$C^*$-algebras and noncommutative mathematics

Jan Hamhalter

Czech Technical University in Prague,
Department of Mathematics, Czech Republic

Czech workshop on applied mathematics in engineering,
November 2011
Quantization program in mathematics

N. Weaver: “The fundamental idea of mathematical quantization is that sets are represented by Hilbert spaces”.

Structure of sets $\rightarrow$ structure of subspaces in a Hilbert space.

functions on sets $\rightarrow$ operators acting on a Hilbert space.
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The structure called \( \ast \)-normed algebra has the following ingredients:

linear space, multiplication, conjugation, norm:

\[
(A, +, \cdot, \ast, \| \cdot \|)
\]

More precisely,

1. \((A, +, \| \cdot \|)\) is a normed linear space
2. Product, \(\cdot\), is associative.
3. \(a(b + c) = ab + bc, \ (b + c)a = ba + ca\).
4. \(a^{**} = a, \ (ab)^* = b^*a^*, \ (\lambda a)^* = \overline{\lambda}a^*, \ (a + b)^* = a^* + b^*\).
In commutative mathematics we work with \textit{algebras of functions} (continuous functions, measurable functions, random variables, holomorphic functions, ...)

\[ K \quad \text{... compact Hausdorff topological space.} \]
\[ A = C(K) \quad \text{... continuous complex functions on } K. \]

- usual arithmetic operations, complex conjugation:
  \[ f(z) \rightarrow f(z) \]

- norm
  \[ \|f(z)\| = \max_{z \in K} |f(z)|. \]
Observe that the following conditions are satisfied for every function algebra (for all $a, b \in A$):

1. $ab = ba$
2. $A$ has a unit with respect to multiplication.
3. $\|ab\| \leq \|a\| \cdot \|b\|.$
4. $\|a\|^2 = \|a^*a\|.$
5. $(A, \| \cdot \|)$ is complete.
Gelfand representation

Gelfand Theorem

For each algebra $A$ satisfying axioms above there is a unique compact Hausdorff space $K$ such that $A$ is isometrically $\ast$-isomorphic to $C(K)$.

Gelfand representation:

$K =$ set of all $\omega : A \to \mathbb{C}$ such that

1. $\omega$ is a linear functional.
2. $\omega(a^* ) = \overline{\omega(a)}$.
3. $\omega(ab) = \omega(a)\omega(b)$.
4. $\omega(1) = 1$.

Topology $=$ pointwise convergence on elements of $A$.

Gelfand transformation is given by an evaluation $a \in A \rightarrow \hat{a} \in C(K)$

$$ \hat{a}(\omega) = \omega(a) \quad \text{for all } \omega \in K.$$
Theorem

Let $X$ and $Y$ be compact spaces. Then $C(X)$ is isometrically $\ast$-isomorphic to $C(Y)$ if, and only if, $X$ and $Y$ are homeomorphic.

$C^\ast$-algebras are generalizations (noncommutative) of compact spaces. It is a basic strategy for Connes noncommutative geometry.
**Matrix algebra**

$M_n$ .... $nxn$ complex matrices.

$a^*$ adjoint matrix of $a$.

$\|a\|^2 = \text{spectral radius } (a^*a)$.

Observe that the following conditions are satisfied for every matrix algebra (for all $a, b \in A$):

1. $A$ is finite dimensional and only multiples of the unit commute with all elements of $A$.
2. $A$ has a unit with respect to multiplication.
3. $\|ab\| \leq \|a\| \cdot \|b\|$.
4. $\|a\|^2 = \|a^*a\|$.
5. $(A, \| \cdot \|)$ is complete.

E.g.

$$\|a^*a\|^2 = \varrho(a^*aa^*a) = \varrho((a^*a)^2) = \varrho^2(a^*a) = \|a\|^4.$$
Theorem

Every algebra $A$ satisfying the axioms above is isometrically $\ast$-isomorphic to the algebra $M_n$ for some $n$. 
We drop conditions (i) in above two cases - so we abandon commutativity and finite dimensionality. This way we get a simultaneous generalization of function and matrix algebras.

**Definition**

Unital $C^*$-algebra is normed $*$-algebra $A$ satisfying the following conditions:

1. $A$ has a unit with respect to multiplication.
2. $\|ab\| \leq \|a\| \cdot \|b\|$. 
3. $\|a\|^2 = \|a^*a\|$. 
4. $(A, \| \cdot \|)$ is complete.

