POLYNOMIAL METHODS FOR ROBUST CONTROL
PART II.7

LMI RELAXATIONS

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BMI - Bilinear Matrix Inequality

$$F(x) = F_0 + \sum_i x_i F_i + \sum_i \sum_j x_i x_j F_{ij} \succeq 0$$

Symmetric matrices F_i, F_{ij} given
Decision variables x_i

Often no quadratic terms x_i^2
in state-space control BMIs

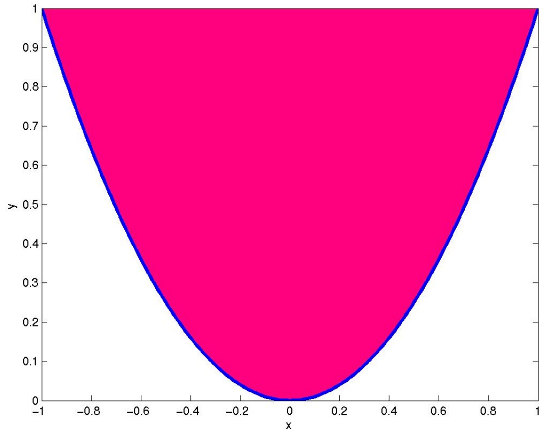
Contrary to LMIs, BMIs may have
non-convex feasible sets

In practice, non-convexity is bad news because:

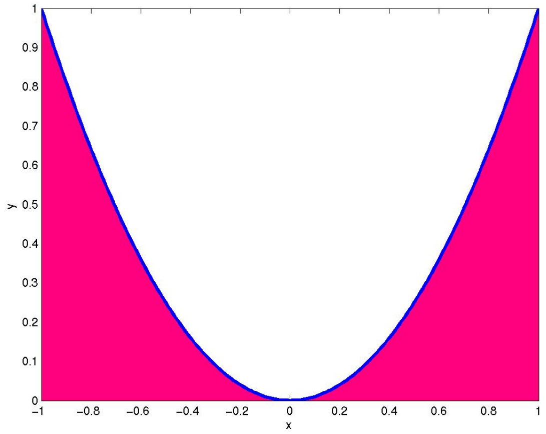
- expensive global opt algos
- no global convergence of local opt algos
- many local (or global) optimizers
- no nice duality results

But we will see that in some cases, suitable reformulations can help restoring convexity

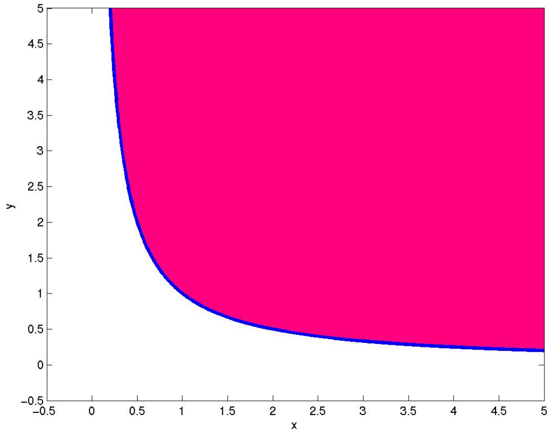
BMI or LMI ?



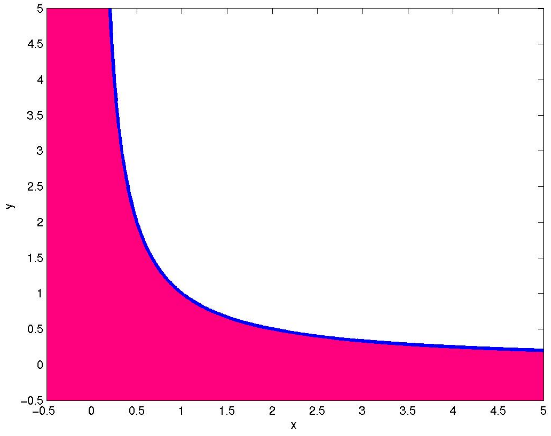
Convex LMI
$x^2 \leq y$



Nonconvex BMI
$x^2 \geq y$



Convex LMI
$xy \geq 1$



Nonconvex BMI
$xy \leq 1$

PMI - Polynomial Matrix Inequality

$$F(x) = \sum_\alpha x^\alpha F_\alpha \succeq 0$$

More general than BMI ?

By appropriate changes of variables any PMI can be written as a BMI

Example

$F(x) = F_0 + F_1 x_1 + F_{12} x_1 x_2^2 + F_{03} x_2^3$

can be written as the BMI

$F(x) = F_0 + F_1 x_1 + F_{12} x_1 x_3 + F_{03} x_2 x_3$

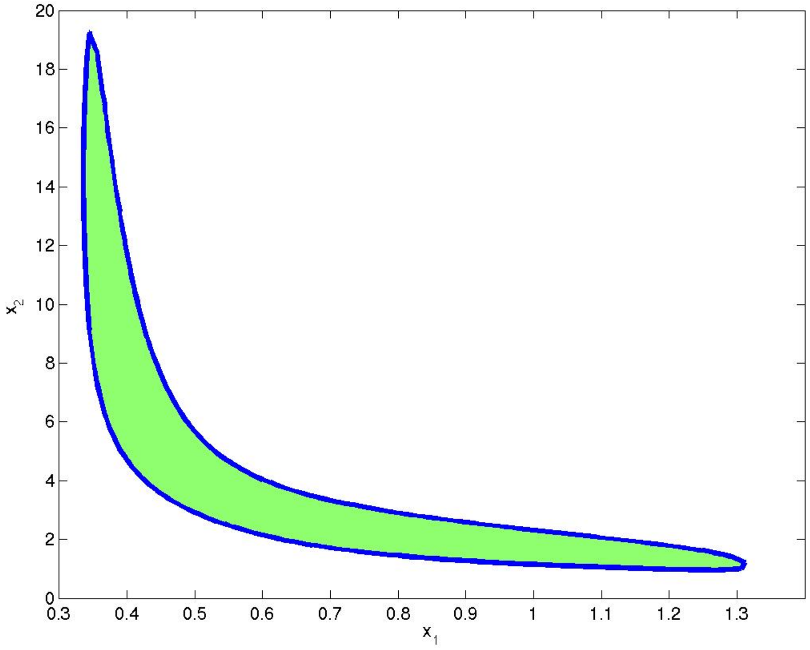
with lifting variable x_3 constrained by $x_3 = x_2^2$

PMIs generally do not occur
in state-space control problems

Later on we will see that PMIs can naturally occur in polynomial control problems

Example of a 2D BMI

$$F(x) = \begin{bmatrix} 10 & 0.5 & 2 \\ 0.5 & -4.5 & 0 \\ 2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} -9 & -0.5 & 0 \\ -0.5 & 0 & 3 \\ 0 & 3 & 1 \end{bmatrix} x_1 + \begin{bmatrix} 1.8 & 0.1 & 0.4 \\ 0.1 & -1.2 & 1 \\ 0.4 & 1 & 0 \end{bmatrix} x_2 + \begin{bmatrix} 0 & 0 & -2 \\ 0 & 5.5 & -3 \\ -2 & -3 & 0 \end{bmatrix} x_1 x_2 \succeq 0$$

