POLYNOMIAL METHODS FOR
ROBUST CONTROL
PART I.4

ROBUST STABILITY ANALYSIS: MULTILINEAR UNCERTAINTY

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Uncertainty structure

In typical applications, uncertainty structure is more complicated than interval or affine

Usually, uncertainty enters highly non-linearly in the closed-loop characteristic polynomial

We distinguish between
• multilinear uncertainty, when each uncertain parameter q_i is linear when other parameters q_j, $i \neq j$ are fixed
• polynomic uncertainty, when coefficients are multivariable polynomials in parameters q_i

We can define the following hierarchy on the uncertainty structures

interval $\subset$ affine $\subset$ multilinear $\subset$ polynomic

Examples of uncertainty structures

Examples

The uncertain polynomial

$$(5q_1 - q_2 + 5) + (4q_1 + q_2 + q_3)s + s^2$$

has affine uncertainty structure

The uncertain polynomial

$$(5q_1 - q_2 + 5) + (4q_1q_3 - 6q_1q_3 + q_3)s + s^2$$

has multilinear uncertainty structure

The uncertain polynomial

$$(5q_1 - q_2 + 5) + (4q_1 - 6q_1 - q_3^2)s + s^2$$

has polynomic (here quadratic) uncertainty structure

The uncertain polynomial

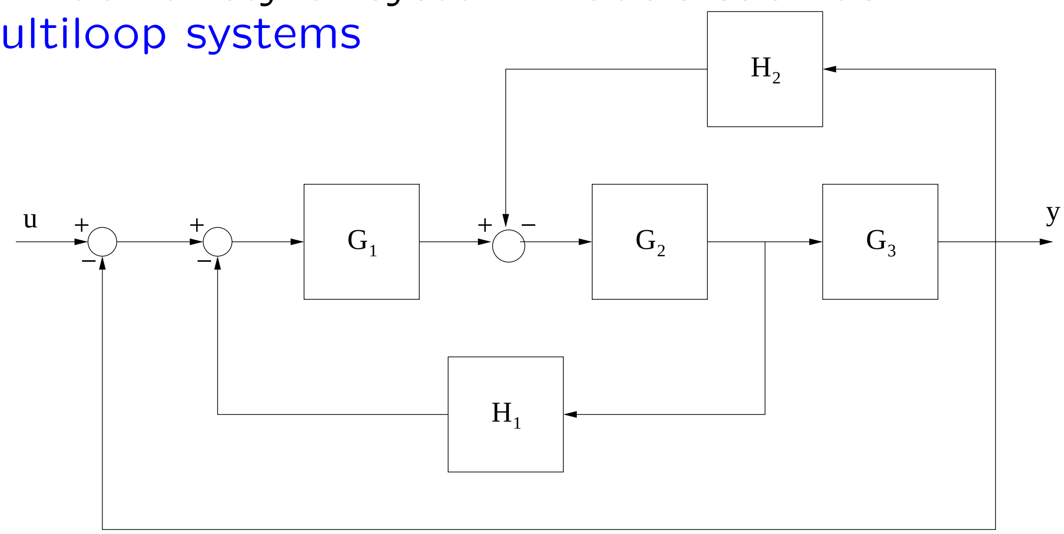
$$(5q_1 - q_2 + 5) + (4q_1 - 6q_1q_3^2 + q_3)s + s^2$$

has polynomic uncertainty structure

Multilinear uncertainty

We will focus on multilinear uncertainty because it arises in a wide variety of system models such as:

- multiloop systems



Closed-loop transfer function

$$\frac{y}{u} = \frac{G_1 G_2 G_3}{1 + G_1 G_2 H_1 + G_2 G_3 H_2 + G_1 G_2 G_3}$$

- state-space models with rank-one uncertainty

$$\dot{x} = A(q)x, \ A(q) = \sum_{i=1}^{n} q_i A_i, \ \text{rank } A_i = 1$$

and characteristic polynomial

$$p(s,q) = \det(sI - A(q))$$

- polynomial MFDs with MIMO interval uncertainty

$$G(s) = A^{-1}(s,q)B(s,q), \quad C(s) = Y(s)X^{-1}(s)$$

and closed-loop characteristic polynomial

$$p(s,q) = \det(A(s,q)X(s) + B(s,q)Y(s))$$

From polynomic to multilinear

Moreover, if a polynomial $p(s,q)$ has polynomic uncertainty structure defined over a polytope Q, then there exists an equivalent polynomial $\tilde{p}(s,\tilde{q})$ with multilinear uncertainty defined over another polytope $\tilde{Q}$

