COURSE ON LMI OPTIMIZATION
WITH APPLICATIONS IN CONTROL
PART II.4

LMIs IN SYSTEMS CONTROL
ROBUST CONTROL DESIGN
POLYNOMIAL METHODS

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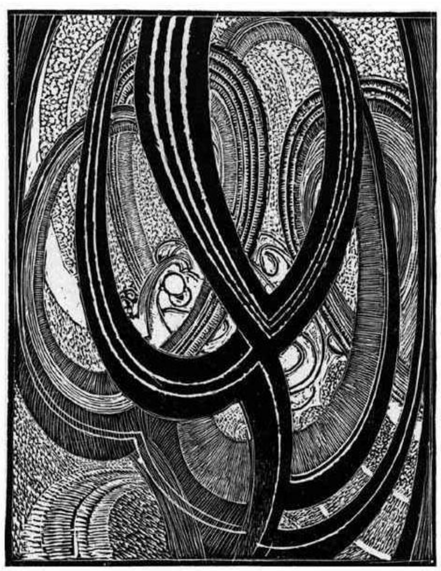


Catalogue from an Exhibition (1924)
František Kupka (1871-1957)

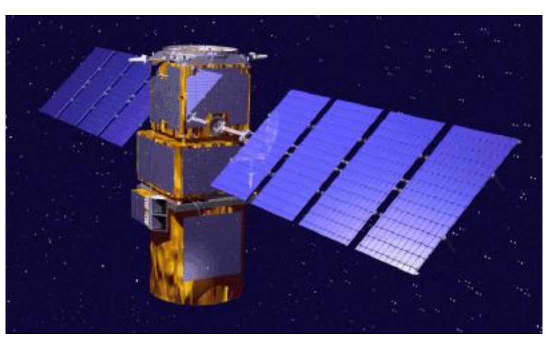
November 2003

Fixed-order robust design: a difficult problem

In this last part of course, we study robust stabilization with a fixed-order controller affected by parametric uncertainty

A difficult problem in general because
- fixed-order controller means non-convexity of the design space
- parametric uncertainty means highly structured uncertainty and NP-hard (combinatorial) complexity

In the literature: a lot of analysis results, but very few design results..



Low-order controllers are required in embedded devices

Once more: uncertainty models

We can consider several uncertainty models, in decreasing order of complexity:

• Polytopic uncertainty, where plant (numerator and/or denominator) polynomial $p(s)$ belongs to a polytope with given vertices

Ex. $p(s) = \lambda_1(s+1)^3 + \lambda_2(s+1)(s+2)^2$ with $\lambda_1 + \lambda_2 = 1$

Very general, but often leads to intractable analysis/design problems

• Interval uncertainty, where each of the plant parameters are assumed to vary independently in given intervals

Ex. $p(s) = 1 + [2,3]s + [-4,1]s^2 + s^3$

Leads to nice analysis results (Kharitonov) but intractable design problems

• Ellipsoidal uncertainty, also called rank-one (LFT), or norm-bounded

Ex. $p(s) = p_0 + p_1 s + p_2 s^2 + s^3$ with $3p_0^2 + p_0 p_1 + 2p_1^2 + p_2^2 \leq 1$

Less structured, but more realistic, naturally arises in the context of parameter estimation for process identification, ellipsoid = covariance matrix

Existing results

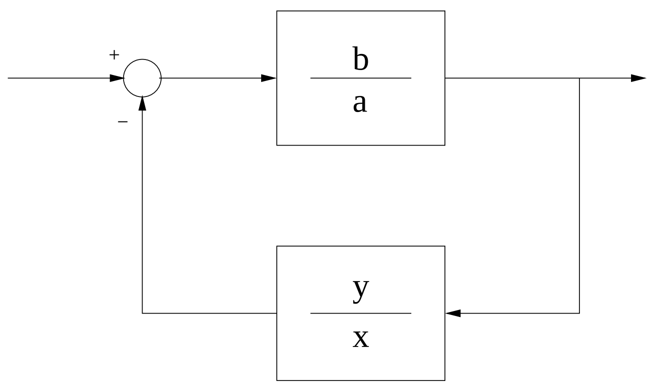
- H_∞ design methods, state-space techniques
- Critical direction (Nyquist), convex optimization with cutting-plane algorithms
- Infinite-dimensional Youla-Kučera parametrization generally leading to high-order controllers
- Use of linear programming with polytopic sufficient conditions for stability

In this course

- Use of polynomial techniques
- Use of LMI optimization
- Controllers of fixed (hence low) order

Nominal Pole placement

We consider the SISO feedback system



Closed-loop transfer function

$$\frac{bx}{ax + by}$$

In the absence of hidden modes (a and b coprime polynomials), pole placement amounts to finding polynomials x and y solving the Diophantine equation (from Diophantus of Alexandria 200-284)

$$ax + by = c$$

where c is a given closed-loop characteristic polynomial capturing the desired system poles

Pole placement: numerical aspects

The polynomial Diophantine equation

$$ax + by = c$$

is linear in unknowns x and y, and denoting

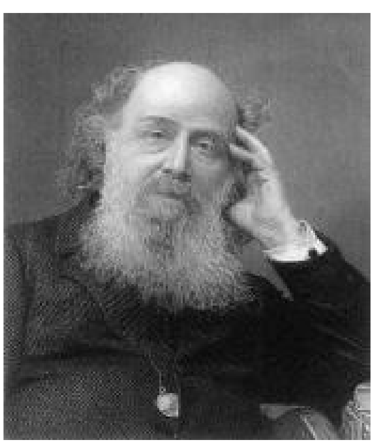
$$a(s) = a_0 + a_1 s + \cdots + a_{d_a} s^{d_a}$$
$$x(s) = x_0 + x_1 s + \cdots + x_{d_b} x^{d_b}$$

etc..

we can identify powers of the indeterminate s to build a linear system of equations

$$\left[\begin{array}{ccc|ccc} a_0 & & & b_0 & & \\ a_1 & \ddots & & b_1 & \ddots & \\ \vdots & & a_0 & \vdots & & b_0 \\ a_{d_a} & & a_1 & b_{d_b} & & b_1 \\ & \ddots & \vdots & & \ddots & \vdots \\ & & a_{d_a} & & & b_{d_b} \end{array}\right] \left[\begin{array}{c} x_0 \\ x_1 \\ \vdots \\ x_{d_x} \\ \hline y_0 \\ y_1 \\ \vdots \\ y_{d_y} \end{array}\right] = \left[\begin{array}{c} c_0 \\ c_1 \\ \vdots \\ c_{d_c} \end{array}\right]$$

The above matrix is called Sylvester matrix, it has a special Toeplitz banded structure that can be exploited when solving the equation



James J Sylvester
(1814 London - 1897 London)

Otto Toeplitz
(1881 Breslau - 1940 Jerusalem)

Pole placement for MIMO systems

Pole placement can be performed similarly for a plant left MFD

$$A^{-1}(s)B(s)$$

with a controller right MFD

$$Y(s)X^{-1}(s)$$

The Diophantine equation to be solved is now over polynomial matrices

$$A(s)X(s) + B(s)Y(s) = C(s)$$

and right hand-side matrix $C(s)$ captures information on invariant polynomials and eigenstructure

For example $C(s)$ may contain H_2 or H_∞ optimal dynamics (obtained with spectral factorization)

Robust pole placement

Now assume that the plant transfer function

$$\frac{b(q)}{a(q)}$$

contains some uncertain parameter q

The problem of robust pole placement will then consists in finding a controller

$$\frac{y}{x}$$

such that the uncertain closed-loop charact. polynomial

$$a(q)x + b(q)y = c(q)$$

is robustly stable

How can we find x, y to ensure robust stability of $c(q)$ for all admissible uncertainty q ?

Coefficients of c are linear in x and y, but we saw that stability conditions are non-linear and highly non-convex in c..

