Optimizing simultaneously over the numerator and denominator polynomials in the Youla-Kučera parametrization

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***Abstract*— Traditionally, when approaching controller design with the Youla-Kučera parametrization of all stabilizing controllers, the denominator of the rational parameter is fixed to a given stable polynomial, and optimization is carried out over the numerator polynomial. In this work, we revisit this design technique, allowing to optimize *simultaneously* over the numerator and denominator polynomials. Stability of the denominator polynomial, as well as fixed-order controller design with H_∞ performance are ensured via the notion of a central polynomial and LMI conditions for polynomial positivity.**

I. Introduction

The Youla-Kučera (YK) parametrization of all stabilizing controllers [8], [17] is particularly useful for controller design because all the closed-loop system transfer functions depend *affinely* in the same parameter that can be optimized over the set of proper stable rational functions [9]. This central idea was used extensively in references [2], [3] which were pioneering works on controller design via convex optimization. In these references, and also in related further works [14], [11], [16] the denominator of the YK parameter is *fixed* and optimization is carried out over the numerator only, allowing for a convex linear matrix inequality (LMI) formulation.

In this paper, we revisit this design technique and we show how to optimize *simultaneously* over of the numerator and denominator polynomials. In order to ensure stability of the denominator polynomial we use an (admittedly potentially conservative) sufficient condition for stability proposed in [5], based on the notion of a central polynomial and LMI conditions for polynomial positivity.

Our main motivation is to extend our previous work [6] on SISO fixed-order H_∞ design. In this paper, we optimized directly in the controller parameter space (numerator and denominator polynomials) by using the above mentioned sufficient stability condition [5]. The approach is valid as soon as the numerator and denominator polynomials of the closed-loop transfer function depend *affinely* in the controller polynomials, which is true for SISO systems but also for MISO or SIMO systems, see [7] for numerical examples. Unfortunately, this is not true anymore for general MIMO systems, for which individual entries of the matrix transfer function generally depends *nonlinearly* in the controller polynomials. This problem can be overcome if, instead of optimizing over the controller polynomials, we optimize over the YK polynomials. Indeed, as already pointed out, the fundamental property of the YK parametrization is that individual numerator and denominator polynomials of the closed-loop matrix transfer function depend *affinely* in the YK numerator and denominator polynomials.

An often mentioned drawback of the YK parametrization is the generally high order of the controller obtained by optimizing over the YK parameter, especially when several design specifications are combined. Moreover, there is only an indirect relationship between the order of the YK parameter and the order of the controller. In this work we show that the order of controller can be enforced while simultaneously optimizing over the YK numerator and denominator polynomial parameters. In other words, we show that the YK parametrization can also prove useful for *fixed-order* controller design.

In this paper we focus exclusively on SISO systems. The extension to MIMO systems will be covered elsewhere. The outline of the paper is as follows. In section II we recall basics of the YK parametrization and we derive a *polynomial* formulation, in contrast with the standard rational formulation. The key idea is to optimize *simultaneously* over the numerator and denominator polynomials of the YK parameter. We show that the essential additional ingredient with respect to our previous work [6] is an additional linear *algebraic constraint* linking the controller polynomials and the YK polynomials. In section III we apply this polynomial formulation to design *fixed-order* controllers. In section IV we apply the polynomial YK parametrization to perform H_∞ design. As in [5] and [6] the key ingredient in the design procedure is the choice of a *central polynomial*, or desired characteristic polynomial.