Example: $B(H)$ — bounded operators on a Hilbert space $H$ endowed with operator norm, composition as multiplication, and conjugation associating to an operator $T$ its adjoint $T^*$ characterised by

\[ < T\xi, \eta > = < \xi, T^*\eta > \quad \text{for all } \xi, \eta \in H. \]
Gelfand-Naimark-Segal Theorem, 1941

For each $C^*$-algebra $A$ there is a Hilbert space $H$ and isometric $\ast$-isomorphism $\pi : A \to B(H)$ such that $\pi(A)$ is a closed $\ast$-subalgebra of $B(H)$. 
How to get an inner product and a representation?

Simple case when $A$ admits a linear functional $\varrho$ such that

1. $\varrho(1) = 1$.
2. $\varrho(x^* x) \geq 0$ for all $x \in A$
3. $\varrho(x^* x) = 0$ implies $x = 0$.

It can be proved:

$$\varrho(b^* a^* ab) \leq \|a^* a\| \varrho(b^* b).$$

Inner product:

$$\langle x, y \rangle = \varrho(y^* x).$$

$H$ — completion of $(A, \langle \cdot, \cdot \rangle)$. 
\( \pi : A \to B(H) \) determined by

\[
\pi(a)b = ab \quad b \in A.
\]

\[
\|ab\|_H = \varrho(b^*a^*ab) \leq \|a^*a\| \cdot \varrho(b^*b) = \|a\|^2 \varrho(b^*b) = \|a\|^2 \|b\|_H.
\]

It implies

\[
\|\pi(a)\| \leq \|a\|.
\]

As \( \pi(a)(1) = a \), and \( \|1\|_H = \varrho(1) = 1 \) we obtain that

\[
\|\pi(a)\| = \|a\|.
\]
Spectroscopic data: \((i, j)\) - pairs consisting of frequency and its respond.
Heisenberg: two dimensional data are forming grupoid \(\Delta\) with multiplication.
Jordan and Bohr - Heisenberg’s multiplication is a matrix multiplication.

\(p\) ... momentum
\(q\) ... position
Commutation relations:

\[ pq - qp = -i \frac{h}{2\pi} I. \]
$p$ and $q$ do not commute. It is not possible to consider finite dimensional matrices:

$$\text{trace}(pq - qp) = 0 \text{ while } \text{trace}\left(-i \frac{\hbar}{2\pi} I\right) \neq 0.$$

Concrete realisation:

- $H = L^2(\mathbb{R})$.
- $qf(x) = xf(x)$
- $p = \frac{\hbar}{2\pi} FqF^{-1}$, where $F$ is the Fourier-Plancherel transform.
A ... $C^*$-algebra with unit $1$

$(\text{Abel}(A), \subset)$ ... set of all abelian $C^*$-subalgebras of $A$ containing the unit $1$, ordered by set theoretic inclusion.

- $\text{Abel}(A)$ is a semilattice: $C_1 \land C_2 = C_1 \cap C_2$; $C_1 \lor C_2$ exists if, and only if, $C_1$ and $C_2$ mutually commute; with a least element $\{\lambda \mathbf{1} | \lambda \in \mathbb{C}\}$.
- $\bigcup_{C \in \text{Abel}(A)} C = \text{normal elements of } A$. 
Let $A$ and $B$ be $C^*$-algebras. A map $\psi : \text{Abel}(A) \to \text{Abel}(B)$ is an order isomorphism if

- $\psi$ is a bijection
- $C_1 \subset C_2 \iff \psi(C_1) \subset \psi(C_2)$ for all $C_1, C_2 \in \text{Abel}(A)$.

If $A, B$ are $\ast$-isomorphic unital $C^*$-algebras, then $\text{Abel}(A)$ and $\text{Abel}(B)$ are isomorphic semilattices. So

$$A \to \text{Abel}(A)$$

is an invariant. However, this invariant is not complete:

A. Connes (1962)

There is a $C^*$-algebra $A$ such that $A$ is not $\ast$-isomorphic to its opposite algebra $A^\circ$.

So in this case $\text{Abel}(A) = \text{Abel}(A^\circ)$, but there is no $\ast$-isomorphism between $A$ and $A^\circ$!
We shall be interested in a detailed relationship between order isomorphisms of the structure of subalgebras and relevant isomorphisms between operator algebras.

**Definition**

Let $A$ and $B$ be $C^*$-algebras. An order isomorphism $\varphi : Abel(A) \rightarrow Abel(B)$ is said to be implemented by a map $\psi : A \rightarrow B$ if

$$\varphi(C) = \psi(C) \quad \text{for all } C \in Abel(A).$$
The question of encoding the $C^*$-structure in $Abel(A)$ is nontrivial even if $A$ is abelian.