Robust pole placement

One possible remedy is a suitable

Convex approximation of the stability region

Then we can perform design with
- linear programming (polytopes)
- quadratic programming (spheres, ellipsoids)
- semidefinite programming (LMIs)

Complexity of design algorithm increases
Conservatism of control law decreases

Robust design via polytopic approximation

MIMO plant with right MFD

$$B(s)A^{-1}(s) = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix}^{-1}$$

with uncertainty in parameter

$$b \in [0.5,\ 1.5]$$

We seek a proper first order controller

$$X^{-1}(s)Y(s) = \begin{bmatrix} s+x_1 & x_2 \\ 0 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} y_1 s + y_2 & y_3 s + y_4 \\ 0 & y_5 \end{bmatrix}$$

assigning robustly the closed-loop polynomial matrix

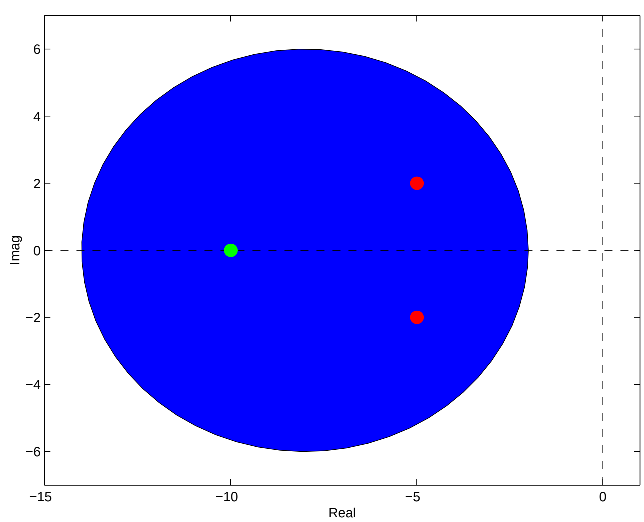
$$C(s) = \begin{bmatrix} s^2 + \alpha s + \beta & \delta(s) \\ 0 & s + \gamma \end{bmatrix}$$

whose coefficients live in the polytope

$$\begin{bmatrix} -14 & 1 & 0 \\ 16 & -2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} > \begin{bmatrix} -196 \\ 56 \\ -4 \\ 2 \\ -14 \end{bmatrix}$$

These specifications amounts to assigning the poles within the disk

$$|s + 8| < 6$$



Robust design via polytopic approximation (2)

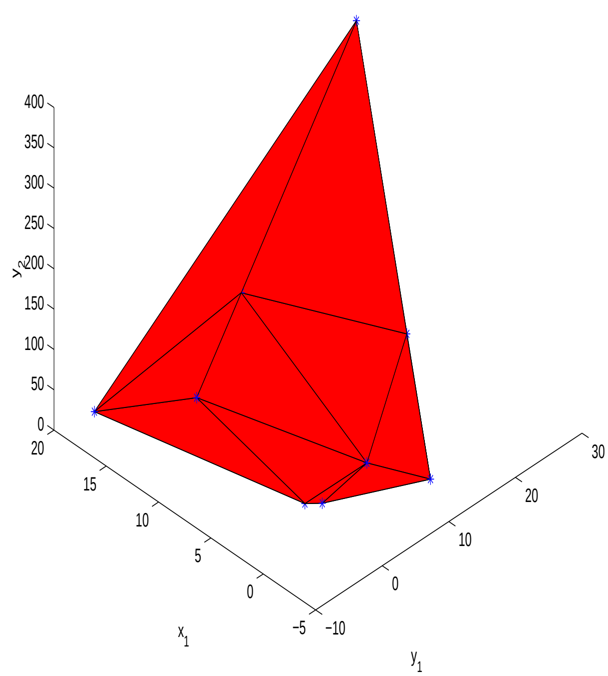
Equating powers of indeterminate s in the polynomial matrix Diophantine equation

$$X(s)A(s) + Y(s)B(s) = C(s)$$

we obtain the design inequalities

$$\begin{bmatrix} -13 & -7 & 0.5 \\ 14 & 8 & -1 \\ -1 & -1 & 0.5 \\ -13 & -21 & 1.5 \\ 14 & 24 & -3 \\ -1 & -3 & 1.5 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \\ y_2 \end{bmatrix} > \begin{bmatrix} -182 \\ 40 \\ -2 \\ -182 \\ 40 \\ -2 \end{bmatrix}$$

characterizing all parameters x_1, y_1 and y_2 of admissible robust controllers



Corresponding polytope with 9 vertices

Robust design via ellipsoidal approximation

Closed-loop characteristic polynomial

$$\begin{aligned} c(s) &= a(s)x(s) + b(s)y(s) \\ &= c_0 + c_1 s + \cdots + c_{d-1}s^{d-1} + s^d \end{aligned}$$

whose coefficients are given by the LSE

$$\begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_{d-1} \end{bmatrix} = \begin{bmatrix} y_0 & & x_0 & \\ y_1 & \ddots & x_1 & \ddots \\ \vdots & y_0 & \vdots & x_0 \\ y_{m-1} & y_1 & x_{m-1} & x_1 \\ y_m & \vdots & 1 & \vdots \\ & \ddots\, y_{m-1} & & \ddots\, x_{m-1} \\ & y_m & & 1 \end{bmatrix} \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{n-1} \\ a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ x_0 \\ x_1 \\ \vdots \\ x_{m-1} \end{bmatrix}$$

$$c = S(x, y)p + e(x)$$

It is assumed that uncertain plant parameters belong to the ellipsoid

$$E_p = \{p \,:\, (p - \bar{p})^\star P(p - \bar{p}) \leq 1\}$$

where $\bar{p}$ is a given nominal plant vector and P is a given positive definite covariance matrix

Robust control problem:

> Find controller coefficients x, y robustly stabilizing plant a, b subject to ellipsoidal uncertainty $p \in E_p$

Robust design via ellipsoidal approximation

From analysis results, recall that solving LMI

$$\begin{array}{ll} \max & \text{trace}\, Q_{11} \\ \text{s.t.} & \lambda H \succ I_d \otimes \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^\star & Q_{22} \end{bmatrix} + \begin{bmatrix} 0 & S_{21}^\star & \cdots & S_{d1}^\star \\ S_{21} & 0 & & S_{d2}^\star \\ \vdots & & \ddots & \vdots \\ S_{d1} & S_{d2} & \cdots & 0 \end{bmatrix} \\ & \lambda > 0, \quad Q_{11} \prec 0, \quad Q_{22} = 1, \quad S_{ij} = -S_{ij}^\star \end{array}$$

and denoting

$$\bar{q} = -Q_{11}^{-1} Q_{12}, \quad Q = -Q_{11}/(Q_{22} - Q_{12}^\star Q_{11}^{-1} Q_{12})$$

yields an ellisoid $E_q = \{q \, : \, (q-\bar{q})^\star Q (q-\bar{q}) \leq 1\}$ such that $q \in E_q$ implies $q(s)$ stable

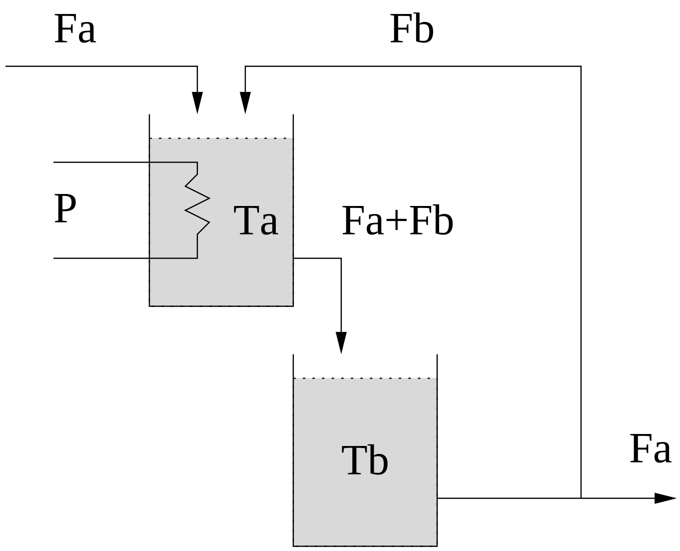
Using LMI conditions for inclusion of an ellipsoid into another we can show that finding x and y such that $q \in E_q$ for all $p \in E_p$ amounts to solving the LMI problem

$$\begin{bmatrix} Q^{-1} & S(x,y) & e(x) - \bar{q} \\ \star & tP & -tP\bar{p} \\ \star & \star & 1 + t(\bar{p}^\star P \bar{p} - 1) \end{bmatrix} \succeq 0$$