II. Youla-Kučera parametrization of stabilizing controllers

Given a strictly proper plant

$$p = \frac{b_n}{a_n}$$

of order n with a_n and b_n coprime polynomials, suppose that we could find a proper stabilizing controller

$$\hat{k} = \frac{\hat{y}_n}{\hat{x}_n}$$

of order m with $\hat{x}_n$ and $\hat{y}_n$ coprime polynomials, in a standard negative feedback configuration, satisfying the pole placement Diophantine equation

$$a_n\hat{x}_n + b_n\hat{y}_n = a_d x_d \quad (1)$$

where stable right hand-side polynomial is a factor of two polynomials such that $\deg a_d = n$ and $\deg x_d = m$. Defining

$$a = \frac{a_n}{a_d}, \quad b = \frac{b_n}{a_d}, \quad \hat{x} = \frac{\hat{x}_n}{x_d}, \quad \hat{y} = \frac{\hat{y}_n}{x_d} \quad (2)$$

as proper stable rational functions with coprime numerator and denominator polynomials, the Diophantine equation becomes the well-known Bézout equation

$$a\hat{x} + b\hat{y} = 1.$$

It follows from the Youla-Kučera (YK) parametrization [8], [17] that all BIBO (bounded-input bounded-output = no impulsive modes) stabilizing controllers are parametrized as

$$k = \frac{y}{x} = \frac{\hat{y} - aq}{\hat{x} + bq}$$

where

$$q = \frac{\hat{q}_n}{\hat{q}_d}$$

is an arbitrary proper stable rational parameter, i.e. such that polynomial $\hat{q}_d$ is *stable* and polynomial $\hat{q}_n$ has equal or lower degree, see [9] for a survey. Writing the controller

$$k = \frac{y_n}{x_n}$$

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in terms of polynomials x_n and y_n, the parametrization becomes

$$(a_d x_d)^2 \begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} b_n x_d & a_d \hat{x}_n \\ -a_n x_d & a_d \hat{y}_n \end{bmatrix} \begin{bmatrix} \hat{q}_n \\ \hat{q}_d \end{bmatrix} \quad (3)$$

where $\hat{q}_d$ is an arbitrary stable polynomial and $\hat{q}_n$ is an arbitrary polynomial of equal or lower degree. Note that the common factor $(a_d x_d)^2$ in relation (3) is cancelled out when expressing k. However, it should be clear later on that this common factor is instrumental to the derivation of polynomial YK parameters.

Starting from relation (3), dividing both hand-sides by polynomial $a_d x_d$, and defining polynomials q_n and q_d such that $\hat{q}_n = a_d q_n$ and $\hat{q}_d = x_d q_d$, we obtain the following simplified equation

$$\underbrace{\begin{bmatrix} a_d x_d & 0 & -b_n & -\hat{x}_n \\ 0 & a_d x_d & a_n & -\hat{y}_n \end{bmatrix}}_{A} \begin{bmatrix} x_n \\ y_n \\ q_n \\ q_d \end{bmatrix} = 0 \quad (4)$$

indicating that polynomials x_n, y_n, q_n and q_d belong to the null-space of a given polynomial matrix A.

Reciprocally, polynomial YK parameters can be retrieved from the controller polynomials via the following relation

$$\begin{bmatrix} q_n \\ q_d \end{bmatrix} = \begin{bmatrix} \hat{y}_n & -\hat{x}_n \\ a_n & b_n \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix}.$$

Note that YK parameters q_n, q_d are polynomial (and not rational) because of the common factor $(a_d x_d)^2$ in relation (3).

Note that there is no loss of generality in assuming that q_n, q_d are polynomials, since if polynomial matrix A has a rational null-space then it has necessarily a polynomial null-space. Note also that polynomial matrix A has full row rank, hence the dimension of the null-space is equal to two. Let

$$N = \begin{bmatrix} x_{1n} & x_{2n} \\ y_{1n} & y_{2n} \\ q_{1n} & q_{2n} \\ q_{1d} & q_{2d} \end{bmatrix} \quad (5)$$

be a minimal basis in the sense of Forney [4] for the null-space of matrix A, i.e. such that

$$AN = 0$$

and the column degrees of N are minimal among all possible null-space bases. Then we can derive the main result of this section.

