & + (q_1 + q_2 + 2.56)s^3 + s^4 \end{aligned}$$

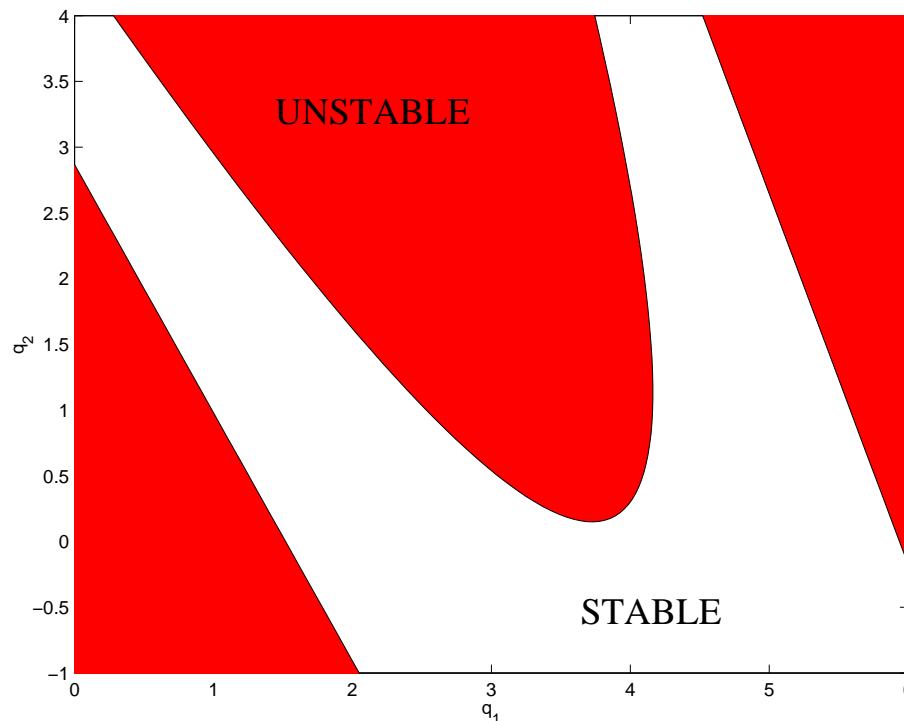
with **multilinear** uncertainty over the polytope  $q_1 \in [0, 1]$ ,  $q_2 \in [0, 3]$ , corresponding to the state-space interval matrix

$$p(s, q) = \det\left(sI - \begin{bmatrix} [-1.5, -0.5] & -12.06 & -0.06 & 0 \\ -0.25 & -0.03 & 1 & 0.5 \\ 0.25 & -4 & -1.03 & 0 \\ 0 & 0.5 & 0 & [-4, 1] \end{bmatrix}\right)$$

The four edges of the uncertainty bounding set are **stable**, however for  $q_1 = 0.5$  and  $q_2 = 1$  polynomial  $p(s, q)$  is **unstable**..

## Non-convexity of stability domain

Main problem: the stability domain in the space of polynomial coefficients  $p_i$  is **non-convex** in general



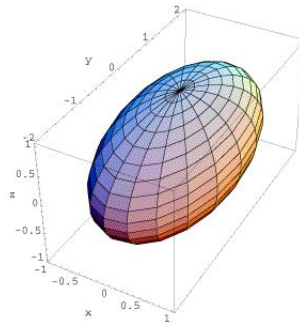
Discrete-time stability domain in  $(q_1, q_2)$  plane for polynomial  $p(z, q) = (-0.825 + 0.225q_1 + 0.1q_2) + (0.895 + 0.025q_1 + 0.09q_2)z + (-2.475 + 0.675q_1 + 0.3q_2)z^2 + z^3$

How can we **overcome** the non-convexity of the stability conditions in the coefficient space ?