Coefficients x, y are such that the controller $y(s)/x(s)$ robustly stabilizes plant $b(s)/a(s)$

Ellipsoidal robust design: example

We consider the two mixing tanks arranged in cascade with recycle stream



The controller must be designed to maintain the temperature T_b of the second tank at a desired set point by manipulating the power P delivered by the heater located in the first tank

The only available measurement is temperature T_b

Ellipsoidal robust design: example (2)

The identification of the nominal plant model is carried out using a standard least-squares method

A second-order discrete-time model

$$p(z) = \frac{b_0 + b_1 z}{a_0 + a_1 z + z^2}$$

is obtained with nominal plant vector

$$\bar{p} = \begin{bmatrix} 0.0038 & 0.0028 & 0.2087 & -1.1871 \end{bmatrix}^\star$$

The positive definite matrix P characterizing the uncertainty ellipsoid

$$E_p = \{p \,:\, (p - \bar{p})^\star P (p - \bar{p}) \leq 1\}$$

is readily available as a by-product of the identification technique

$$P = 10^5 \begin{bmatrix} 2.4179 & 0.0568 & 0.0069 & 0 \\ 0.0568 & 2.4121 & 0.0045 & 0.0062 \\ 0.0069 & 0.0045 & 0.0015 & 0.0014 \\ 0 & 0.0062 & 0.0014 & 0.0015 \end{bmatrix}$$

Ellipsoidal robust design: example (3)

Solving the LMI analysis problem we obtain first an inner ellipsoidal approximation

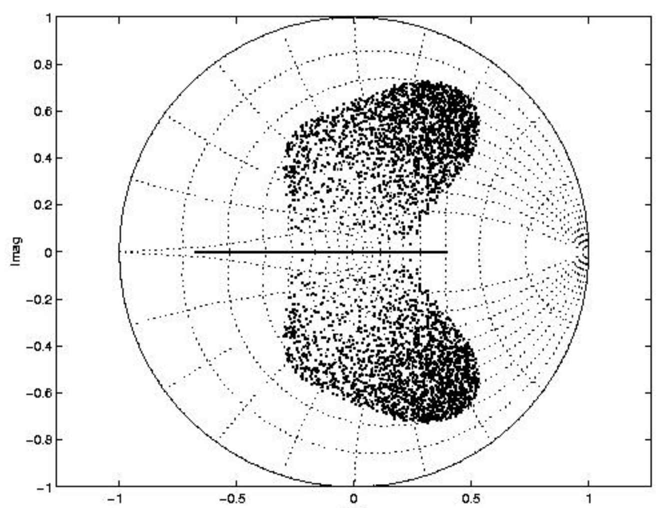
$$E_q = \{q \,:\, (q-\bar{q})^\star Q (q-\bar{q}) \leq 1\}$$

of the non-convex stability region, with

$$Q = \begin{bmatrix} 2.3378 & 0 & 0.5397 \\ 0 & 2.1368 & 0 \\ 0.5397 & 0 & 1.7552 \end{bmatrix} \quad \bar{q} = \begin{bmatrix} 0 \\ 0.1235 \\ 0 \end{bmatrix}$$

Then we solve the design LMI to obtain the first-order robustly stabilizing controller

$$\frac{y(z)}{x(z)} = \frac{0.3377+166.0z}{0.6212+z}$$



Robust closed-loop root-locus for random admissible ellipsoidal uncertainty

Strict positive realness

Let

$$\mathcal{D} = \{s \;:\; \begin{bmatrix} 1 \\ s \end{bmatrix}^\star \underbrace{\begin{bmatrix} a & b \\ b^\star & c \end{bmatrix}}_{H} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0\}$$

be a stability region in the complex plane where Hermitian matrix H has inertia $(1, 0, 1)$

Let $\partial\mathcal{D}$ denote the 1-D boundary of $\mathcal{D}$

Standard choices are

$$H = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad H = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

for the left half-plane and the unit disk resp.

We say that a rational matrix $G(s)$ is strictly positive real (SPR for short) when

$$\text{Re}\, G(s) \succ 0 \quad \text{for all} \quad s \in \partial\mathcal{D}$$

Stability and strict positive realness

Consider two square polynomial matrices of size n and degree d

$$\begin{aligned} N(s) &= N_0 + N_1 s + \cdots + N_d s^d \\ D(s) &= D_0 + D_1 s + \cdots + D_d s^d \end{aligned}$$

Polynomial matrix $N(s)$ is stable iff there is a stable polynomial $D(s)$ such that rational matrix $N(s)D^{-1}(s)$ is strictly positive real

Proof

From the definition of SPRness, $N(s)D^{-1}(s)$ SPR with $D(s)$ stable implies $N(s)$ stable

Conversely, if $N(s)$ is stable then the choice $D(s) = N(s)$ makes rational matrix $N(s)D^{-1}(s) = I$ obviously SPR

Thus we obtain a sufficient stability condition for a polynomial matrix

It turns out that this condition can be characterized by an LMI, as shown in the next slide

SPRness as an LMI

Let $N = [N_0 \quad N_1 \cdots N_d]$, $D = [D_0 \quad D_1 \cdots D_d]$ and

$$\Pi = \begin{bmatrix} I & & & 0 \\ & \ddots & & \vdots \\ & & I & 0 \\ 0 & I & & \\ \vdots & & \ddots & \\ 0 & & & I \end{bmatrix}$$

Given a stable $D(s)$, $N(s)$ ensures SPRness of $N(s)D^{-1}(s)$ iff there exists a matrix $P = P^\star$ of size dn such that

$$D^\star N + N^\star D - H(P) \succ 0$$

where

$$H(P) = \Pi^\star(S \otimes P)\Pi = \Pi^\star \begin{bmatrix} aP & bP \\ b^\star P & cP \end{bmatrix} \Pi$$

Proof

Similar to the proof on positivity of a polynomial, based on the decomposition as a sum-of-squares

LMI stability condition

Polynomial matrix $N(s)$ is stable iff there is a polynomial matrix $D(s)$ and a matrix P satisfying the LMI

$$\begin{array}{c} D^\star N + N^\star D - H(P) \succ 0 \\ P \succ 0 \end{array}$$

When fixing reference matrix $D(s)$ we obtain another sufficient LMI condition for stability of a polynomial matrix

- New convex inner approximation

of stability domain

- Shape described by an LMI
- More general than polytopes and ellipsoids

Polynomial $D(s)$ will be referred to as the

central polynomial

Second-degree discrete-time polynomial

Consider the discrete polynomial

$$n(z) = n_0 + n_1 z + z^2$$

We will study the shape of the LMI stability region for the following central polynomial

$$d(z) = z^2$$

We can show that non-strict feasibility of the LMI is equivalent to existence of a matrix P satisfying

$$\begin{array}{rcl} p_{00} + p_{11} + p_{22} & = & 1 \\ p_{10} + p_{01} + p_{21} + p_{22} & = & n_1 \\ p_{20} + p_{02} & = & n_0 \\ P & \succeq & 0 \end{array}$$

which is an LMI in the primal SDP form

Second-degree discrete-time polynomial (2)