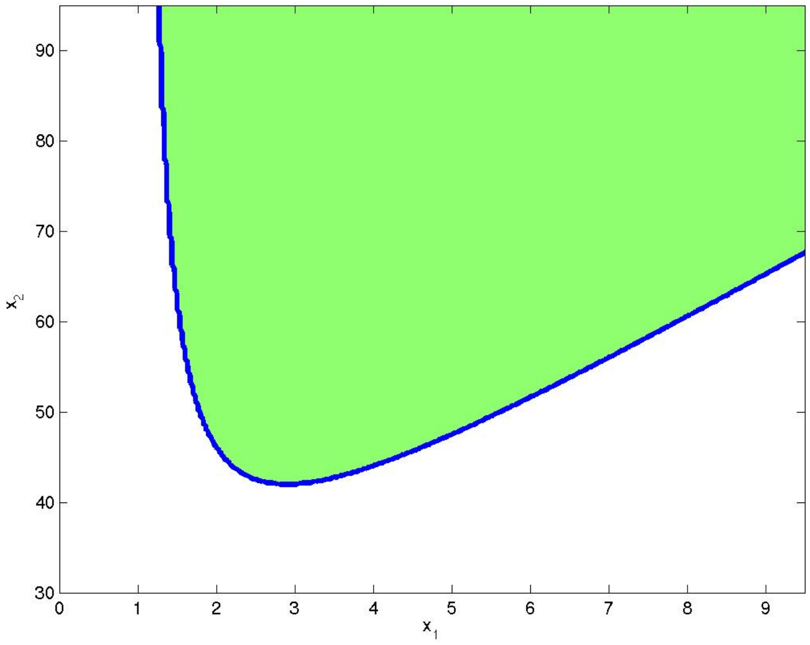


This boomerang is definitely non-convex..

Another example of a 2D BMI

$$\begin{bmatrix} x_2(-13-5x_1+x_2) & x_2 & 0 \\ x_2 & x_1 & 0 \\ 0 & 0 & x_1(-13-5x_1+x_2)-x_2 \end{bmatrix} \succ 0$$

coming from a static output feedback problem



Feasible set is apparently convex..

Can we detect or exploit convexity ?

Converting BMI into LMI

Sometimes a BMI problem can be reformulated as an LMI problem

For example the static output feedback BMI

$$\begin{bmatrix} x_2(-13-5x_1+x_2) & x_2 & 0 \\ x_2 & x_1 & 0 \\ 0 & 0 & x_1(-13-5x_1+x_2)-x_2 \end{bmatrix} \succ 0$$

can be reformulated as the LMI

$$\begin{bmatrix} -1+x_1 & 1 \\ 1 & -1-\frac{5}{18}x_1+\frac{1}{18}x_2 \end{bmatrix} \succ 0$$

Can we systematically detect whether such a reformulation is possible ?

Can we design a systematic reformulation algorithm ?

History of BMIs

Interest in BMIs originated in control
Mid 1990s, Safonov's team

Typical BMI: static output feedback

$$(A + BKC)^T P + P(A + BKC) \prec 0, \quad P \succ 0$$

More intricate BMI arise for reduced order controller design, H_2, H_∞ performance, see last part of the course

Main criticisms:
- too general
- no good algorithm

..in sharp contrast with LMIs..

Linearizing changes of variables

State-feedback design BMI

$$(A+BK)^T P + P(A+BK) \prec 0, \qquad P \succ 0$$

Non-convex in this original formulation

Using the change of variables

$$Q = P^{-1} \qquad Y = KP^{-1}$$

(more details later on) we obtain an
equivalent convex LMI

$$AQ + QA^T + BY + Y^T B \prec 0, \qquad Q \succ 0$$

Same trick works for dynamic output feedback
Technically more involved

BMI as a rank-one LMI

Defining liftings

$$x_{ij} = x_i x_j$$

the BMI

$$F_0 + \sum_i x_i F_i + \sum_i \sum_j x_i x_j F_{ij} \succeq 0$$

can be written as an LMI

$$F_0 + \textstyle\sum_i x_i F_i + \sum_i \sum_j x_{ij} F_{ij} \succeq 0$$

$$X = \begin{bmatrix} 1 & x_1 & x_2 & \\ x_1 & x_{11} & x_{12} & \\ x_2 & x_{12} & x_{22} & \\ & & & \ddots \end{bmatrix} \succeq 0$$

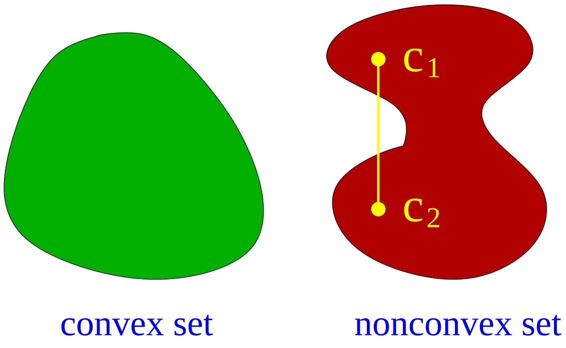
with an additional rank constraint

$$\boxed{\text{rank } X = 1}$$

All the non-convexity is concentrated in this rank constraint

Handling nonconvexity

We have seen that additional variables, or liftings can prove useful in describing convex sets with LMIs



But LMI are also frequently used to cope with non-convex sets and/or BMIs !

This chapter is dedicated to the joint use of

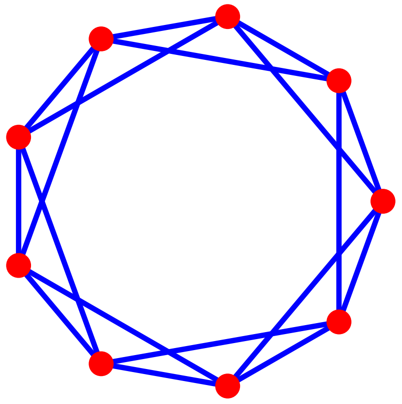
- convex LMI relaxations, and
- additional variables = liftings

Combinatorial optimization

Typical combinatorial optimization problem

$$\begin{array}{ll} \text{min} & x^T Q x \\ \text{s.t.} & x_i \in \{-1, 1\} \end{array}$$

Examples: MAXCUT, knapsack..



Antiweb AW_9^2 graph

Basic non-convex constraints

$$\boxed{x_i^2 = 1}$$

Exponential # of points = difficult problem

LMI relaxation

Basic idea..

For each i replace non-convex constraint

$$\boxed{x_i^2 = 1}$$

with relaxed convex constraint

$$x_i^2 \leq 1$$

which is an LMI constraint

$$\boxed{\begin{bmatrix} 1 & x_i \\ x_i & 1 \end{bmatrix} \succeq 0}$$

Not bad idea, but we can do better..