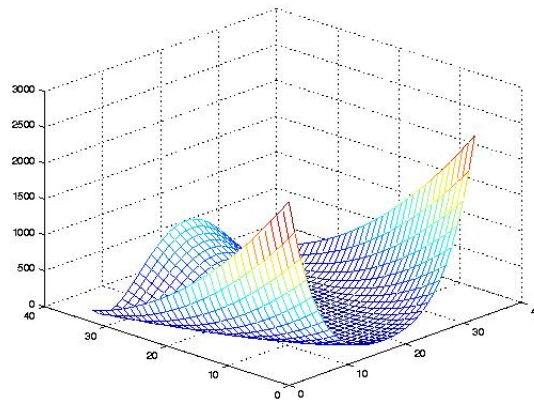
## Handling non-convexity

Basically, we can pursue two approaches:

- we can **approximate** the non-convex stability domain with a convex domain (segment, polytope, sphere, ellipsoid, LMI)



- we can address the non-convexity with the help of **non-convex optimization** (global or local optimization)



## Approximation of the stability domain

From the tools of **robust stability analysis**, we can build around a nominally stable polynomial

- stability **segments** (eigenvalue criterion)
- stability **boxes** (Kharitonov's theorem)
- stability **polytopes** (Edge theorem)

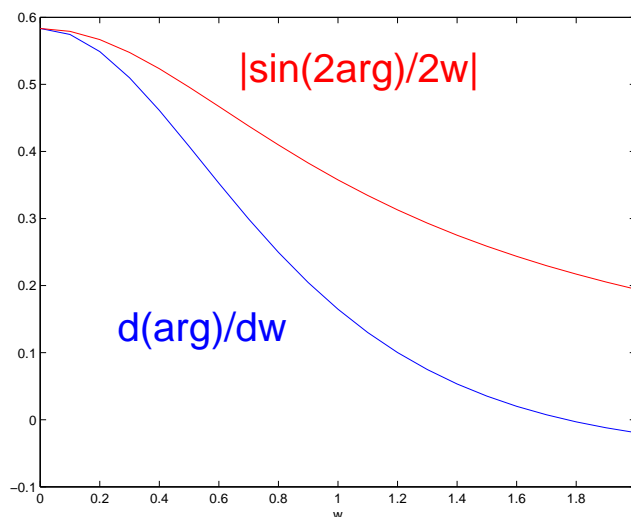
There exists other results, such as **Rantzer's growth condition**: a polynomial  $g(s)$  is a convex direction iff

$$\frac{d}{d\omega} \arg g(j\omega) \leq \left| \frac{\sin 2 \arg g(j\omega)}{2\omega} \right|, \quad \omega > 0$$

It means that given any stable  $f(s)$  such that  $f(s) + g(s)$  is stable then the whole segment  $[f(s), g(s)]$  is stable

Example

$g(s) = 24 + 14s - 13s^2 - 2s^3 + s^4$  is a growth direction



## Stability polytopes

Largest **hyper-rectangle** around a nominally stable polynomial

$$p(s) + r \sum_{i=0}^n [-\varepsilon_i, \varepsilon_i] s^i$$

obtained with the **eigenvalue criterion** applied on the 4 **Kharitonov polynomials**

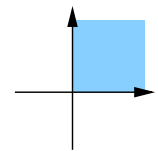
In general, there is no **systematic way** to obtain more general stability polytopes, namely because of computational complexity

(no analytic formula for the volume of a polytope)

Well-known candidates:

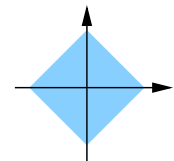
- ct LHP: outer approximation (necessary stab cond)

**positive cone**  $p_i > 0$



- dt unit disk: inner approximation (sufficient stab cond)

**diamond**  $|p_0| + |p_1| + \dots + |p_{n-1}| < 1$



## Stability region (second degree)

**Necessary** stab cond in dt: convex hull of stability domain is a polytope whose  $n + 1$  vertices are polynomials with roots  $+1$  or  $-1$