Lemma 1: Given an open-loop plant with denominator and numerator polynomials a_n and b_n, let $\hat{x}_n$ and $\hat{y}_n$ be denominator and numerator polynomials of a controller placing the closed-loop poles at $a_d x_d$, i.e. satisfying Diophantine equation (1). Extract a minimal polynomial basis N as in (5) for the null-space of the polynomial matrix A defined in (4). Then all the stabilizing controllers with denominator and numerator polynomials x_n and y_n can be generated for this plant as follows

$$\begin{bmatrix} x_n \\ y_n \end{bmatrix} = \begin{bmatrix} x_{1n} & x_{2n} \\ y_{1n} & y_{2n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

where λ_1 and λ_2 are polynomials such that the YK denominator polynomial $q_d = \lambda_1 q_{1d} + \lambda_2 q_{2d}$ is stable. The corresponding YK numerator polynomial is given by $q_n = \lambda_1 q_{1n} + \lambda_2 q_{2n}$.

III. Fixed-order controller design

Generally, the YK parametrization of all stabilizing controllers results in high-order controllers when fixing the denominator polynomial of the YK parameter. In this section, we show that Lemma 1 can be used to overcome this well-known drawback of the YK parameterization.

In [5] a convex inner approximation of the set of stable polynomials was proposed, based on the notion of a *central polynomial* and LMI conditions for polynomial positivity. For the paper to be self-contained, in the sequel we briefly describe this convex set in the special case of continuous-time (left half-plane, or Hurwitz) stability. Let $c(s) = c_0 + c_1 s + \cdots + c_\delta s^\delta$ be a given polynomial of degree δ with real coefficient vector $\bar{c} = [c_0 \ c_1 \ \cdots \ c_\delta]$ of length $\delta + 1$. Define matrices

$$\Pi_1 = \begin{bmatrix} 1 & & 0 \\ & \ddots & \vdots \\ & 1 & 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} 0 & 1 & \\ \vdots & & \ddots \\ 0 & & 1 \end{bmatrix}$$

of size δ-by-$(\delta+1)$. Then the following lemma [5] gives a sufficient LMI condition ensuring stability of a polynomial $d(s) = d_0 + d_1 s + \cdots + d_\delta s^\delta$ with real coefficient vector $\bar{d} = [d_0 \ d_1 \ \cdots \ d_\delta]$.

Lemma 2: Given a central polynomial $c(s)$ of degree δ, polynomial $d(s)$ of degree δ is stable if there exists a real symmetric matrix Q of size δ-by-δ such that

$$F_c(d, Q) = \bar{c}'\bar{d} + \bar{d}'\bar{c} - \varepsilon \bar{c}'\bar{c} + \Pi_1' Q \Pi_2 + \Pi_2' Q \Pi_1 \succeq 0$$

where $\succeq 0$ means positive semidefinite and ε is a given arbitrarily small positive scalar.

Defining

$$\mathcal{S}_c = \{d \, : \, \exists Q \, : \, F(d, Q) \succeq 0\}$$

as the projection of the LMI set $F(d, Q) \succeq 0$ onto the d-subspace, the above lemma ensures that $\mathcal{S}_c$ belongs to the non-convex set of coefficients of stable polynomials. Note that $\mathcal{S}_c$ is convex since it is the projection of a convex LMI set. Admittedly, the polynomial stability condition $d(s) \in \mathcal{S}_c$ is only sufficient, hence potentially conservative, and strongly depends on the choice of the central polynomial $c(s)$.

Corollary 1: A stabilizing controller of fixed order m with denominator and numerator polynomials x_n and y_n can be found if there exists polynomials λ_1 and λ_2 satisfying the (linear, hence convex) algebraic constraint

$$\deg \left(\begin{bmatrix} x_{1n} & x_{2n} \\ y_{1n} & y_{2n} \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} \right) = m \quad (6)$$

and the (convex) LMI constraint

$$\lambda_1 q_{1d} + \lambda_2 q_{2d} \in \mathcal{S}_c. \quad (7)$$

Example 1: For the numerical examples in this paper we used the Polynomial Toolbox for Matlab [13] to solve polynomial equations, and the SeDuMi LMI solver [15] interfaced with YALMIP [12].