Infeasibility of primal LMI is equivalent to the existence of a vector satisfying the dual LMI

$$y_0 + n_1 y_1 + n_0 y_2 < 0$$
$$Y = \begin{bmatrix} y_0 & y_1 & y_2 \\ y_1 & y_0 & y_1 \\ y_2 & y_1 & y_0 \end{bmatrix} \succeq 0$$

The eigenvalues of Toeplitz matrix Y are

$$y_0 - y_2 \quad \text{and} \quad (2y_0 + y_2 \pm \sqrt{y_2^2 + 8y_1^2})/2$$

so it is positive definite iff y_1 and y_2 belong to the interior of a bounded parabola scaled by y_0

The corresponding values of n_0 and n_1 belong to the interior of the envelope generated by the curve

$$(2\lambda_2 - 1)n_0 + (2\lambda_1 - 1)\sqrt{\lambda_2} n_1 + 1 > 0 \quad 0 \leq \lambda_i \leq 1$$

Second-degree discrete-time polynomial (3)

The implicit equation of the envelope is

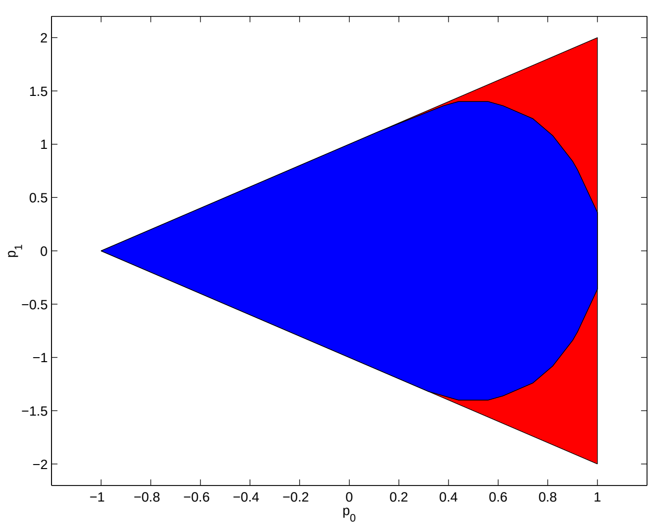
$$(2n_0 - 1)^2 + (\frac{\sqrt{2}}{2}n_1)^2 = 1$$

a scaled circle

The LMI stability region is then the union of the interior of the circle with the interior of the triangle delimited by the two lines

$$n_0 \pm n_1 + 1 = 0$$

tangent to the circle, with vertices $[-1,\ 0]$, $[1/3,\ 4/3]$ and $[1/3,\ -4/3]$



Application to robust stability analysis

Assume that $N(s,\lambda)$ is a polynomial matrix with multi-linear dependence in a parameter vector λ belonging to a polytope Λ

Denote by $N_i(s)$ the vertices obtained by enumerating each vertex in Λ

Polytopic polynomial matrix $N(s,\lambda)$ is robustly stable if there exists a matrix D and matrices P_i satisfying the LMI

$$\begin{array}{c} D^\star N_i + N_i^\star D - H(P_i) \succ 0 \\ P_i \succ 0 \quad \forall i \end{array}$$

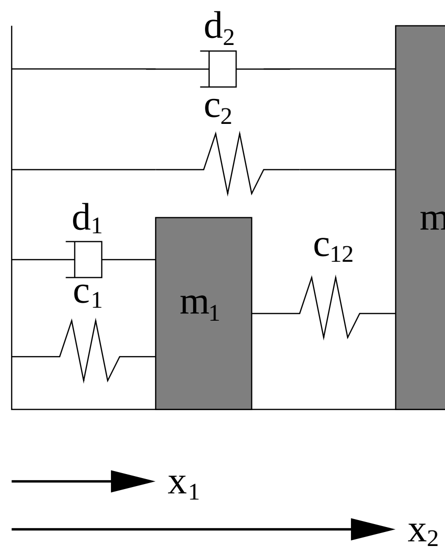
Proof

Since the LMI is linear in D - matrix of coefficients of polynomial matrix $D(s)$ - it is enough to check the vertices to prove stability in the whole polytope

Robust stability of polynomial matrices

Example

Consider the following mechanical system



It is described by the polynomial MFD

$$\begin{bmatrix} m_1 s^2 + d_1 s + c_1 + c_{12} & -c_{12} \\ -c_{12} & m_2 s^2 + d_2 s + c_2 + c_{12} \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ u(s) \end{bmatrix}$$

System parameters $\lambda = [\, m_1 \quad d_1 \quad c_1 \quad m_2 \quad d_2 \quad c_2 \,]$ belong to the uncertainty hyper-rectangle

$\Lambda = [1,\, 3] \times [0.5,\, 2] \times [1,\, 2] \times [2,\, 5] \times [0.5,\, 2] \times [2,\, 4]$

and we set $c_{12} = 1$

This mechanical system is passive so it must be open-loop stable (when $u(s) = 0$) independently of the values of the masses, springs, and dampers

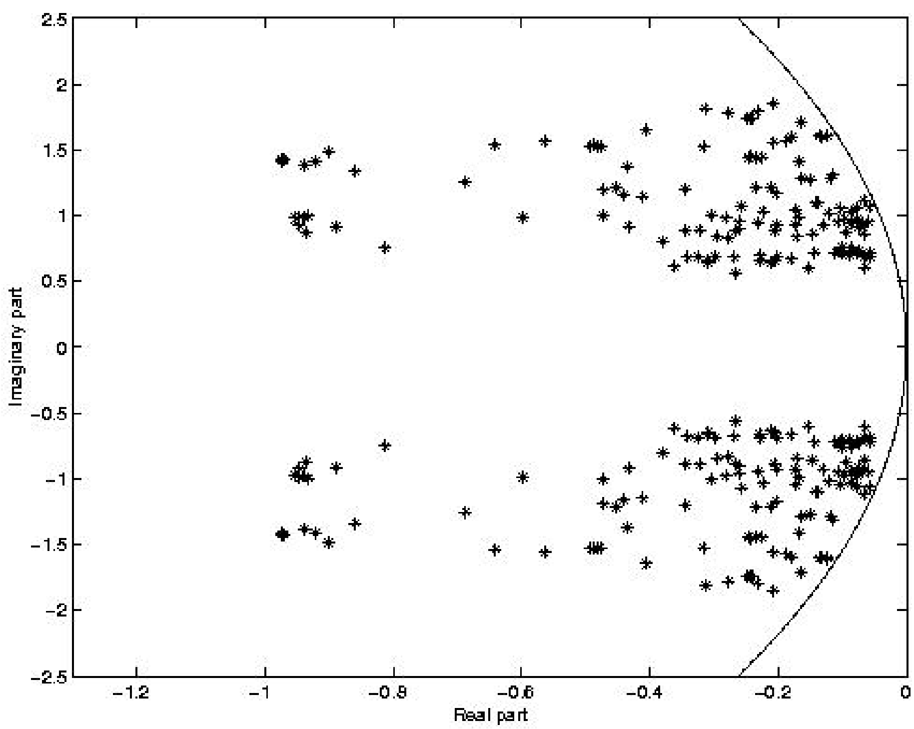
Robust stability of polynomial matrices

However, it is a non-trivial task to know whether the open-loop system is robustly D-stable in some stability region $\mathcal{D}$ ensuring a certain damping. Here we choose the disk of radius 12 centered at -12

$$\mathcal{D} = \{s : (s + 12)^2 < 12^2\}$$

The robust stability analysis problem amounts to assessing whether the second degree polynomial matrix in the MFD has its zeros in $\mathcal{D}$ for all admissible uncertainty in a polytope with $m = 2^6 = 64$ vertices