(Better) LMI relaxation

Replace all non-convex constraints

$$x_i^2 = 1, \quad i = 1, 2, \ldots, n$$

with relaxed LMI constraint

$$X = \begin{bmatrix} 1 & x_1 & x_2 & \cdots & x_n \\ x_1 & 1 & x_{12} & & x_{1n} \\ x_2 & x_{12} & 1 & & x_{2n} \\ \vdots & & & \ddots & \vdots \\ x_n & x_{1n} & x_{2n} & \cdots & 1 \end{bmatrix} \succeq 0$$

where x_{ij} are additional variables = liftings

Always less conservative than previous relaxation because $X \succeq 0$ implies

$$\begin{bmatrix} 1 & x_i \\ x_i & 1 \end{bmatrix} \succeq 0$$

for each $i = 1, 2, \ldots, n$

Rank constrained LMI

In the original problem

$$\begin{array}{rcll} g^\star & = & \text{min} & x^TQx \\ & & \text{s.t.} & x_i^2 = 1 \end{array}$$

let $X = xx^T$ and then

$$x^TQx = \operatorname{trace} Qxx^T = \operatorname{trace} QX$$

and

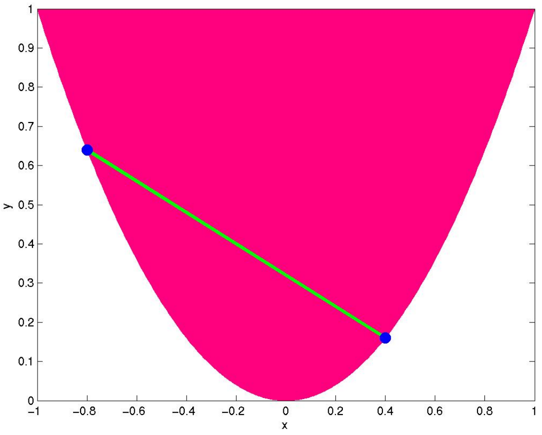
$$x_i^2 = X_{ii} = 1$$

so that the problem can be written as a rank constrained LMI

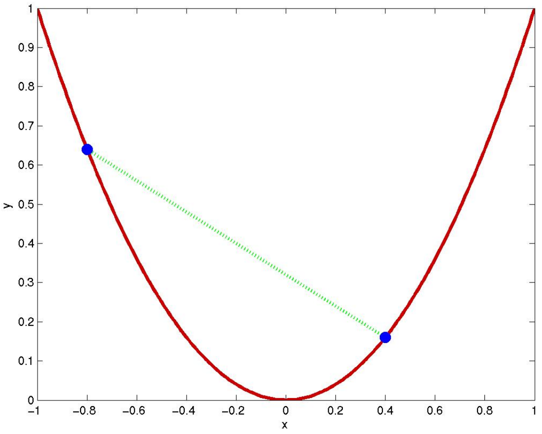
$$\boxed{\begin{array}{rcll} g^\star & = & \text{min} & \operatorname{trace} QX \\ & & \text{s.t.} & X_{ii} = 1 \\ & & & X \succeq 0 \\ & & & \operatorname{rank} X = 1 \end{array}}$$

Example of rank constrained LMI

$$X = \begin{bmatrix} y & x \\ x & 1 \end{bmatrix}$$



Convex set $X \succeq 0$ $(x^2 \leq y)$



Non-convex set $X \succeq 0$, rank $X = 1$ $(x^2 = y)$

Relaxing the rank constraint

All the nonconvexity is concentrated into the rank constraint, so we just drop it !

The obtained LMI relaxation is called Shor's relaxation

$$\begin{array}{rcll} p^\star & = & \min & \mathrm{trace}\, QX \\ & & \text{s.t.} & X_{ii} = 1 \\ & & & X \succeq 0 \end{array}$$

Naum Zuselevich Shor (Inst Cybernetics, Kiev) in the 1980s was among the first to recognize the relevance of this approach

Since the feasible set is relaxed = enlarged we get a lower bound for the original non-convex optimization problem

$$p^\star \leq g^\star$$

Shor's relaxation

Systematic approach: can be applied to general polynomial optimization problems

Example:

$$x_1^2 x_2 = x_1 \quad \left\{ \begin{array}{c} x_1^2 = x_3 \\ x_3 x_2 = x_1 \end{array} \right. \quad \left\{ \begin{array}{c} X_{11} = X_{30} \\ X_{32} = X_{10} \\ X \succeq 0 \\ \text{rank}\, X = 1 \end{array} \right. \quad \left\{ \begin{array}{c} X_{11} = X_{30} \\ X_{32} = X_{10} \\ X \succeq 0 \end{array} \right.$$

Algorithm:

- introduce lifting variables to reduce polynomials to quadratic and linear terms
- build the rank-one LMI problem
- solve the LMI problem by relaxing the non-convex rank constraint

Relaxed LMI via duality

Consider again the original problem

$$\begin{array}{ll} \text{min} & x^T Q x \\ \text{s.t.} & x_i^2 = 1 \end{array}$$

and build Lagrangian

$$\begin{array}{rcl} L(x,y) & = & x^T Q x - \sum_i y_i (x_i^2 - 1) \\ & = & x^T (Q - Y) x + \text{trace}\, Y \end{array}$$

where Y is a diagonal matrix and $Q - Y \succeq 0$ must be enforced to ensure that Lagrangian is bounded below

Associated dual problem reads

$$\boxed{\begin{array}{ll} \text{max} & \text{trace}\, Y \\ \text{s.t.} & Q - Y \succeq 0 \\ & Y \text{ diagonal} \end{array}}$$

This is an LMI problem !

Relaxed LMI via duality

The dual LMI problem

$$\begin{array}{ll} \text{max} & \text{trace}\, Y \\ \text{s.t.} & Q \succeq Y \\ & Y \text{ diagonal} \end{array}$$

has for dual the primal LMI problem

$$\begin{array}{ll} \text{min} & \text{trace}\, QX \\ \text{s.t.} & X_{ii} = 1 \\ & X \succeq 0 \end{array}$$

which is Shor's original LMI relaxation !

More generally it can be shown that

LMI rank dropping

=

Lagrangian relaxation

Example of LMI relaxation

Original nonconvex 0-1 quadratic problem

$$\begin{array}{rcll} g^\star & = & \min & 2x_1x_2 + 4x_1x_3 + 6x_2x_3 \\ & & \text{s.t.} & x_i^2 = 1 \end{array} \qquad Q = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 2 & 3 & 0 \end{bmatrix}$$

Primal and dual LMI solutions

$$X = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix} \qquad Y = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix}$$

yield lower bound

$$p^\star = \operatorname{trace} QX = d^\star = \operatorname{trace} Y = -8$$

(strong duality holds here)

Since $\operatorname{rank} X = 1$ we recover here the optimum

$$x = [1 \ \ 1 \ \ -1]^T$$

such that $X = xx^T$ and hence

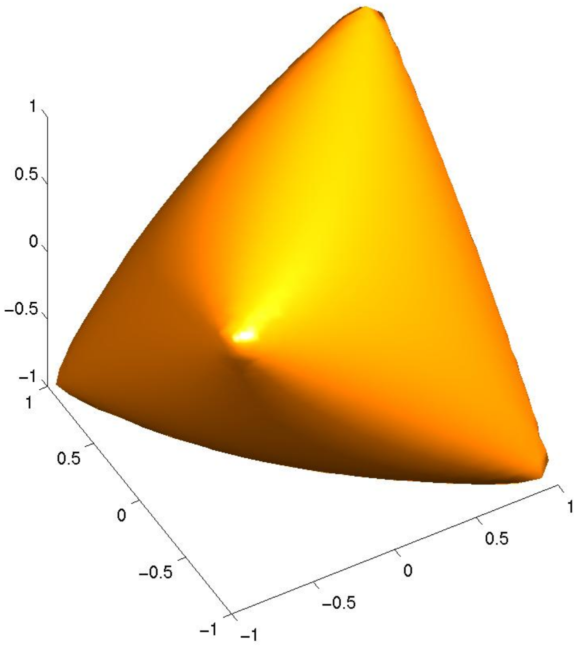
$$g^\star = p^\star = d^\star$$

the relaxation is exact !