Consider the open-loop plant

$$p = \frac{b_n}{a_n} = \frac{1}{s(s^2 + s + 10)}$$

of order $n = 3$ corresponding to a system with a flexible mode. By non-singularity of the Sylvester matrix, a system of order n can always be stabilized by a proper controller of order $m = n - 1$. Such a controller can be obtained by placing the poles at arbitrary locations. As appropriate denominator polynomials in (2) we choose e.g. $a_d = (s+1)^3$ and $x_d = (s+1)^2$ to place all the poles at -1 with a controller of order $m = 2$. Solving the Diophantine equation (1) we obtain the controller

$$\hat{k} = \frac{\hat{y}_n}{\hat{x}_n} = \frac{1 + 45s - 26s^2}{-4 + 4s + s^2}.$$

A minimal polynomial basis (5) for the polynomial matrix A in (4) is given by

$$N = \left[\begin{array}{cc} 0 & 1 \\ -1 & -26 \\ -4+4s+s^2 & -103+149s \\ -1 & -26+10s+s^2+s^3 \end{array}\right].$$

In virtue of Lemma 1, all the stabilizing controllers can be recovered from polynomials λ_1 and λ_2 such that YK polynomial $q_d = -\lambda_1 + \lambda_2(-26+10s+s^2+s^3)$ is stable.

It turns out that from the first two rows of matrix N, a controller of order $m = 0$ can be obtained by restricting parameters λ_1 and λ_2 to be constant. A simple root-locus argument can thus be invoked to show that polynomial q_d is stable if $-36 < \lambda_1 < -26$ and $\lambda_2 = 1$. For example with $\lambda_1 = -30$ we obtain controller and YK parameter polynomials $x_n = 1$, $y_n = 4$, $q_n = 17+29s-30s^2$ and $q_d = 4+10s+s^2+s^3$. This static controller

$$k = \frac{y_n}{x_n} = 4$$

assigns the characteristic polynomial at $a_n x_n + b_n y_n = q_d$, with roots at -0.4099 and $-0.2950 \pm j3.1098$.

IV. H_∞ DESIGN

The closed-loop sensitivity function is expressed as

$$\begin{aligned} S &= \frac{1}{1+kp} = ax = a\hat{x} + abq = \frac{a_n x_n}{a_n x_n + b_n y_n} \\ &= \frac{a_n(b_n q_n + \hat{x}_n q_d)}{q_d(a_n \hat{x}_n + b_n \hat{y}_n)} = \frac{a_n(a_d x_d x_n)}{q_d(a_d x_d)} = \frac{a_n x_n}{q_d} \end{aligned}$$

where, in virtue of relation (4), polynomial x_n satisfies the algebraic constraint

$$a_d x_d x_n = b_n q_n + \hat{x}_n q_d$$

which is linear in YK polynomials q_n and q_d. Similarly, the complementary sensitivity function is given by

$$\begin{aligned} T = 1 - S &= \frac{kp}{1+kp} = by \\ &= b\hat{y} - abq = \frac{b_n y_n}{a_n x_n + b_n y_n} = \frac{b_n y_n}{q_d} \end{aligned}$$

where polynomial y_n satisfies the algebraic constraint

$$a_d x_d y_n = -a_n q_n + \hat{y}_n a_d$$

which is also linear in YK polynomials q_n and q_d.

More generally, it can be shown that every closed-loop transfer function can be expressed as a ratio of polynomials which are *linear* functions of the YK polynomials q_n and q_d. The optimization procedure of [6] can thus be applied: H_∞ bounds on scalar transfer functions whose numerator and denominator polynomials depend affinely on design polynomials q_n and q_d can be expressed as an LMI in these design polynomials. Let us denote this LMI constraint by $\mathcal{H}$. The additional feature here is that design polynomials q_n and q_d must satisfy the additional linear *algebraic constraints* (4). As in section III, stability of polynomial q_d can be ensured with the help of a central polynomial c and the corresponding inner convex LMI approximation $\mathcal{S}_c$ of the set of stable polynomials.

Corollary 2: Closed-loop H_∞ performance can be guaranteed if design polynomials q_n and q_d satisfy the (convex) LMI constraint

$$q_n,\ q_d \in \mathcal{H},$$

the (convex) LMI constraint

$$q_d \in \mathcal{S},$$

and the (linear, hence convex) algebraic constraint (4).