Example

Consider the uncertain polynomial

$$\begin{aligned} p(s,q) &= (6q_1q_2+17)+(4q_1^2+q_2^2+15)s \\ &\quad +(3q_1^3+q_1^2q_2+q_1q_2+3q_1+10)s^2+s^3 \end{aligned}$$

with polynomic uncertainty over the polytope $|q_1| \leq 1$ and $|q_2| \leq 2$

We expand the uncertainty space by defining new variables $\tilde{q}_i$ such that $\tilde{q}_1\tilde{q}_2\tilde{q}_3 = q_1^3$ and $\tilde{q}_4\tilde{q}_5 = q_2^2$, so that the equivalent uncertain polynomial

$$\begin{aligned} \tilde{p}(s,\tilde{q}) &= (6\tilde{q}_1\tilde{q}_4+17)+(4\tilde{q}_1\tilde{q}_2+\tilde{q}_4\tilde{q}_5+15)s \\ &\quad +(3\tilde{q}_1\tilde{q}_2\tilde{q}_3+\tilde{q}_1\tilde{q}_2\tilde{q}_4+\tilde{q}_1\tilde{q}_4+3\tilde{q}_1+10)s^2+s^3 \end{aligned}$$

has multilinear uncertainty over the polytope $|\tilde{q}_1 = \tilde{q}_2 = \tilde{q}_3| \leq 1$ and $|\tilde{q}_4 = \tilde{q}_5| \leq 2$

Lack of edge results

We have seen that there is a lack of vertex results for affine uncertainty, so that we had to develop edge results

Unfortunately, there is a lack of edge results for multilinear uncertainty

Example
Consider the uncertain polynomial

$$\begin{array}{rcl} p(s,q) & = & (4.032q_1q_2 + 3.773q_1 + 1.985q_2 + 1.853) \\ & & +(1.06q_1q_2 + 4.841q_1 + 1.561q_2 + 3.164)s \\ & & +(q_1q_2 + 2.06q_1 + 1.561q_2 + 2.871)s^2 \\ & & +(q_1 + q_2 + 2.56)s^3 + s^4 \end{array}$$

with multilinear uncertainty over the polytope $q_1 \in [0,\ 1]$, $q_2 \in [0,\ 3]$, corresponding to the state-space interval matrix

$$p(s,q) = \det(sI - \begin{bmatrix} [-1.5,\ -0.5] & -12.06 & -0.06 & 0 \\ -0.25 & -0.03 & 1 & 0.5 \\ 0.25 & -4 & -1.03 & 0 \\ 0 & 0.5 & 0 & [-4,\ 1] \end{bmatrix})$$

The four edges of the uncertainty bounding set are stable, however for $q_1 = 0.5$ and $q_2 = 1$ polynomial $p(s,q)$ is unstable..

The mapping theorem

To handle multilinear uncertainty we will use a special overbounding family of polynomials, as captured by the following mapping theorem

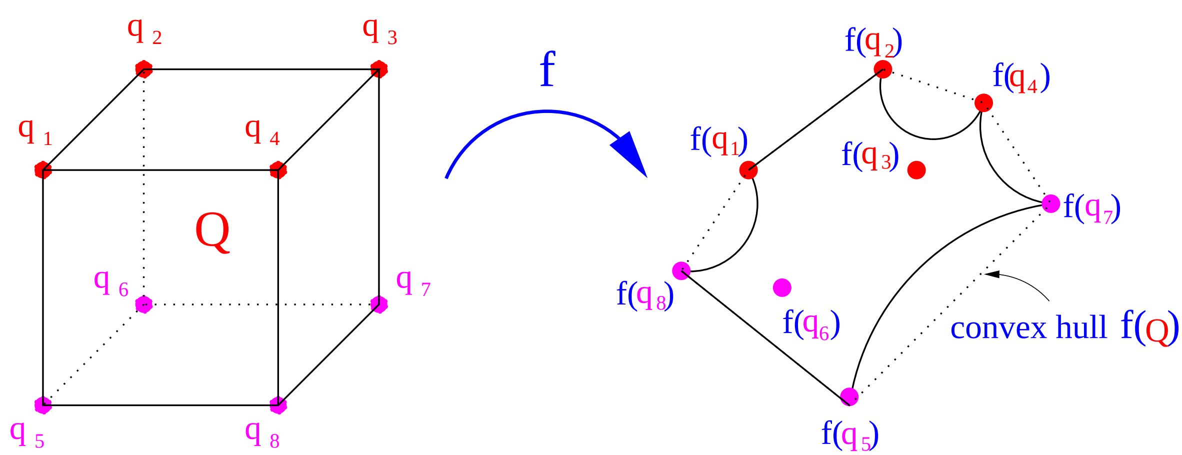
Let Q be a box with vertices q_i
Let f be a multilinear function with range