LMI problem is feasible – vertex zeros shown below



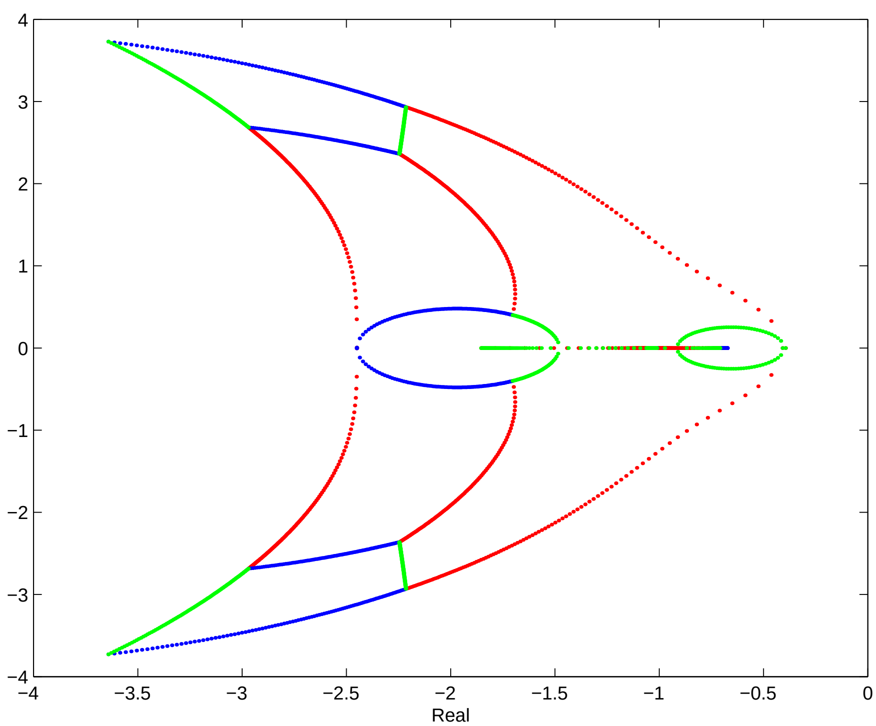
Polytope of polynomials

We can also check robust stability of polytopes of polynomials without using the edge theorem or the graphical value set

Example
Continuous-time polytope of degree 3 with 3 vertices

$$\begin{aligned} n_1(s) &= 28.3820 + 34.7667s + 8.3273s^2 + s^3 \\ n_2(s) &= 0.2985 + 1.6491s + 2.6567s^2 + s^3 \\ n_3(s) &= 4.0421 + 9.3039s + 5.5741s^2 + s^3 \end{aligned}$$

The LMI problem is feasible, so the polytope is robustly stable – see robust root locus below



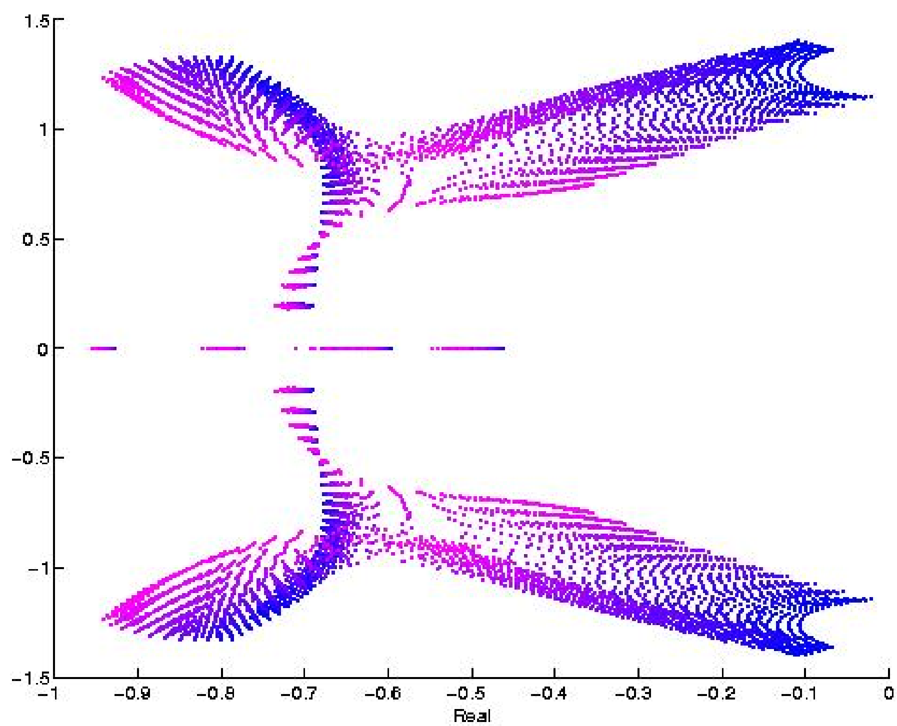
Interval polynomial matrices

Similarly, we can assess robust stability of interval polynomial matrices, an NP-hard problem in general

Example
Continuous-time interval polynomial matrix of degree 2 with $2^3 = 8$ vertices

$$\begin{bmatrix} [7.7 - 2.3s + 4.3s^2, \; 3.7 + 2.7s + 4.3s^2] & [-3.1 - 6s - 2.2s^2, \; -4.1 - 7s - 2.2s^2] \\ 3.6 + 6.4s + 4.3s^2 & [3.2 + 11s + 8.2s^2, \; 16 + 12s + 8.2s^2] \end{bmatrix}$$

LMI is feasible so the matrix is robustly stable
See robust root locus below



State-space systems

One advantage of our approach is that state-space results can be obtained as simple by-products, since stability of a constant matrix A is equivalent to stability of the pencil matrix

$$N(s) = sI - A$$

Matrix A is stable iff there exists a matrix F and a matrix P solving the LMI

$$\begin{bmatrix} F^\star A + A^\star F - aP & -A^\star - F^\star - bP \\ -A - F - b^\star P & 2I - cP \end{bmatrix} \succ 0$$
$$P \succ 0$$

Proof

Just take $D(s) = sI - F$ and notice that the LMI can be also written more explicitly as

$$\begin{bmatrix} -F^\star \\ I \end{bmatrix} \begin{bmatrix} -A & I \end{bmatrix} + \begin{bmatrix} -A^\star \\ I \end{bmatrix} \begin{bmatrix} -F & I \end{bmatrix} - \begin{bmatrix} aP & bP \\ b^\star P & cP \end{bmatrix} > 0$$

Robust stability of state-space systems

We recover the new LMI stability conditions obtained by Geromel and de Oliveira in 1999

Nice decoupling between Lyapunov matrix P and additional variable F allows for construction of parameter-dependent Lyapunov matrix

Assume that uncertain matrix $A(\lambda)$ has multilinear dependence on polytopic uncertain parameter λ and denote by A_i the corresponding vertices

Matrix $A(\lambda)$ is robustly stable if there exists a matrix F and matrices P_i solving the LMI

$$\left[\begin{array}{cc} F^\star A_i + A_i^\star F - aP_i & -A_i^\star - F^\star - bP_i \\ -A_i - F - b^\star P_i & 2I - cP_i \end{array}\right] \succ 0$$
$$P_i \succ 0 \quad \forall i$$

Proof

Consider the parameter-dependent Lyapunov matrix $P(\lambda)$ built from vertices P_i

Robust design

Assume now that system matrix $C(s,\lambda)$ comes from a polynomial Diophantine equation

$$C(s,\lambda) = A(s,\lambda)X(s) + B(s,\lambda)Y(s)$$

where system matrices A and B are subject to multi-linear polytopic uncertainty λ

In order to ensure robust SPRness of the rational matrix $D^{-1}(s)C(s,\lambda)$ central polynomial $D(s)$ must be close to the nominal closed-loop matrix, such as

$$D(s) = C(s,\lambda_0)$$

where λ_0 is the nominal parameter vector

A sensible simple choice of $D(s)$ is therefore the nominal closed-loop denominator polynomial matrix, obtained by any standard design method (pole assignment, LQ, H_∞)