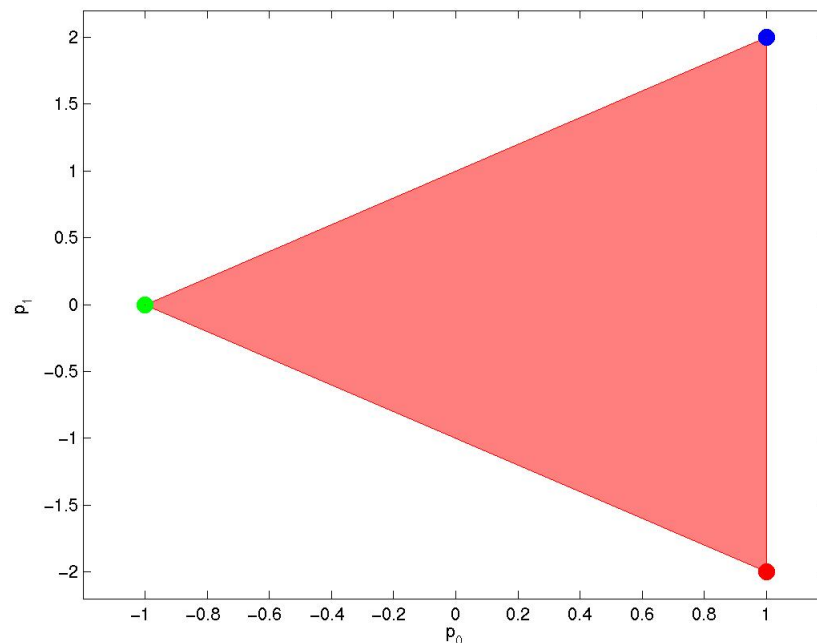
Example

When  $n = 2$ : triangle with vertices

$$(z + 1)(z + 1) = 1 + 2z + z^2$$

$$(z + 1)(z - 1) = -1 + z^2$$

$$(z - 1)(z - 1) = 1 - 2z + z^2$$

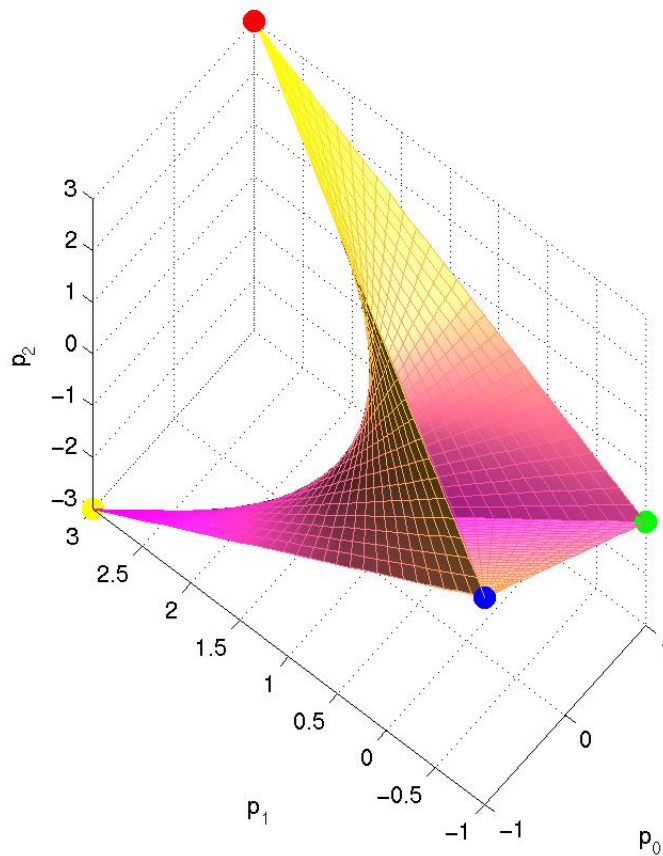


## Stability region (third degree)

Example

Third degree dt polynomial: two hyperplanes and a **non-convex** hyperbolic paraboloid with a saddle point at  $p(z) = p_0 + p_1z + p_2z^2 + z^3 = z(1 + z^2)$

$$\begin{aligned}(z + 1)(z + 1)(z + 1) &= 1 + 3z + 3z^2 + z^3 \\(z + 1)(z + 1)(z - 1) &= -1 - z + z^2 + z^3 \\(z + 1)(z - 1)(z - 1) &= 1 - z - z^2 + z^3 \\(z - 1)(z - 1)(z - 1) &= -1 + 3z - 3z^2 + z^3\end{aligned}$$



