Course on LMI optimization with applications in control

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Labs

1 LMI optimization

1.1 Minimum volume ellipsoid containing given points

Use Matlab function `ginput` to enter interactively a set of points in the plane. Write a YALMIP code to compute the minimum volume ellipsoid containing the points. For simplicity, instead of optimizing over the determinant of the positive definite matrix shaping the ellipsoid, optimize over its trace.

1.2 Constrained polynomial interpolation

Write a YALMIP code to compute a fourth-degree polynomial $p(x)$ vanishing at $x = 1$ and $x = 2$ and such that $p(x) \geq -1/2$ for all $x \in \mathbb{R}$.

1.3 Polynomial factorization

Write the primal-dual LMI formulation of the polynomial optimization problem

$$\begin{array}{ll} \min & p(x) = x^4 - 2x^3 - 3x^2 + 4x + 4 \\ \text{s.t.} & x \in \mathbb{R}. \end{array}$$

Solve the dual (moment) LMI problem with YALMIP. Is polynomial $p(x)$ globally nonnegative ?

From the primal (sum-of-squares) LMI derive a reduced LMI formulation with only one decision variable. Show that the reduced LMI can be solved explicitly as an eigenvalue problem.

Derive a rank-one factorization of the primal LMI matrix, and then a square-root factor $q(x)$ such that $p(x) = q^2(x)$.

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2 LMI in control

2.1 Lyapunov stability and duality

We want to know whether matrix

$$A = \begin{bmatrix} -1 & 2 \\ 3 & 4 \end{bmatrix}.$$

is continuous-time stable, i.e. if all its eigenvalues lie in the open left half-plane.

Use the theorem of alternatives to formulate and solve the LMI problem dual to Lyapunov's LMI. Derive the unstable eigenvector of matrix A from the rank-one decomposition of the dual matrices.

Same problem with

$$A = \begin{bmatrix} -1 & -2 \\ -3 & 4 \end{bmatrix}.$$

2.2 Quadratic and robust stability analysis

Check quadratic stability of the continuous-time system

$$\dot{x} = \left(\begin{bmatrix} -4 & 4 \\ -5 & 0 \end{bmatrix} + \delta \begin{bmatrix} -2 & 2 \\ -1 & 4 \end{bmatrix} \right) x$$

affected by one uncertain parameter $\delta \in \mathbb{R}$ such that $|\delta| \leq \mu$, where μ is given. Try $\mu = 0.5$, $\mu = 0.75$ and $\mu = 1$.

Check robust stability with an affine parameter-dependent Lyapunov function for the same system. Try $\mu = 1$, $\mu = 1.5$ and $\mu = 2$.

Graphically, determine the largest value of μ ensuring robust stability, referred to as the robust stability radius.

Finally, apply the eigenvalue criterion on the characteristic polynomial of the system to find numerically the robust stability radius.

2.3 Static output feedback design

We would like to solve the static output feedback design problem for the following data:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 13 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 5 & -1 \\ -1 & -1 & 0 \end{bmatrix}$$

with

$$K = \left[\begin{array}{cc} k_1 & k_2 \end{array}\right].$$

The characteristic polynomial is

$$\begin{array}{rcl} q(s,k) & = & \det\left(sI - A - BKC\right) \\ & = & k_2 + (-13 - 5k_1 + k_2)s + k_1 s^2 + s^3 \\ & = & q_0(k) + q_1(k)s + q_2(k)s^2 + q_3(k)s^3 \end{array}$$

The Hermite matrix corresponding to the left half-plane $\mathcal{D} = \{s \in \mathbb{C} : s + s^\star < 0\}$, after permutation of odd and even rows and columns, is given by

$$\begin{array}{rcl} H(k) & = & \left[\begin{array}{ccc} q_0q_1 & q_0q_3 & 0 \\ q_0q_3 & q_2q_3 & 0 \\ 0 & 0 & q_1q_2 - q_0q_3 \end{array}\right] \\ & = & \begin{bmatrix} -13k_2 - 5k_1k_2 + k_2^2 & k_2 & 0 \\ k_2 & k_1 & 0 \\ 0 & 0 & -13k_1 - k_2 - 5k_1^2 + k_1k_2 \end{bmatrix}. \end{array}$$

It is positive definite if and only if $k_1 > 0$, $-13-5k_1+k_2 > 0$ and $-13k_1-k_2-5k_1^2+k_1k_2 > 0$, which corresponds to the interior of a hyperbolic branch in the positive orthant.

Notice that the stability region in the parameter space k_1, k_2 is convex is this case. It is actually the convex branch of a hyperbola that can equivalently be described by the LMI

$$\left[\begin{array}{cc} -1 + k_1 & 1 \\ 1 & -1 - \frac{5}{18}k_1 + \frac{1}{18}k_2 \end{array}\right] \succ 0$$

which was not apparent by inspecting the original Hermite BMI. In other words, in this particular case, the SOF problem boils down to solving a convex LMI problem in the parameter space.

Write a YALMIP script to solve this LMI and find e.g. the minimum Euclidean norm static output feedback K.

Write a YALMIP script to solve the original Hermite BMI with PENBMI.