Example of LMI relaxation

LMI relaxation of ±1 constraints

$$X = \begin{bmatrix} 1 & X_{12} & X_{13} \\ X_{12} & 1 & X_{23} \\ X_{13} & X_{23} & 1 \end{bmatrix} \succeq 0\}$$



So we optimize the linear objective function

$$\text{trace}\, QX = 2X_{12} + 4X_{13} + 6X_{23}$$

and the optimal is a vertex $[1 \;\; -1 \;\; -1]$

Beyond Shor's relaxation

Recent work (2000) to narrow relaxation gap

- gradually adding lifting variables
- hierarchy of nested LMI relaxations
- theoretical proof of convergence

Dual point of views:

- theory of moments (Lasserre)
- sum-of-squares decompositions (Parrilo)

Tradeoff between conservatism and computational effort

Higher order LMI relaxations
Illustration

Non-convex quadratic problem

$$\begin{array}{ll} \text{min} & g_0(x) = -2x_1^2 - 2x_2^2 + 2x_1x_2 + 2x_1 + 6x_2 - 10 \\ \text{s.t.} & g_1(x) = -x_1^2 + 2x_1 \geq 0 \\ & g_2(x) = -x_1^2 - x_2^2 + 2x_1x_2 + 1 \geq 0 \\ & g_3(x) = -x_2^2 + 6x_2 - 8 \geq 0. \end{array}$$

LMI relaxation built by replacing each monomial $x_1^i x_2^j$ with lifting variable y_{ij}

For example, quadratic expression

$$g_2(x) = -x_1^2 - x_2^2 + 2x_1x_2 + 1 \geq 0$$

is replaced with linear expression

$$-y_{20} - y_{02} + 2y_{11} + 1 \geq 0$$

Lifting variables y_{ij} satisfy non-convex relations such as $y_{10}y_{01} = y_{11}$ or $y_{20} = y_{10}^2$

LMI relaxations: illustration (2)

Relax these non-convex relations by enforcing LMI constraint

$$M_1(y) = \left[\begin{array}{c|cc} 1 & y_{10} & y_{01} \\ \hline y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{array}\right] \succeq 0$$

Moment matrix of first order relaxing monomials of degree up to 2

You have recognized Shor's relaxation !

First LMI (=Shor's) relaxation of original global optimization problem is given by

$$\begin{array}{ll} \text{min} & -2y_{20} - 2y_{02} + 2y_{11} + 2y_{10} + 6y_{01} - 10 \\ \text{s.t.} & -y_{20} + 2y_{10} \geq 0 \\ & -y_{20} - y_{02} + 2y_{11} + 1 \geq 0 \\ & -y_{02} + 6y_{01} - 8 \geq 0 \\ & M_1(y) \succeq 0 \end{array}$$

LMI relaxations: illustration (3)

To build second LMI relaxation, we must increase size of moment matrix so that it captures expressions of degrees up to 4

Second order moment matrix reads

$$M_2(y) = \left[\begin{array}{c|cc|ccc} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ \hline y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ \hline y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{array}\right] \succeq 0$$

Constraints are localized on moment matrices, meaning that original constraint

$$g_1(x) = -x_1^2 + 2x_1 \geq 0$$

becomes localizing matrix constraint

$$M_1(g_1 y) = \left[\begin{array}{c|cc} -y_{20} + 2y_{10} & -y_{30} + 2y_{20} & -y_{21} + 2y_{11} \\ \hline -y_{30} + 2y_{20} & -y_{40} + 2y_{30} & -y_{31} + 2y_{21} \\ -y_{21} + 2y_{11} & -y_{31} + 2y_{21} & -y_{22} + 2y_{12} \end{array}\right] \succeq 0$$

LMI relaxations: illustration (3)

Second LMI feasible set included in first LMI feasible set, thus providing a tighter relaxation

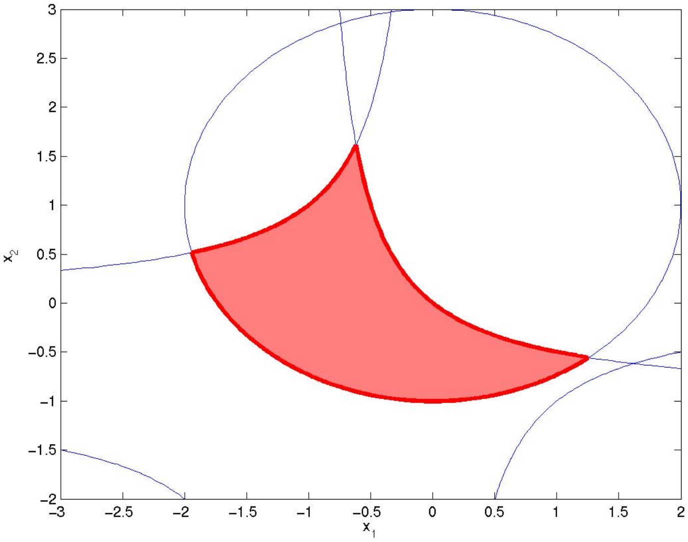
$$\begin{array}{ll} \text{min} & -2y_{20} - 2y_{02} + 2y_{11} + 2y_{10} + 6y_{01} - 10 \\ \text{s.t.} & M_1(g_1 y) \succeq 0, \;\; M_1(g_2 y) \succeq 0, \;\; M_1(g_3 y) \succeq 0 \\ & M_2(y) \succeq 0 \end{array}$$

Similary, we can build up 3rd, 4th, 5th LMI relaxations..