$$f(Q) = \{f(q) \; : \; q \in Q\}$$

Then it follows that

$$\text{convex hull } f(Q) = \text{convex hull } \{f(q_i)\}$$

Geometric interpretation for a 3-D box Q and a 2-D multilinear function f



Any $f(Q)$ which curves outward is not realizable

Can be proved by induction or with the help of support functions in linear programming

Applications of mapping theorem

If $p(s,q)$ is an uncertain polynomial with coefficient vector $a(q)$ depending multilinearly on a parameter q living in a box Q with vertices q_i then

$$\text{convex hull}\, a(Q) = \text{convex hull}\, \{a(q_i)\}$$

The same property holds for the value set

$$\text{convex hull}\, p(z,Q) = \text{convex hull}\, \{p(z,q_i)\}$$

hence the zero exclusion condition

$$0 \notin \text{convex hull}\, \{p(z,q_i)\}$$

is a sufficient condition for robust stability

When testing robust stability of polynomials with multilinear uncertainty if this condition is not satisfied (i.e. zero belongs to the convex hull of the value set) then we cannot conclude without intensive gridding..

Value set with multilinear uncertainty

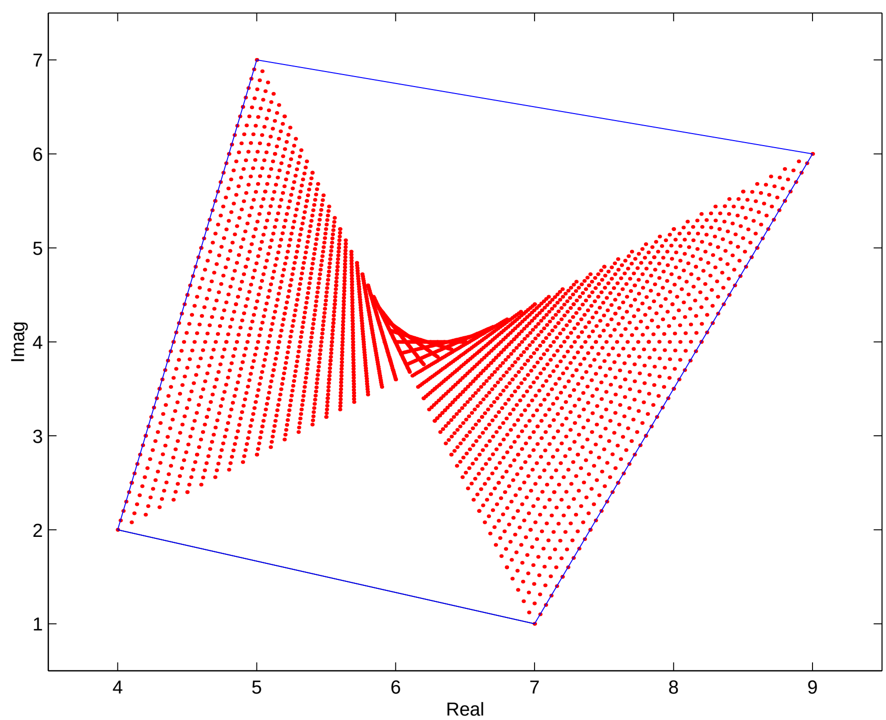
Example

Consider the multilinear polynomial

$$(5+2q_1-q_2+3q_1q_2)+(7-6q_1-5q_2+10q_1q_2)s$$

where $q_1, q_2 \in [0,\ 1]$

Its actual value set at $s = j$ (obtained by gridding) and its convex hull are shown below