Reactor

Consider the stirred tank reactor model described by non-linear state-space equations

$$
\begin{aligned}
\xi_1 &= (\xi_2 + 0.5)\exp(E\xi_1/(\xi_1+2)) - (2+u)(\xi_1+0.25) \\
\xi_2 &= 0.5 - \xi_2 - (\xi_2+0.5)\exp(E\xi_1/(\xi_1+2))
\end{aligned}
$$

where E is a parameter related to the activation energy

During the life of the reactor, some representative values of E are 20, 25 and 30

Using only ξ_1 for feedback we obtain a polytopic system with 3 linearized vertex transfer functions

$$
\begin{aligned}
b_1(s)/a_1(s) &= (0.5 - 0.25s)/(11 - 5s + s^2) \\
b_2(s)/a_2(s) &= (-0.5 - 0.25s)/(-2.25 - 2.25s + s^2) \\
b_3(s)/a_3(s) &= (-0.5 - 0.25s)/(-3.5 - 3.5s + s^2).
\end{aligned}
$$

Our objective is to stabilize the whole polytopic system with a single static output feedback controller $y(s)/x(s)$

Reactor

For the choice

$$d(s) = (s+1)^2$$

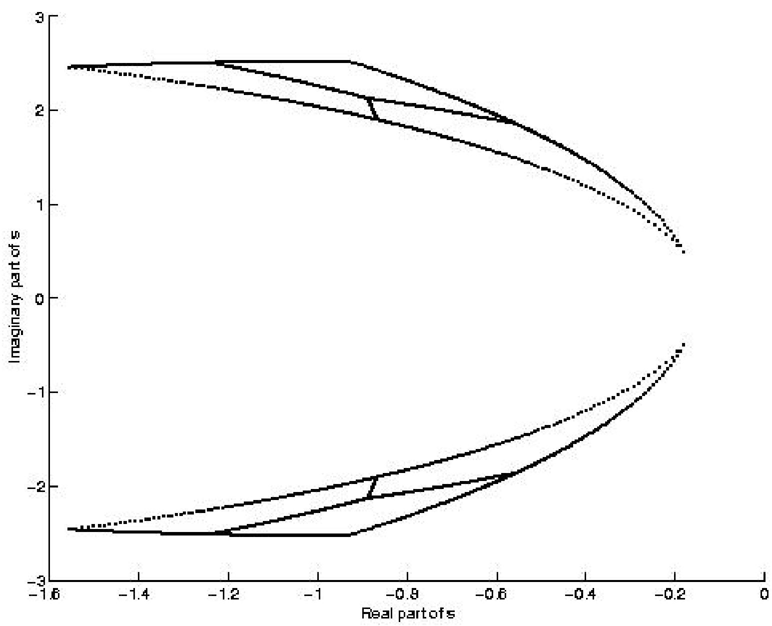
as a central polynomial, solving the LMI yields the robustly stabilizing controller

$$y(s)/x(s) = k = -21.4409$$

With the eigenvalue criterion we can check that k is simultaneous stabilizing iff

$$-22 < k < -20$$

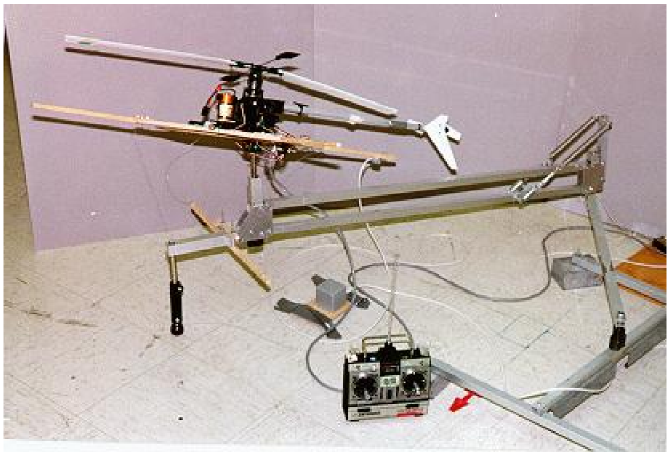
Note however that the problem solved here is more difficult since we must stabilize the whole polytope, not only its vertices



Edges of root locus

Helicopter

Consider a helicopter toy used in a university lab



whose simplified linearized model is given by the interval transfer function

$$\frac{[20,\ 60]}{s(0.1s+1)(s+[0,\ 1])}$$

We are seeking a second-order controller $y(s)/x(s)$ robustly stabilizing the interval plant

Using some design method we obtain a nominally stabilizing controller (for the central parameters 40 and 0.5)

$$\frac{y^0}{x^0} = \frac{2+2.2s+2.2s^2}{10s+s^2}$$

yielding the central polynomial

$$d = a(q^0)x^0 + b(q^0)y^0 = 80 + 88s + 93s^2 + 11s^3 + 2.05s^4 + 0.10s^5$$

Solving the LMI we get the robustly stabilizing controller

$$\frac{y}{x} = \frac{0.9876 + 0.3045s + 1.5747s^2}{1.6136 + 6.2396s + s^2}$$

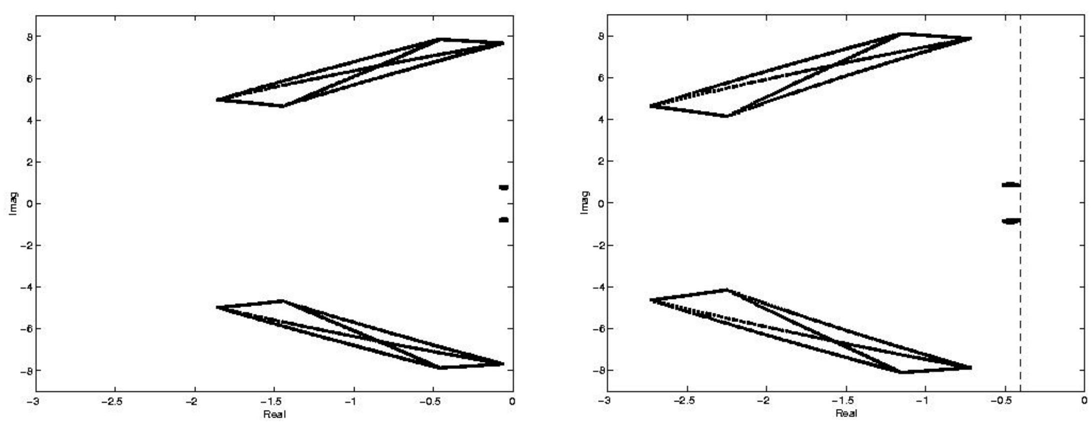
Helicopter

In our experiments we attempt to minimize the Euclidean norm of the vector of controller coefficients (this is still an LMI problem, see previous course)

We obtain a controller of norm 6.79, but from the root locus edges (see bottom left) stability is ensured only marginally, so we shift the stability region to the left

$$\mathcal{S} = \{s : \mathrm{Re}\, s < -0.4\} \quad S = \begin{bmatrix} 0.8 & 1 \\ 1 & 0 \end{bmatrix}$$

Solving the new LMI problem with the stability margin, we obtain a controller of norm 11.9 greater than the previous one, since more effort is needed to shift the poles farther from the imaginary axis (see bottom right)



Without stability margin With stability margin

Edges of robust root locii

Oblique wing aircraft

We consider the model of an experimental oblique wing aircraft

The linearized transfer function

$$\frac{[90,\ 166] + [54,\ 74]s}{[-0.1,\ 0.1] + [30.1,\ 33.9]s + [50.4,\ 80.8]s^2 + [2.8,\ 4.6]s^3 + s^4}$$

features 6 uncertain parameters, and must be stabilized with a PI controller

$$\frac{y(s)}{x(s)} = K_p + \frac{K_i}{s}$$

One can check easily that the choice $K_p = 1$ and $K_i = 1$ stabilizes the vertex plant

$$\frac{90 + 54s}{-0.1 + 30s + 50s^2 + 2.8s^3 + s^4}$$

Oblique wing aircraft (2)