Geometric illustration

Non-convex quadratic problem with linear objective function

$$\begin{array}{ll} \text{max} & x_2 \\ \text{s.t.} & 3 - 2x_2 - x_1^2 - x_2^2 \geq 0 \\ & -x_1 - x_2 - x_1 x_2 \geq 0 \\ & 1 + x_1 x_2 \geq 0 \end{array}$$

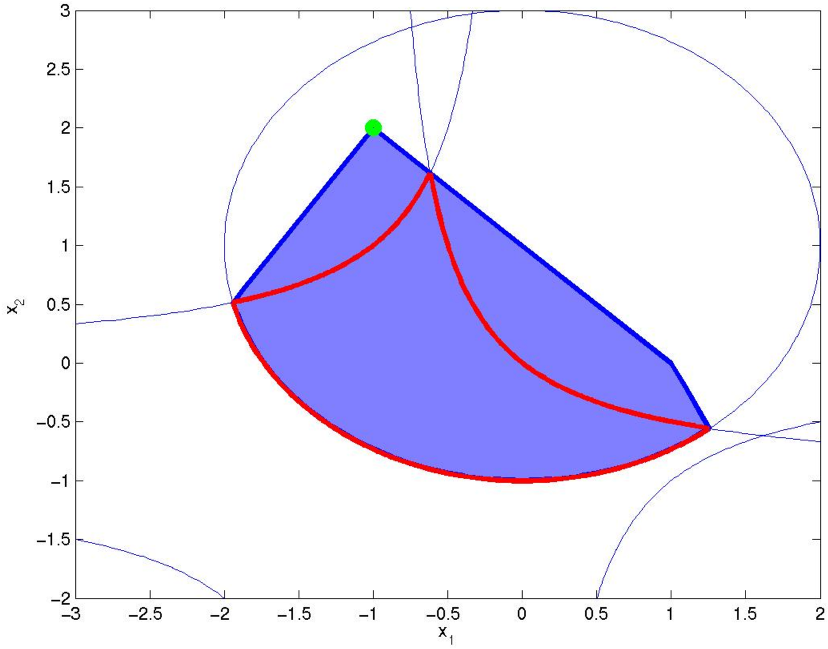


Non-convex feasible set delimited by circular and hyperbolic arcs

Geometric illustration (2)

First LMI relaxation given by

$$\begin{array}{ll} \max & y_{01} \\ \text{s.t.} & \begin{bmatrix} 1 & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{bmatrix} \succeq 0 \\ & 3 - 2y_{01} - y_{20} - y_{02} \geq 0 \\ & -y_{10} - y_{01} - y_{11} \geq 0 \\ & 1 + y_{11} \geq 0 \end{array}$$



Projection of the LMI feasible set onto the plane y_{10}, y_{01} of first-order moments

LMI optimum = 2 = upper-bound on global optimum

Geometric illustration (3)

To build second LMI relaxation, the moment matrix must capture expressions of degrees up to 4

$$M_2^2(y) = \left[\begin{array}{c|cc|ccc} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ \hline y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ \hline y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{array}\right]$$

Constraints are also lifted and relaxed with the help of localization matrices

Second LMI provides tighter relaxation

$$\begin{array}{ll} \text{max} & y_{01} \\ \text{s.t.} & \begin{bmatrix} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succeq 0 \end{array}$$

$$\begin{bmatrix} 3-2y_{01}-y_{20}-y_{02} & 3y_{10}-2y_{11}-y_{30}-y_{12} & 3y_{01}-2y_{02}-y_{21}-y_{03} \\ 3y_{10}-2y_{11}-y_{30}-y_{12} & 3y_{20}-2y_{21}-y_{40}-y_{22} & 3y_{11}-2y_{12}-y_{31}-y_{13} \\ 3y_{01}-2y_{02}-y_{21}-y_{03} & 3y_{11}-2y_{12}-y_{31}-y_{13} & 3y_{02}-2y_{03}-y_{22}-y_{04} \end{bmatrix} \succeq 0$$

$$\begin{bmatrix} -y_{10}-y_{01}-y_{11} & -y_{20}-y_{11}-y_{21} & -y_{11}-y_{02}-y_{12} \\ -y_{20}-y_{11}-y_{21} & -y_{30}-y_{21}-y_{31} & -y_{21}-y_{12}-y_{22} \\ -y_{11}-y_{02}-y_{12} & -y_{21}-y_{12}-y_{22} & -y_{12}-y_{03}-y_{13} \end{bmatrix} \succeq 0$$

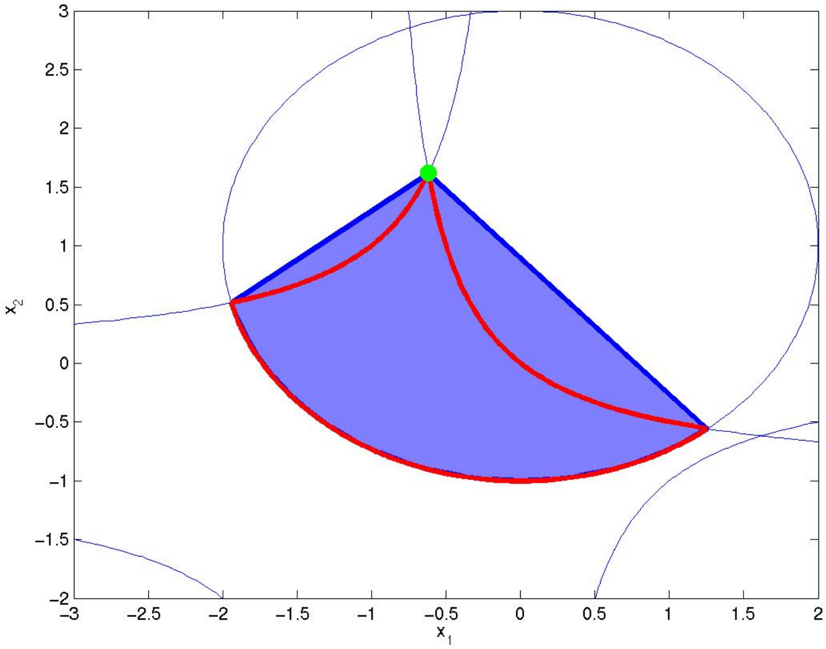
$$\begin{bmatrix} 1+y_{11} & y_{10}+y_{21} & y_{01}+y_{12} \\ y_{10}+y_{21} & y_{20}+y_{31} & y_{11}+y_{22} \\ y_{01}+y_{12} & y_{11}+y_{22} & y_{02}+y_{13} \end{bmatrix} \succeq 0$$

Geometric illustration (4)

Optimal value of 2nd LMI relaxation = 1.6180 = global optimum within numerical accuracy

Numerical certificate = moment matrix has rank one

First order moments $(y^*_{10}, y^*_{01}) = (-0.6180, 1.6180)$ provides optimal solution of original problem



Polynomial multipliers

Polynomial optimization problem

$$\begin{array}{rcl} g^\star & = & \min \;\; g_0(x) \\ & & \text{s.t.} \;\; g_i(x) \geq 0,\; i = 1,\ldots,m \end{array}$$

where $g_i(x)$ are real-valued multivariate polynomials in vector indeterminate $x \in \mathbb{R}^n$

Non-convex problem in general (includes 0-1 or quadratic problems) = NP-hard

If $g^\star$ is the global optimum, polynomial

$$\boxed{g_0(x) - g^\star = q_0(x) + \sum_{i=1}^{m} g_i(x) q_i(x)}$$

is globally positive (non-negative)
when multipliers $q_i(x)$ are positive..

Recall Lagrangian when building dual..