## Stability hyper-spheres

Largest **hyper-sphere** around a nominally stable polynomial

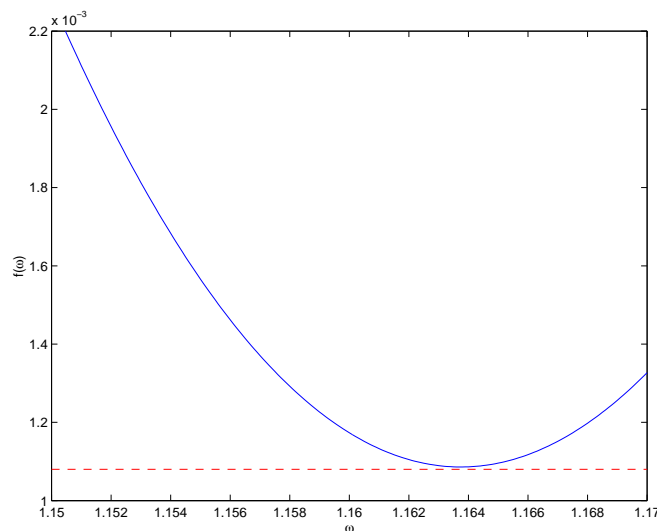
$$p(s) + \sum_{i=0}^n q_i s^i, \|q\| \leq r$$

has radius

$$r_{\max} = \min\{|p_0|, |p_n|, \inf_{\omega>0} \sqrt{\frac{(\operatorname{Re} p(j\omega))^2}{1 + \omega^4 + \omega^8 \dots} + \frac{(\operatorname{Im} p(j\omega))^2}{\omega^2 + \omega^6 + \dots}}\}$$

Example

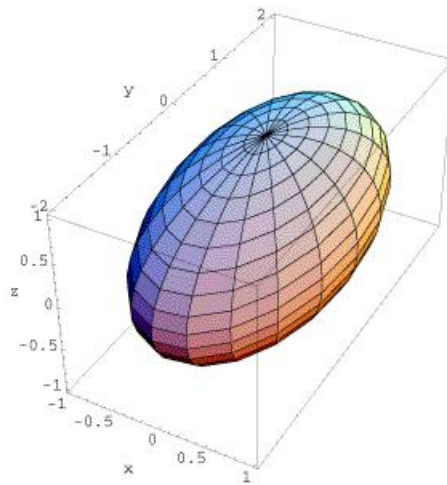
$$(2 + q_0) + (1.4 + q_1)s + (1.5 + q_2)s^2 + (1 + q_3)s^3, \|q\| \leq r$$



$$r_{\max} = \min\{2, 1, \inf_{\omega>0} f(\omega)\} = 1.08 \cdot 10^{-3}$$

## Stability ellipsoids

A **weighted** and **rotated** hyper-sphere is an **ellipsoid**



We are interested in **inner** ellipsoidal approximations of stability domains

$$E = \{p : (p - \bar{p})^* P (p - \bar{p}) \leq 1\}$$

where

- $p$  coef vector of polynomial  $p(s)$
- $\bar{p}$  center of ellipsoid
- $P$  positive definite matrix

## Hermite stability criterion



Charles Hermite (1822 Dieuze - 1901 Paris)

The polynomial  $p(s) = p_0 + p_1s + \dots + p_ns^n$  is **stable** if and only if

$$H(x) = \sum_i \sum_j p_i p_j H_{ij} \succ 0$$

where matrices  $H_{ij}$  are given and depend on the root clustering region only