Example 2: Consider as in Example 1 an open-loop plant with a flexible mode, with $a_n = s(s^2+s+10)$ and $b_n = 1$. Recall that initial second-degree controller polynomials $\hat{x}_n = -4+4s+s^2$ and $\hat{y}_n = 1+45s-26s^2$ were found by assigning the characteristic polynomial at $a_d x_d = (s+1)^5$, an arbitrary choice.

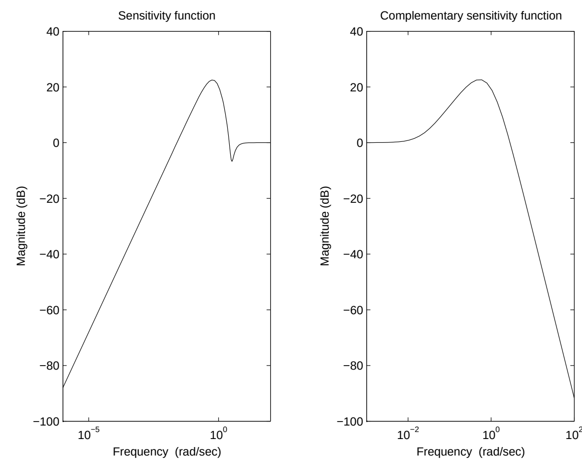

Fig. 1. Flexible mode controlled by initial controller. Closed-loop frequency-domain characteristics with large peaks.

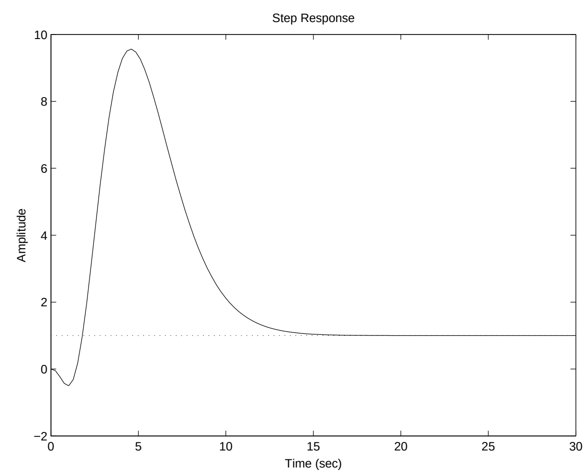

Fig. 2. Flexible mode controlled by initial controller. Closed-loop response to a step with unacceptable overshoot.

In Figure 1 we see that both sensitivity function S and complementary sensitivity function T feature large peaks with $\|S\|_\infty \approx 22.5$ dB and $\|T\|_\infty \approx 22.7$ dB. Recall that appropriate values are $\|S\|_\infty \approx 3$ dB and $\|T\|_\infty \approx 1$ dB [10], [1]. As a result, the closed-loop response to a step shown in Figure 2 shows an unacceptable overshoot as well as a small undershoot.

Mixing the methodology introduced in Corollaries 1 and 2 we can design a static H_∞ controller. For the central polynomial $c(s) = (s+0.5)(s^2+s+10)$ (mirroring stable open-loop poles, and with an additional controller pole) and by enforcing an upper bound on the H_∞-norm of the sensitivity function S, we solve the LMI problem for the YK polynomials $q_n = 7.3505 + 38.6495s - 27.588s^2$, $q_d = 1.5876 + 10.000s + 1.0000s^2 + 1.0000s^3$ and we obtain the static controller $K = y_n/x_n = 1.5876$ with closed-loop performance $\|S\|_\infty \approx 1.1$ dB and $\|T\|_\infty \approx 0$ dB. The closed-loop response to a step has an acceptable shape, but the settling time is quite large. Other attempts to reduce the settling time by enforcing a faster controller pole in $c(s)$ resulted in larger values of $\|S\|_\infty$ and undesirable oscillations. We obtained similar results with first-order controllers.