Mendivill (1999)

Let $X$ and $Y$ be compact Hausdorff spaces. If $Abel(C(X))$ is isomorphic to $Abel(C(Y))$, then $X$ and $Y$ are homeomorphic.
Theorem, 2011

Let $A$ and $B$ be commutative unital $C^*$-algebras. For any order isomorphism

$$
\varphi : \text{Abel}(A) \to \text{Abel}(B)
$$

there is a $\ast$-isomorphism

$$
\psi : A \to B
$$

such that $\psi$ implements $\varphi$.

Moreover, if $A$ has not dimension two, then there is only one $\ast$-isomorphism $\psi$ implementing $\varphi$. 
Remark: If $A = \mathbb{C} \oplus \mathbb{C}$, then identity map and the $\ast$-isomorphism exchanging the direct summands both implement the identical automorphism of $Abel(A)$.

**Conclusion:** For abelian algebras there is a one-to-one correspondence between the order isomorphisms of the structure of abelian subalgebras and $\ast$-isomorphisms of global algebras in the sense of implementation.
Le $A$ and $B$ be $C^*$-algebras. A linear map $\psi : A \rightarrow B$ is a Jordan $\ast$-homomorphism if

1. $\psi(x^2) = \psi(x)^2$ for all self-adjoint elements $x \in A$.  
2. $\psi(x^* ) = \psi(x)^*$ for all $x \in A$.

Jordan $\ast$-isomorphism is a bijective Jordan $\ast$-homomorphism.

For showing that a continuous map preserves abelian subalgebras it suffices to assume linearity only on abelian subalgebra. It motivates:

**Definition**

Let $A$ be $C^*$-algebra and $X$ a linear space. A map $\psi : A \rightarrow X$ is called **quasi-linear** if the following conditions are satisfied:

1. $\psi(\lambda x) = \lambda \psi(x)$ for all $\lambda \in \mathbb{C}$, $x \in A$.  
2. $\psi(x + y) = \psi(x) + \psi(y)$ for all $x, y \in A$ such that $xy = yx$. 
A map $\psi : A \to B$ between $C^*$-algebras $A$ and $B$ is called a quasi linear Jordan $\ast$-isomorphism if the following holds:

1. $\psi$ is a quasi-linear bijection,
2. $\psi(x^2) = \psi(x)^2$ for all self-adjoint $x \in A$,
3. $\psi(x^*) = \psi(x)^*$ for all $x \in A$.

Example of order isomorphism

Let $\psi : A \to B$ be a quasi linear $\ast$-isomorphism. Then the map

$$\varphi : \text{Abel}(A) \to \text{Abel}(B)$$

defined by

$$\varphi(C) = \psi(C) \quad C \in \text{Abel}(A)$$

is an order isomorphism.
Question
Is every order isomorphism implemented by a unique quasi-Jordan $\ast$-isomorphism?

Another obstacle for unicity: $A = B = M_2(\mathbb{C})$:
Identity order isomorphism can be implemented by many quasi Jordan isomorphisms.
Theorem, 2011

Let $A$ and $B$ be unital $C^*$-algebras. Suppose that $A$ is not isomorphic either to $\mathbb{C}^2$ or to $M_2(\mathbb{C})$. Let

$$\varphi : Abel(A) \to Abel(B)$$

be an order isomorphism. Then there is exactly one quasi Jordan $\ast$-isomorphism $\psi : A \to B$ implementing $\varphi$.

Quasi-linearity problem: When is a quasi-linear map between $C^*$-algebras linear?
Definition

A $C^*$-subalgebra of $B(H)$ is called a von Neumann algebra if $A'' = A$, where $A' = \{x \in B(H) \mid xy = yx \text{ for all } y \in A\}$

Generalized Gleason Theorem: The answer is yes for von Neumann algebras without Type $I_2$ direct summand and bounded maps.

Credits: Aaarnes, Gunson, Christensen, Paszckiewicz, Yeadon, Bunce, Wright (see J.Hamhalter: Quantum Measure Theory, (2003)).
Corollary

Any order isomorphism with domain $\text{Abel}(M)$, where $M$ is a von Neumann algebra without Type $I_2$ direct summand, $\dim M \geq 3$, is implemented by a unique Jordan $\ast$-isomorphism.
In the spirit of Bohr, quantum system can be viewed only through classical system. Formalization of this idea:

- Quantum system: Self-adjoint operators in a $C^*$-algebra $A$
- Structure of classical subsystems: $Abel(A)$

Intuitionistic quantum logic is based on the domain $Abel(A)$
There has been a very rapid development in this area recently.