Examples for  $n = 3$ :

continuous-time stability

$$H(p) = \begin{bmatrix} 2p_0p_1 & 0 & 2p_0p_3 \\ 0 & 2p_1p_2 - 2p_0p_3 & 0 \\ 2p_0p_3 & 0 & 2p_2p_3 \end{bmatrix}$$

discrete-time stability

$$H(p) = \begin{bmatrix} p_3^2 - p_0^2 & p_2p_3 - p_0p_1 & p_1p_3 - p_0p_2 \\ p_2p_3 - p_0p_1 & p_2^2 + p_3^2 - p_0^2 - p_1^2 & p_2p_3 - p_0p_1 \\ p_1p_3 - p_0p_2 & p_2p_3 - p_0p_1 & p_3^2 - p_0^2 \end{bmatrix}$$

## Inner ellipsoidal approximation

Our objective is then to find  $\bar{p}$  and  $P$  such that the ellipsoid

$$E = \{p : (p - \bar{p})^* P (p - \bar{p}) \leq 1\}$$

is a convex inner approximation of the actual non-convex stability region

$$S = \{p : H(p) \succ 0\}$$

that is to say

$$E \subset S$$

Naturally, we will try to **enlarge the volume** of the ellipsoid as much as we can

The Hermite matrix defining stability region  $S$  can be written as

$$H(p) = (I_n \otimes p^*) \tilde{H} (I_n \otimes p) \succ 0$$

where the big and sparse matrix  $\tilde{H}$  depends on the stability region only

## Preliminaries

The quadratic inequality defining stability ellipsoid  $E$  can be written as

$$P(p) = p^* \tilde{P} p \geq 0$$

where

$$\begin{aligned} \tilde{P} &= \begin{bmatrix} P_{11} & P_{12} \\ P_{12}^* & P_{22} \end{bmatrix} & P_{22} & \text{scalar} \\ \bar{p} &= -P_{11}^{-1} P_{12} \\ P &= -P_{11} / (P_{22} - P_{12}^* P_{11}^{-1} P_{12}) \end{aligned}$$

Ellipsoid  $E$  is non-empty and bounded iff

$$P_{11} \prec 0 \quad P_{22} - P_{12}^* P_{11}^{-1} P_{12} > 0$$

the later inequality is **non-convex** so we

- either specify a vector  $p_0$  in  $E$

$$p_0^* \tilde{P} p_0 = 1$$

- or specify the center  $\bar{p}$  of  $E$

$$P_{11} = -P \prec 0 \quad P_{12} = P \bar{p} \quad P_{22} = 1 - \bar{p}^* P \bar{p}$$

## LMI inner ellipsoidal approximation

If there exists some symmetric block matrix

$$S = \begin{bmatrix} 0 & S_{21}^* & \cdots & S_{n1}^* \\ S_{21} & 0 & & S_{n2}^* \\ \vdots & & \ddots & \vdots \\ S_{n1} & S_{n2} & \cdots & 0 \end{bmatrix}$$

made up of skew-symmetric blocks  $S_{ij} = -S_{ij}^*$  and a positive scalar  $\lambda$  such that the LMI

$$\lambda \tilde{H} \succ I_n \otimes \tilde{P} + S$$

is satisfied, then

$$\begin{aligned} \lambda H(x) &= \lambda (I_n \otimes p)^* \tilde{H} (I_n \otimes p) \\ &\succ (I_n \otimes p)^* (I_n \otimes \tilde{P} + S) (I_n \otimes p) \\ &= I_n \otimes P(p) \end{aligned}$$