In order to improve the closed-loop performance, we thus seek a second-order controller. With an initial choice of $c(s) = (s+1)^3(s^2+s+10)$ for the central polynomial (mirroring stable open-loop poles and with additional controller poles at -1), and after a series of trial and attempts consisting in moving appropriately the controller poles in $c(s)$ and lowering the upper-bound on $\|S\|_\infty$ – see [6], [7] for details – we come up with the central polynomial $c(s) = (s+4)^2(s+3)(s^2+s+10)$ and the YK polynomials $q_n = 891.70+680.48s-467.07s^2-3.8198s^3-0.55993s^4$, $q_d = 214.59+340.12s+178.91s^2+54.913s^3+12.556s^4+s^5$ yielding the second-order controller

$$K = \frac{y_n}{x_n} = \frac{214.59+6.5461s+29.993s^2}{33.357+11.556s+s^2}$$

with closed-loop performance $\|S\|_\infty \approx 1.5$ dB and $\|T\|_\infty \approx 0$ dB, see Figure 3. The closed-loop response to a step shown in Figure 4 is now acceptable.

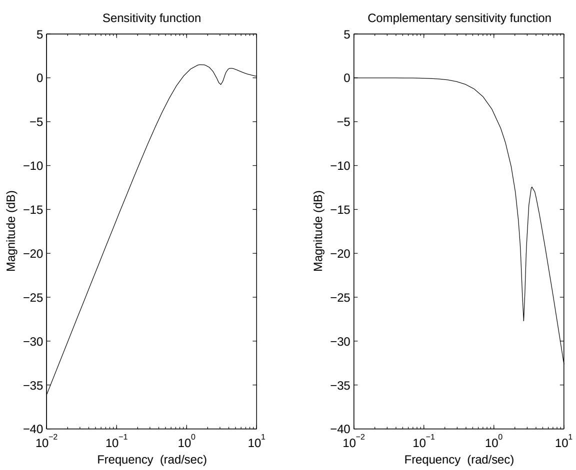

Fig. 3. Flexible mode controlled by optimized second-order controller. Closed-loop frequency-domain characteristics.

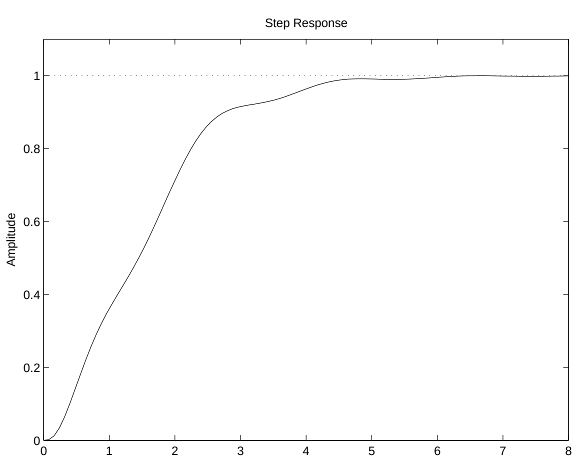

Fig. 4. Flexible mode controlled by optimized second-order controller. Closed-loop response to a step.

V. Conclusion

We have shown that optimizing *simultaneously* over the numerator and denominator polynomials of the rational YK parameter provides the designer with a greater flexibility. The essential difference with respect to our previous work [6] is an additional linear *algebraic constraint* linking the controller polynomials and the YK polynomials. As in [6], stability of the denominator polynomial is ensured with the sufficient condition proposed in [5], meaning that the key ingredient in the design procedure is the choice of the so-called *central polynomial* around which the closed-loop dynamics must be optimized.

With the help of numerical examples, it was shown that the approach is suitable for fixed-order and H_∞ controller design, along the lines initiated in [6]. It is therefore possible to control at will the growth in the controller order, hence overcoming an often mentioned difficulty of the YK parametrization.

The results reported here must be considered as preliminary since the ultimate objective, as pointed out in the introduction, is to extend the SISO, MISO or SIMO results of [6], [7] to genuine MIMO systems. This extension will be covered elsewhere.

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