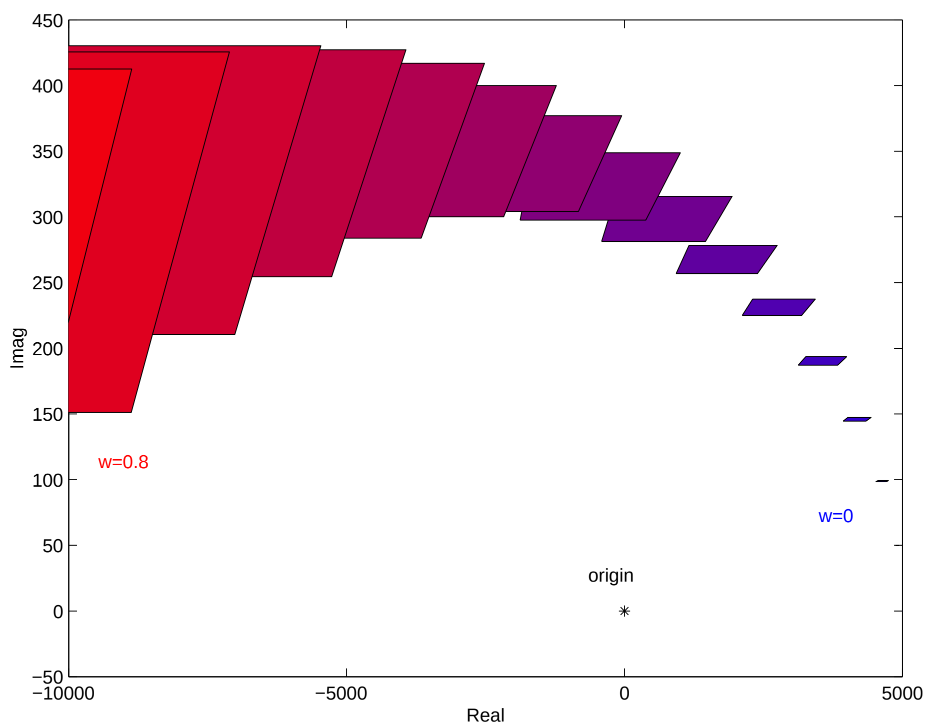
Robust stability analysis with multilinear uncertainty

Example

Consider the crane system with uncertain rope length $l \in [8,\ 16]$, crab mass $m_C \in [100,\ 2000]$ and multilinear closed-loop characteristic polynomial

$$5000+1000s+(20100+5000l+10m_C)s^2+100ls^3+lm_Cs^4$$

Nominal stability at $l = 8$ and $m_C = 1000$ is easily checked, so we can build the convex hull of the value set = convex hull of value set of the 4 vertex polynomials = polygon with 4 vertices at each frequency



The convex hull of the value set excludes zero hence the polynomial is robustly stable