With the choice

$$d(s) = sa_1(s) + (1+s)b_1(s)$$

as the central polynomial, the LMI problem is infeasible

But when considering another vertex plant

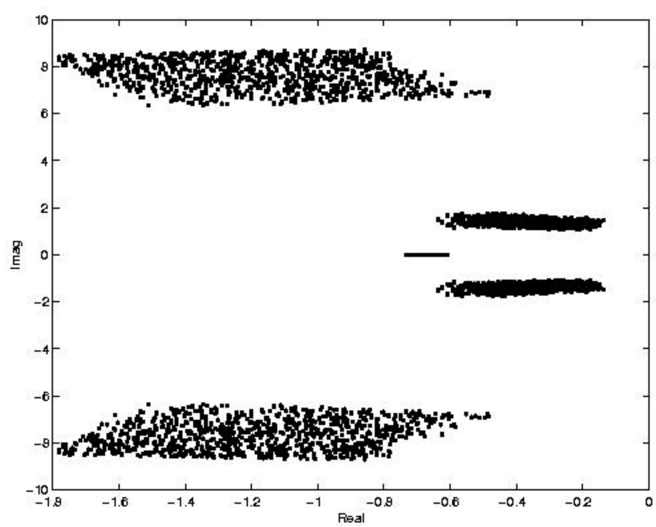
$$\frac{b_2(s)}{a_2(s)} = \frac{90+54s}{-0.1+30s+50s^2+4.6s^3+s^4}$$

the choice

$$d(s) = sa_2(s) + (1+s)b_2(s)$$

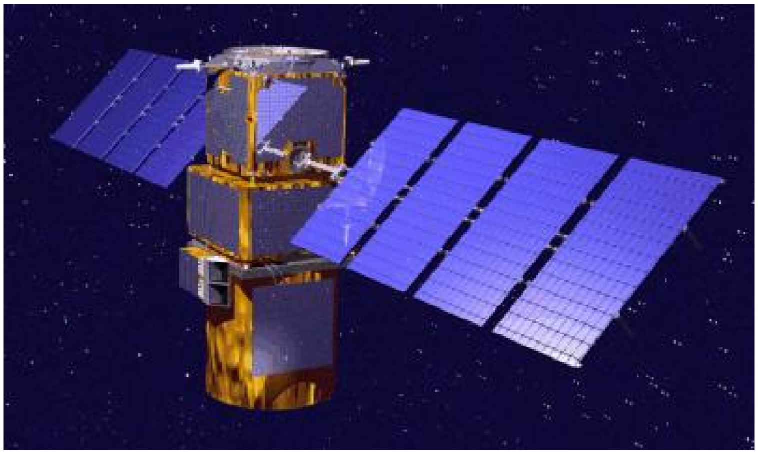
now proves successful, the LMI is solved and we obtain the robustly stabilizing PI controller

$$\frac{y(s)}{x(s)} = 0.8634 + \frac{0.6454}{s}$$



Satellite

We consider a satellite control problem where the satellite model is two masses with the same inertia connected by a spring with torque constant q_1 and viscous damping constant q_2



The transfer function between the control torque and the satellite angle is given by

$$\frac{b(s,q)}{a(s,q)} = \frac{q_1 + q_2 s + s^2}{s^2(2q_1 + 2q_2 s + s^2)}$$

where uncertain parameters are assumed to vary within the bounds

$$q_1 \in [0.09,\ 4],\ q_2 \in [0.04\sqrt{q_1}10,\ 0.2\sqrt{q_2}10]$$

With the choice

$$d(s) = (s+1)^6$$

as a central polynomial, the LMI design method cannot stabilize the whole polytope

Satellite (2)

However we can stabilize each vertex $a_i(s)$, $b_i(s)$ individually with controller polynomials $x_i(s)$, $y_i(s)$

The roots of the corresponding closed-loop polynomials $c_i(s) = a_i(s)x_i(s) + b_i(s)y_i(s)$ are given below

i	roots of $c_i(s)$
1	$-0.4520 \pm j1.8129$, $-0.0365 \pm j0.8954$, $-0.0019 \pm j0.2533$
2	$-0.6161 \pm j1.3829$, $-0.2190 \pm j0.6745$, $-0.0237 \pm j0.2136$
3	-1.0484, $-0.0917 \pm j2.1598$, $-0.0425 \pm j0.5019$, -0.0004
4	$-0.2220 \pm j2.0695$, $-0.1695 \pm j0.5354$, $-0.0709 \pm j0.0045$

The third vertex has poles nearest to the imaginary axis, and with the choice

$$d(s) = c_3(s)$$

as a central polynomial, the LMI is found feasible and we obtain the robustly stabilizing controller

$$\frac{y(s)}{x(s)} = \frac{0.0181 + 0.0170s - 1.4048s^2}{3.6082 + 1.0237s + s^2}$$

Sometimes it can be tricky to be find the central polynomial (provided one exists = the system is robustly stabilizable)..

Robot

We consider the problem of designing a robust controller for the approximate ARMAX model of a PUMA robotic disk grinding process

From the results of identification and because of the nonlinearity of the robot, the coefficients of the numerator of the plant transfer function change for different positions of the robot arm. We consider variations of up to 20% around the nominal value of the parameters

The fourth-order discrete-time model is given by

$$\frac{b(z^{-1},q)}{a(z^{-1},q)} = \frac{\begin{pmatrix} (0.0257+q_1)+(-0.0764+q_2)z^{-1} \\ +(-0.1619+q_3)z^{-2}+(-0.1688+q_4)z^{-3} \end{pmatrix}}{1-1.914z^{-1}+1.779z^{-2}-1.0265z^{-3}+0.2508z^{-4}}$$

where

$|q_1| \leq 0.00514, |q_2| \leq 0.01528, |q_3| \leq 0.03238, |q_4| \leq 0.03376$

Robot

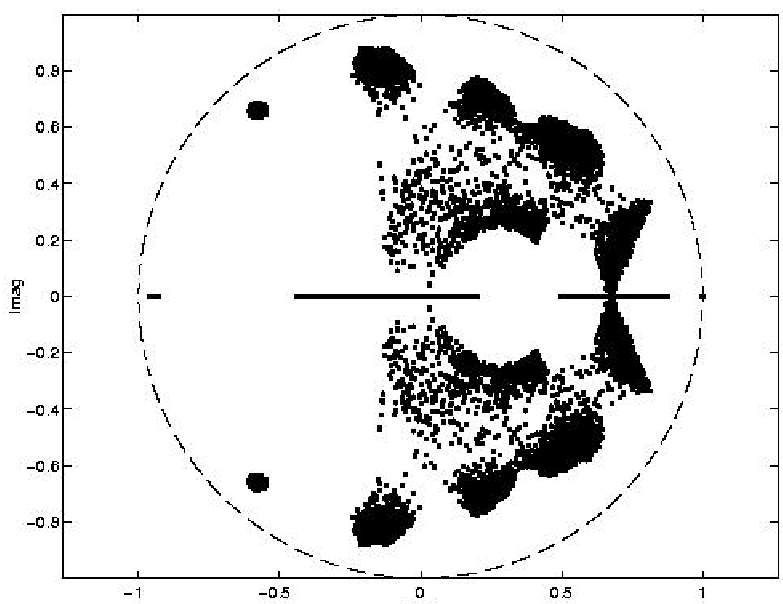
Closed-loop polynomial

$$d(z,q) = z^{12}[(1-z^{-1})a(z^{-1},q)x(z^{-1}) + z^{-5}b(z^{-1},q)y(z^{-1})]$$

where the term $1 - z^{-1}$ is introduced in the controller denominator to maintain the steady state error to zero when parameters are changed

With the input central polynomial $d(z) = z^{19}$ the LMI returns the seventh-order robust controller

$$\frac{y(z^{-1})}{x(z^{-1})} = \frac{\begin{pmatrix} -0.2863 + 0.2928z^{-1} + 0.0221z^{-2} \\ -0.1558z^{-3} + 0.0809z^{-4} + 0.1420z^{-5} \\ -0.1254z^{-6} + 0.0281z^{-7} \end{pmatrix}}{\begin{pmatrix} 1 + 1.1590z^{-1} + 0.9428z^{-2} \\ +0.4996z^{-3} + 0.3044z^{-4} + 0.4881z^{-5} \\ +0.4003z^{-6} + 0.3660z^{-7} \end{pmatrix}}$$