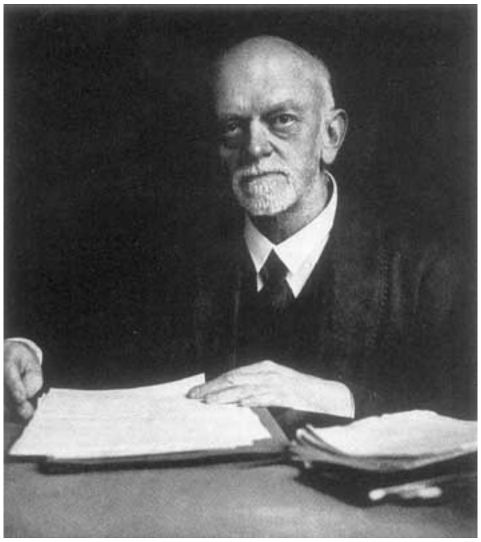
Multipliers $q_i(x)$ are now polynomials !
How can we enforce their positivity ?

SOS polynomials

How can we ensure that a real valued polynomial is globally non-negative ?

$$p(x) \geq 0,\ \forall x \in \mathbb{R}^n$$

Hilbert's 17th pb about algebraic sum-of-squares decompositions of rational functions (Intl Congress of Mathematicians, Paris, 1900)



David Hilbert
(1862 Königsberg - 1943 Göttingen)

SOS polynomials

A form is a homogeneous polynomial
= all monomials have same degree

An obvious condition for a polynomial or form $p(x)$ to be non-negative is that is a sum-of-squares (SOS) of other polynomials

$$p(x) = \sum_i q_i^2(x)$$

Unfortunately, not every non-negative polynomial or form is SOS

$$p(x) \text{ SOS} \Longrightarrow p(x) \geq 0$$

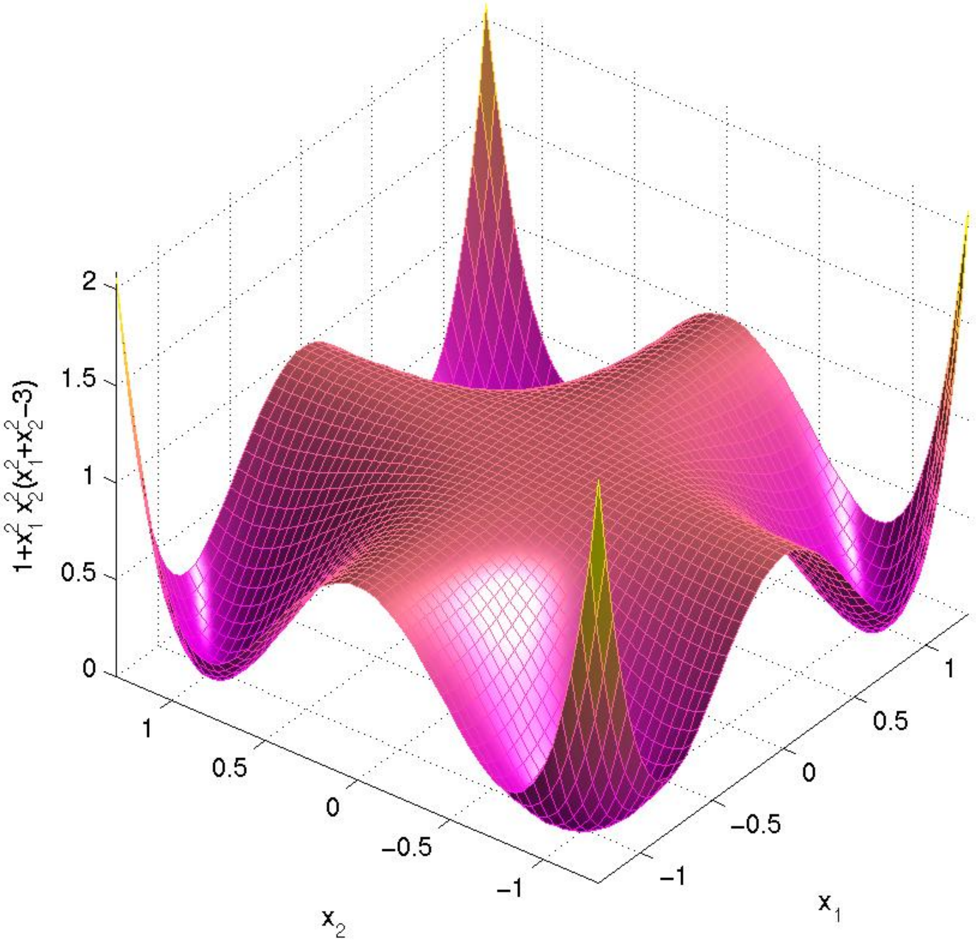
Sufficient non-negativity condition only..

Motzkin's polynomial

Counterexample:

$$p(x) = 1 + x_1^2 x_2^2 (x_1^2 + x_2^2 - 3)$$

cannot be written as an SOS but it is globally non-negative (vanishes at $|x_1| = |x_2| = 1$)



SOS polynomials

Let n denote the number of variables and d the degree

Non-negativity and SOS are sometimes equivalent:

$n = 2$	bivariate forms univariate polynomials (dehomogen)
$d = 2$	quadratic forms
$n = 3$, $d = 4$	quartic forms of 3 variables

In all other cases, the set of SOS polynomials (a cone) is a subset of the set of non-negative polynomials

Checking whether a polynomial is non-negative is NP-hard when $d \geq 4$

Note however that the set of SOS polynomials is dense in the set of polynomials nonnegative over the n-dimensional box $[-1, 1]^n$

Most importantly

The cone of SOS polynomials
is lifted-LMI representable

as we will see in the sequel..

LMI formulation of SOS polynomials

Polynomial

$$p(x) = \sum_{\alpha} p_\alpha x^\alpha$$

of degree $|\alpha| \leq 2d$ ($\alpha =$ vector of powers of indeterminates x) is SOS iff

$$\boxed{p(x) = z^T X z \quad X \succeq 0}$$

where z is a vector with all monomials with degree not greater than d

Cholesky factorization

$$X = Q^T Q$$

such that

$$\begin{aligned} p(x) &= z^T Q^T Q z = \|Qz\|_2 = \textstyle\sum_i (Qz)_i^2 \\ &= \textstyle\sum_i q_i^2(x) \end{aligned}$$

Number of squares $q_i^2(x) =$ rank X

LMI formulation of SOS polynomials

Comparing monomial coefficients in expression

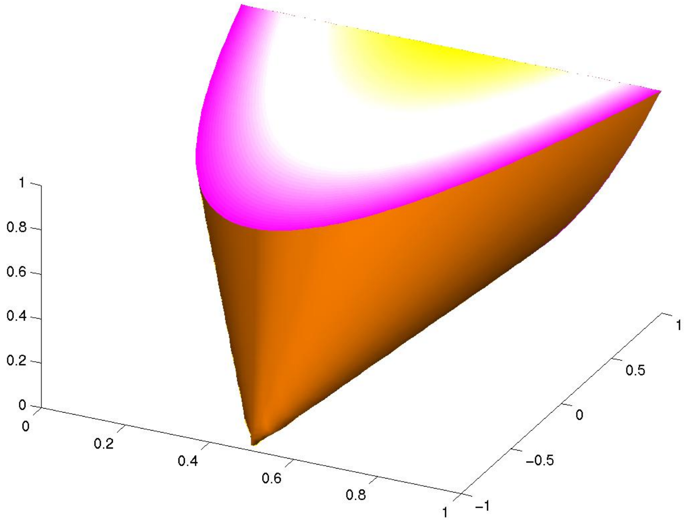
$$p(x) = z^T X z = \sum_\alpha p_\alpha x^\alpha \geq 0$$

we get an LMI

$$\boxed{\begin{array}{l} \text{trace}\, H_\alpha X = p_\alpha \quad \forall \alpha \\ X \succeq 0 \end{array}}$$

where H_α are Hankel-like matrices

SOS polynomials described by an intersection between a subspace and the PSD cone



SOS example

Consider the homogeneous form

$$\begin{aligned} p(x) &= 2x_1^4 + 5x_2^4 + 2x_1^3x_2 - x_1^2x_2^2 \\ &= z^T X z \end{aligned}$$