Nasty value set

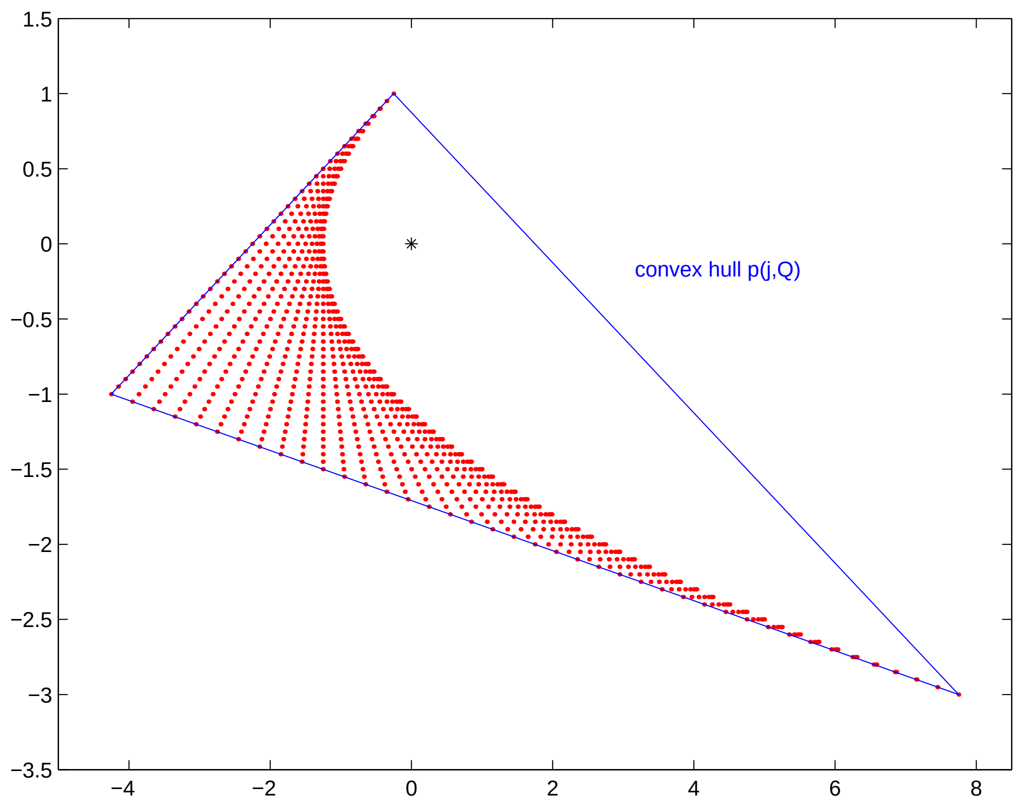
Example

Consider now the multilinear polynomial

$$p(s,q) = 3+2s+(0.25+2q_1+2q_2)s^2+0.5(q_1+q_2)s^3+q_1q_2s^4$$

with uncertainty box $Q = [1,\ 5] \times [1,\ 5]$

We build the convex hull $p(j,Q)$ of the value set at $s = j$

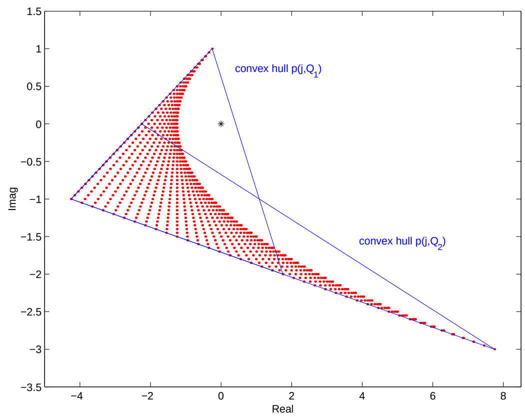


The convex hull contains the origin so we cannot conclude about robust stability

From the actual value set obtained by gridding it turns out however that the polynomial is robustly stable

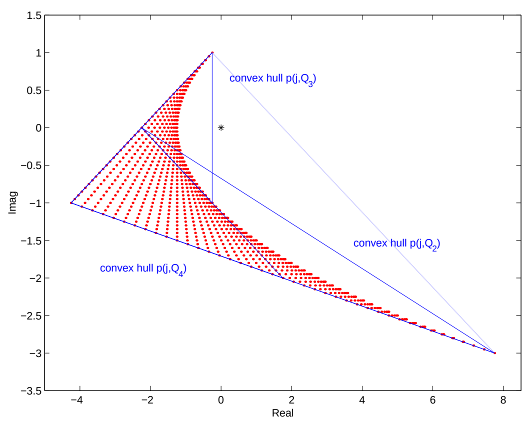
Splitting uncertainty set

Now splitting the uncertainty interval Q into $Q_1 = [1,\ 3] \times [1,\ 5]$ and $Q_2 = [3,\ 5] \times [1,\ 5]$ we build two convex hulls $p(j, Q_1)$ and $p(j, Q_2)$



The convex hull of $p(j, Q_1)$ contains the origin so once more we cannot conclude about robust stability

Now splitting Q_1 into $Q_3 = [1,\ 3] \times [1,\ 3]$ and $Q_4 = [1,\ 3] \times [3,\ 5]$ we construct three convex hulls $p(j, Q_3)$, $p(j, Q_4)$ and $p(j, Q_2)$



The origin is now excluded, so the polynomial is robustly stable

We can build a systematic algorithm to split the uncertainty intervals but the computational complexity is exponential in the number of parameters