Second-order systems

Second-order linear system

$$\begin{aligned}(A_0 + A_1 s + A_2 s^2)x &= Bu \\ y &= Cx\end{aligned}$$

to be controlled by PD output-feedback controller

$$u = -(F_0 + F_1 s)y$$

Applications: large flexible space structures, earthquake engineering, mechanical multi-body systems, damped gyroscopic systems, robotics control, vibration in structural dynamics, flows in fluid mechanics, electrical circuits

320m long Millenium footbridge over river Thames in London

PD controller

Closed-loop system behavior captured by zeros of quadratic polynomial matrix

$$N(s) = (A_0 + BF_0C) + (A_1 + BF_1C)s + A_2s^2$$

Zeros of $N(s)$ must be located in some stability region $\mathcal{D}$ characterized as before by matrix H

Uncertainty can affect A_0 (stiffness)
A_1 (damping) and A_2 (mass)

Given A_0, A_1, A_2, B, C
find F_0, F_1
ensuring robust pole placement

Robust LMI stability condition

- Norm-bounded (unstructured) uncertainty

$$N(s) + \Delta M(s) \quad \sigma_{\max}(\Delta) \leq \delta$$

LMI robust stability condition on $N(s)$

$$\begin{bmatrix} D^\star N + N^\star D - H(P) - \gamma D^\star D & \delta M^\star \\ \delta M & \gamma I \end{bmatrix} \succ 0$$

- Polytopic (structured) uncertainty

$$N(s) = \sum_i \lambda^i N^i(s) \quad \sum_i \lambda^i = 1 \quad \lambda^i \geq 0$$

Vertex LMI robust stability conditions:

$$DN^i + (N^i)^\star D - H(P^i) \succ 0, \quad i = 1, 2, \ldots$$

Parameter-dependent Lyapunov matrix
$P(\lambda) = \sum_i \lambda^i P^i$

Robust design

Once central polynomial matrix $D(s)$ is fixed, robust stability condition is LMI in $N(s)$, so extension to design is straightforward

Easy incorporation of structural constraints on controller coefficient matrices F_0, F_1:

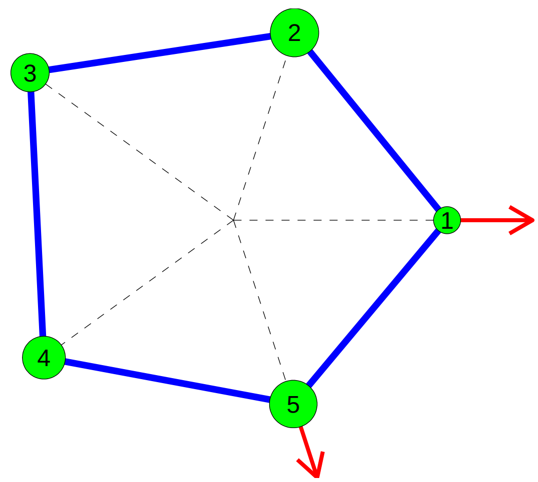
- minimization of 2-norm (SOCP)
- enforcing some entries to zero (LP)
- maximization of uncertainty radius (SDP)

Key point is choice of central polynomial matrix

Good policy: set $D(s)$ to some nominal system matrix obtained by some standard design method (pole placement, LQ, H_2 or H_∞), then try to optimize around $D(s)$

Example: five masses

Five masses linked by elastic springs
controlled by two external forces



$$A_0 = \begin{bmatrix} 2.565 & 1.080 & 0 & 0 & 1.089 \\ 0.6038 & 0.8206 & 0.4766 & 0 & 0 \\ 0 & 0.6009 & 1.504 & 0.4808 & 0 \\ 0 & 0 & 0.4300 & 1.114 & 0.5131 \\ 0.6190 & 0 & 0 & 0.4626 & 0.8352 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1.964 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1.116 & 0 \end{bmatrix} \qquad \begin{array}{c} A_1 = 0_5 \\ A_2 = I_5 \\ C = I_5 \end{array}$$

Purely imaginary open-loop poles $\pm i1.783$, $\pm i1.380$, $\pm i1.145$, $\pm i0.5675$ and $\pm i0.3507$

Nominal PD controller F_0^0, F_1^0 obtained with LQ design
Resulting central polynomial matrix

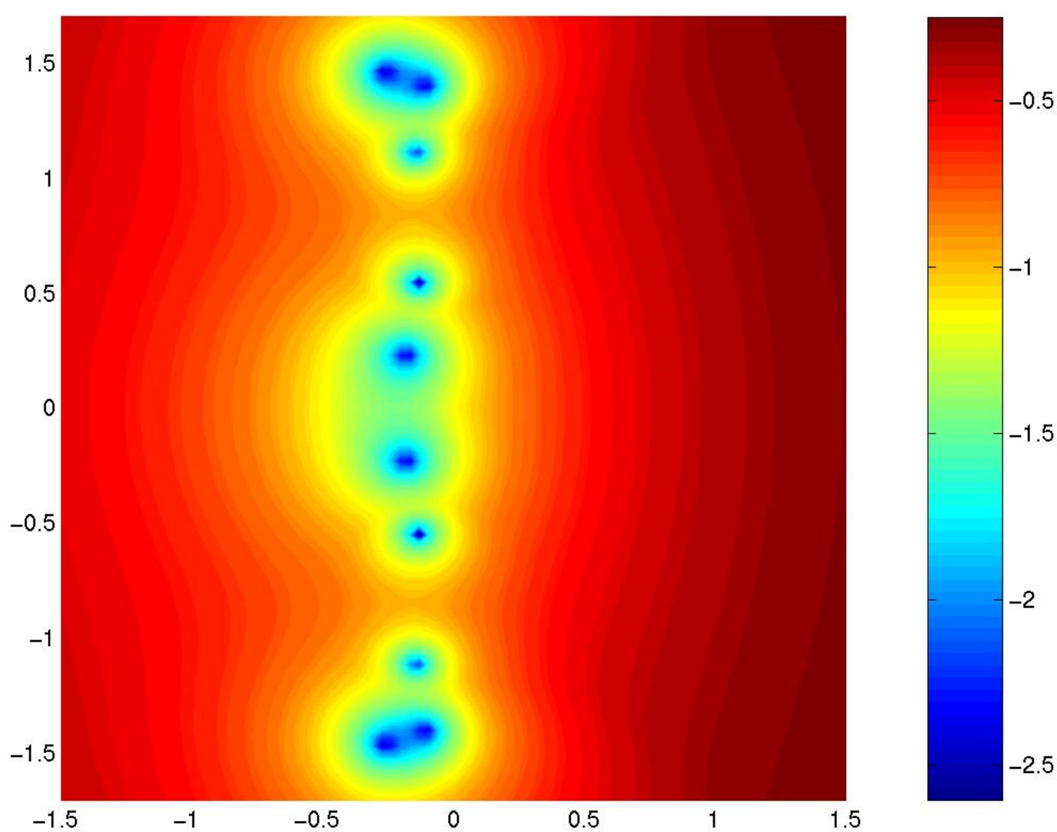
$$D(s) = (A_0 + BF_0^0C) + (A_1 + BF_1^0C)s + A_2s^2$$

Stability region $\mathcal{D} = \{s : \mathrm{Re}\, s < -0.1\}$

Five mass example (2)

Minimizing the norm of feedback matrices F_0, F_1 over the design LMI yields

$$\|[F_0 \quad F_1]\| = 0.7537 < \|[F_0^0 \quad F_1^0]\| = 1.462$$



Closed-loop pseudospectrum of the five masses example