With monomial vector

$$z = \begin{bmatrix} x_1^2 \\ x_2^2 \\ x_1x_2 \end{bmatrix}$$

a general bivariate form of degree 4 reads

$$\begin{aligned} z^T X z = \; & X_{11}x_1^4 + X_{22}x_2^4 + 2X_{31}x_1^3x_2 \\ & + 2X_{32}x_1x_2^3 + (X_{33} + 2X_{21})x_1^2x_2^2 \end{aligned}$$

$p(x)$ SOS iff there exists $X \succeq 0$ such that

$$\begin{array}{ll} X_{11} = 2 & X_{22} = 5 \\ 2X_{31} = 2 & 2X_{32} = 0 \\ X_{33} + 2X_{21} = -1 & \end{array}$$

SOS example

One particular solution is

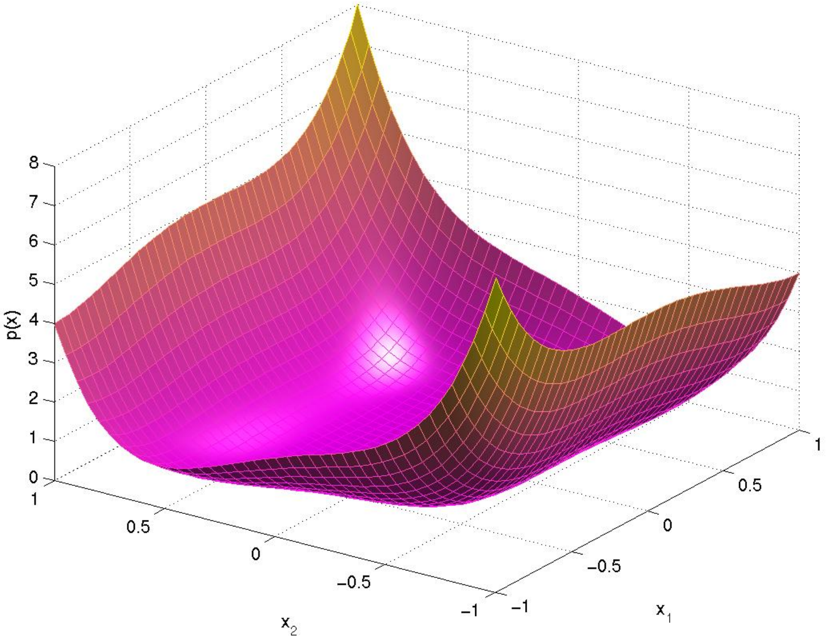
$$X = \begin{bmatrix} 2 & -3 & 1 \\ -3 & 5 & 0 \\ 1 & 0 & 5 \end{bmatrix} = Q^T Q$$

with Cholesky factor

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 3 \end{bmatrix}$$

So $p(x)$ is the sum of rank $X = 2$ squares

$$p(x) = \tfrac{1}{2}(2x_1^2 - 3x_2^2 + x_1 x_2)^2 + \tfrac{1}{2}(x_2^2 + 3x_1 x_2)^2$$



Finding polynomial multipliers

Returning to our global optimization problem

$$\begin{array}{rcl} g^\star & = & \min \;\; g_0(x) \\ & & \text{s.t.} \;\; g_i(x) \geq 0, \; i = 1, \ldots, m \end{array}$$

the problem of finding SOS polynomial multipliers $q_i(x)$ such that

$$p(x) = g_0(x) - g^\star = q_0(x) + \sum_{i=1}^{m} g_i(x) q_i(x)$$

can be formulated as an LMI as soon as the degrees of the $q_i(x)$ are fixed

Depending on parity let deg $p(x) = 2k - 1$ or $2k$ - then the LMI problem of finding an SOS $p(x)$ is referred to as the

LMI relaxation of order k

Hierarchy of LMI relaxations

The LMI relaxation of order k reads

$$\begin{array}{rcll} d_k^\star & = & \min & \sum_\alpha (g_0)_\alpha y_\alpha \\ & & \text{s.t.} & M_k(y) = \sum_\alpha A_\alpha y_\alpha \succeq 0 \\ & & & M_{k-d_i}(g_i y) = \sum_\alpha A_\alpha^{g_i} y_\alpha \succeq 0 \quad \forall i \end{array}$$

with $y_0 = 1$ (normalization)
d_i is half the degree of $g_i(x)$
$M_k(y)$ is the moment matrix
$M_{k-d_i}(g_i y)$ are the localization matrices

The dual LMI

$$\begin{array}{rcll} p_k^\star & = & \max & \sum_\alpha \text{trace}\, A_0 X + \sum_i \text{trace}\, A_0^{g_i} X_i \\ & & \text{s.t.} & \text{trace}\, A_\alpha X \\ & & & + \sum_i \text{trace}\, A_\alpha^{g_i} X_i = (g_0)_\alpha \quad \forall \alpha \neq 0 \end{array}$$

corresponds to $p(x)$ SOS

Hierarchy of LMI relaxations

If feasible set $g_i(x) \geq 0$ is compact, and under mild additional assumptions, Lasserre could use results by Putinar (on SOS representations of positive polynomials) and Curto/Fialkow (on flat extension of moment matrices) to prove in 2000 that