therefore

$H(p) \succ 0$  for all  $p$  such that  $P(p) \geq 0$

and the inclusion  $E \subset S$  is ensured

## Stability ellipsoids

### Example

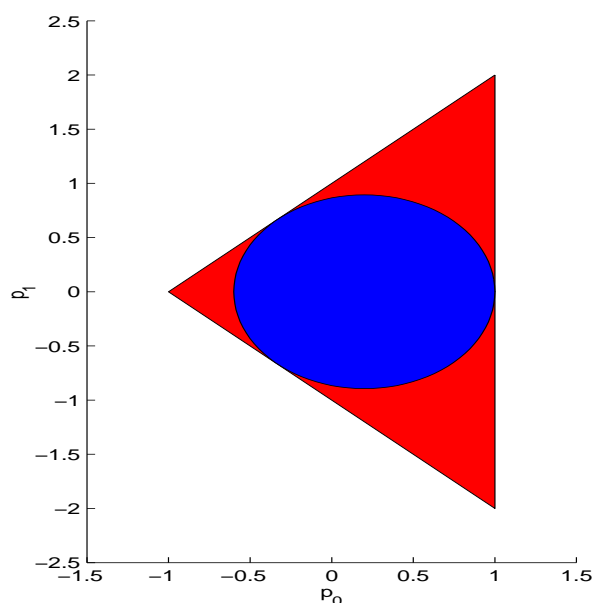
Discrete-time second degree polynomial

$$p(z) = p_0 + p_1 z + z^2$$

We solve the LMI problem and we obtain

$$P = \begin{bmatrix} 1.5625 & 0 \\ 0 & 1.2501 \end{bmatrix} \quad \bar{p} = \begin{bmatrix} 0.2000 \\ 0 \end{bmatrix}$$

which describes an ellipse  $E$  inscribed in the exact triangular stability domain  $S$



## Stability ellipsoids

### Example

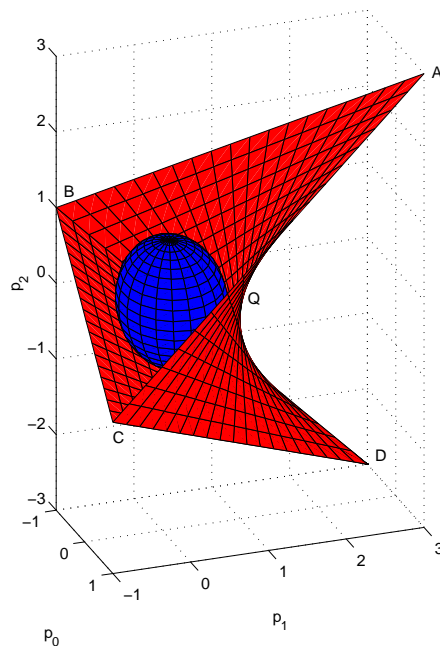
Discrete-time third degree polynomial

$$p(z) = p_0 + p_1z + p_2z^2 + z^3$$

We solve the LMI problem and we obtain

$$P = \begin{bmatrix} 2.3378 & 0 & 0.5397 \\ 0 & 2.1368 & 0 \\ 0.5397 & 0 & 1.7552 \end{bmatrix} \quad \bar{x} = \begin{bmatrix} 0 \\ 0.1235 \\ 0 \end{bmatrix}$$

which describes a **convex** ellipse  $E$  inscribed in the exact stability domain  $S$  delimited by the **non-convex** hyperbolic paraboloid



Very simple scalar convex sufficient stability condition

$$2.4166p_0^2 + 2.2088p_1^2 + 1.8143p_2^2 - 0.5458p_1 + 1.1158p_0p_2 \leq 1$$

## Volume of stability ellipsoid

In the discrete-time case, the well-known sufficient stability condition defines a diamond

$$D = \{p : |p_0| + |p_1| + \cdots + |p_{n-1}| < 1\}$$

For different values of degree  $n$ , we compared **volumes** of exact stability domain  $S$ , ellipsoid  $E$  and diamond  $D$

	$n = 2$	$n = 3$	$n = 4$	$n = 5$
Stability domain $S$	4.0000	5.3333	7.1111	7.5852
Ellipsoid $E$	2.2479	1.4677	0.7770	0.3171
Diamond $D$	2.0000	1.3333	0.6667	0.2667

$E$  is “larger” than  $D$ , yet very small wrt  $S$

In the last part of this course, we will propose better **LMI** inner approximations of the stability domain