$$p_k^\star = d_k^\star \leq g^\star$$

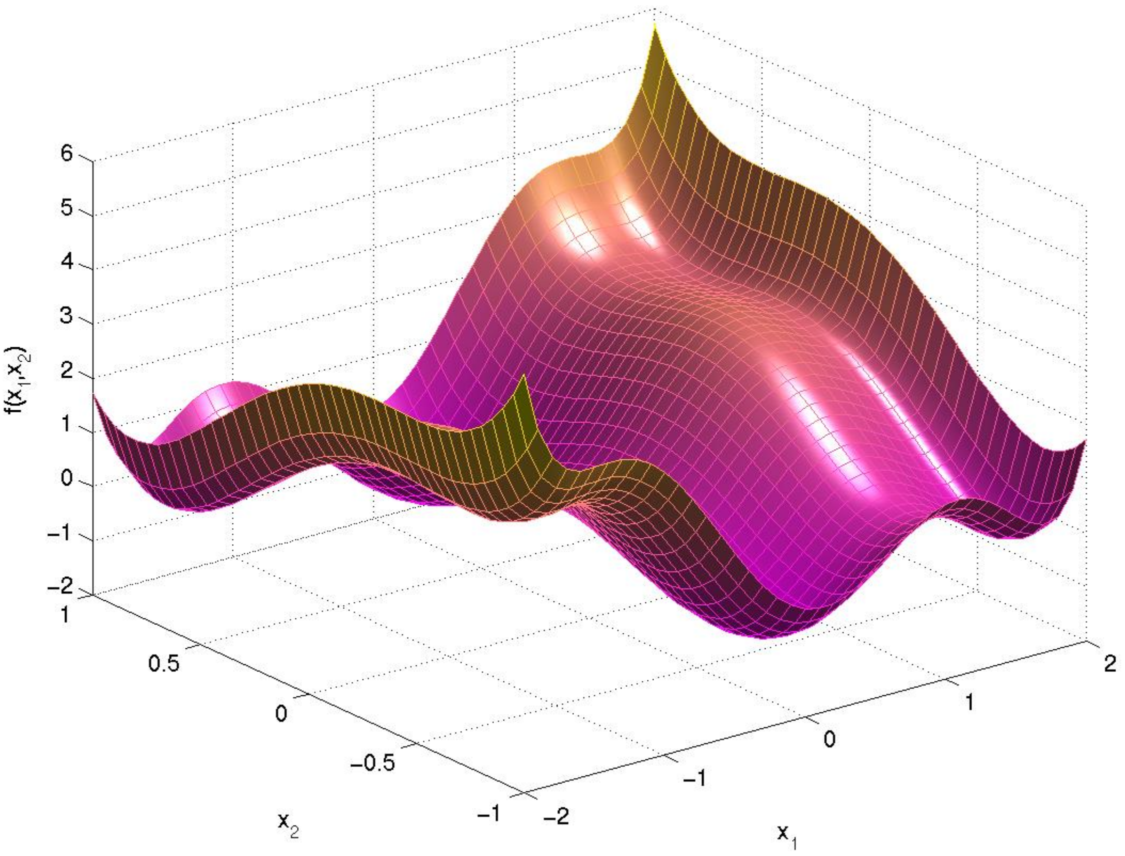
with asymptotic convergence guarantee

$$\boxed{\lim_{k\to\infty} p_k^\star = g^\star}$$

Moreover, in practice, convergence is fast:
$p_k^\star$ is very close to $g^\star$ for small k

Camelback function

For the well-known six-hump camelback function



with two global optima and six local optima, the global optimum is reached at the first LMI relaxation ($k = 1$) without any problem splitting

LMI hierarchy: example

Quadratic problem

$$\begin{array}{ll} \min & -2x_1 + x_2 - x_3 \\ \text{s.t.} & x_1(4x_1 - 4x_2 + 4x_3 - 20) + x_2(2x_2 - 2x_3 + 9) \\ & \qquad +x_3(2x_3 - 13) + 24 \geq 0 \\ & x_1 + x_2 + x_3 \leq 4, \quad 3x_2 + x_3 \leq 6 \\ & 0 \leq x_1 \leq 2, \quad 0 \leq x_2, \quad 0 \leq x_3 \leq 3. \end{array}$$

Number of LMI variables (M) and size of relaxed LMI problem (N) increase quickly with relaxation order:

Relaxation	LMI opt	M	N
1	-6.0000	9	24
2	-5.6923	34	228
3	-4.0685	83	1200
4	-4.0000	164	4425
5	-4.0000	285	12936
6	-4.0000	454	32144

..yet fourth LMI relaxation solves globally the problem

Solving BMIs with LMI relaxations

Two approaches: scalarization or polynomial matrix inequality (PMI) relaxations

Scalarization:
- scalarize using Descartes' rule of signs
- polynomials with generally large degree

PMI relaxations:
- keep the matrix structure
- no degree growth
- theory for matrix polynomial SOS

Theory is ready, but..
..experimentally at a very preliminary level

Numerical aspects (conditioning, solution extraction) must be further studied

Robust stability analysis

Linear system of order n

$$\boxed{\dot{x} = A(q)x}$$

with polytopic uncertainty

$$A(q) = A_0 + \sum_{i=1}^{m} q_i A_i \quad \forall q \in \mathcal{Q} \subset \mathbb{R}^m$$

Particular "easy" cases:

- $\mathcal{Q}$ = box and rank $A_i = 1$: Kharitonov
- rank $A_i = 1$: Edge Theorem

Otherwise checking robust stability of this system is generally NP-hard

Hermite stability criterion

Uncertain linear system robustly stable iff

$$\boxed{H(q) \succ 0 \quad \forall q \in \mathcal{Q} \subset \mathbb{R}^m}$$

where $H(q)$ Hermite matrix bilinear in coeffs $h_i(q)$ of charact poly

$$h(s,q) = \det\left(sI_n - A(q)\right) = \sum_{i=1}^{n} h_i(q) s^i$$

Hermite matrix acts as a parameter-dependent Lyapunov matrix of degree $\leq 2nm$ in q

Charles Hermite (1822 Dieuze - 1901 Paris)

Assessing robust stability

Solve the optimization problem

$$\begin{array}{rcll} p^\star & = & \min_q & \det H(q) \\ & & \text{s.t.} & q \in \mathcal{Q} \end{array}$$

minimization of an m-variable polynomial over a polytope

Uncertain linear system
robustly stable iff $p^\star > 0$

Hierarchy of LMI relaxations

$$\begin{array}{rcll} \underline{p}_k & = & \min_y & f(y) \\ & & \text{s.t.} & M_k(y) \succeq 0 \\ & & & M_k(g_i, y) \succeq 0 \quad i = 1, 2 \ldots \end{array}$$

with $M_k(y)$ moment matrix and
$M_k(g_i, y)$ polytope localization matrices

LMI hierarchy yields converging
lower and upper bounds on $p^\star$

Interval matrix stability

Consider the interval matrix

$$A(q) = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & q_3 \\ 0 & -0.7115 & q_4 \end{bmatrix}$$

where

$$\begin{aligned} q \in \mathcal{Q} &= [-2.4780,\ -1.4471] \times [-0.0518,\ -0.0194] \\ &\quad \times [2.0000,\ 3.4347] \times [-0.0026,\ -0.0012] \end{aligned}$$

Minimizing p with LMI relaxation of order 3 and feasibility radius 10^3 on the vector of moments yields inconclusive result $-171 \leq p^\star \leq 0.1505$

LMI relaxation of order 4 yields certified global optimum

$$\boxed{p^\star = 0.1505}$$

attained at $q^\star = [-1.4471, -0.0194, 2.0000, -0.0012]$ hence proving robust stability

LMI relaxations: conclusions

LMI relaxations prove useful to solve general non-convex polynomial optimization problems

Shor's relaxation = rank dropping = Lagrangian relaxation = first order LMI relaxation

A hierarchy of successive LMI relaxations can be built with additional lifting variables and constraints

Theoretical guarantee of asymptotic convergence to global optimum without any problem splitting (no branch and bound scheme)