Positive Polynomial Matrices and Improved LMI Robustness Conditions

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Abstract

Recently several new LMI conditions for stability of linear systems have been proposed, introducing additional slack variables to reduce the gap between conservative convex quadratic stability conditions and intractable non-convex robust stability conditions. In this paper we show that these improved LMI conditions can be derived with the help of some basic results on positive polynomial matrices. The approach allows us to derive in a unifying way results in the state-space and polynomial frameworks. Applications to robust stability analysis and robust stabilization of systems with multi-linear parametric uncertainty are fully described.

Keywords

Linear Systems, Robustness, LMI, State-space Methods, Polynomial Methods

1 Introduction

The main criticism formulated by control engineers against modern robust analysis and design methods for linear systems concerns the lack of efficient, easy-to-use and systematic numerical tools. Indeed, a lot of analysis techniques and most of the design techniques for uncertain systems boil down to non-convex bilinear matrix inequality (BMI) problems [Mesbahi, 2000], for which no polynomial-time algorithm has been proposed so far. This is especially true when analyzing robust stability of systems affected by highly structured (parametric) uncertainty, or when seeking a robust control law of low complexity (fixed- or low-order controller design).

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In the control literature, one can distinguish several fundamentally different approaches to circumvent these difficulties. One can either resort to (a) global optimization, with a guarantee of convergence, at the price of a considerable computational burden that may prove prohibitive, see e.g. [de Gaston, 1988, Psarris, 1995, Apkarian, 2000]; (b) randomized or probabilistic methods that allow to quantify the amount of confidence one can have in the result, see e.g. [Calafiore, 2000]; (c) iterative heuristics based on convex optimization, without guarantee of global convergence, but that can prove very efficient in practice and can solve problems at a low computational cost, see e.g. [El Ghaoui, 1997, Iwasaki, 1999]; (d) non-iterative sufficient robustness conditions, that rely on convex optimization too, but can prove overly conservative [Iwasaki, 1998, Scherer, 2001]. In this paper, we focus on the latter approach.

Within this scope, new LMI conditions for stability of linear systems have recently been introduced that partially alleviates some degrees of conservatism. The conditions, that can be traced back to [Geromel, 1998], were originally proposed for discrete-time systems only [de Oliveira, 1999]. Afterwards, they have been extended to more general stability regions [Peaucelle, 2000] and other problems in robust control [Shaked, 2001]. The improved LMI conditions feature additional slack variables allowing some decoupling between the system state-space matrices and the Lyapunov matrix proving robust stability. Based on the same idea, similar conditions have also been used to assess robust stability of polynomial matrices [Henrion, 2001a], an important step when studying uncertain systems in the polynomial framework [Kučera, 1979]. Numerical experiments reported in [Bachelier, 1999] show that the improved LMI conditions significantly narrow the gap between conservative LMI quadratic stability conditions and intractable non-convex robust stability conditions. In [de Oliveira, 2002] the additional slack variables are interpreted as Lagrange multipliers arising when relaxing robust stability tests.

The purpose of this paper is to provide another interpretation of the additional variables, and to show how the improved LMI conditions can be readily derived from basic results on strictly positive real functions and positive polynomial matrices. The approach is sufficiently general to cover in a unified fashion continuous-time and discrete-time systems in the state-space and polynomial frameworks. As a result, we obtain several new results for robust stability analysis and design in the polynomial framework.

2 Positivity and Stability of Polynomial Matrices

Let

$$\mathcal{D} = \{s \in \mathbb{C} : \begin{bmatrix} 1 \\ s \end{bmatrix}^\star \underbrace{\begin{bmatrix} a & b \\ b^\star & c \end{bmatrix}}_{H} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0\}$$

be a stability region in the complex plane, where the star denotes transpose conjugate and Hermitian matrix $H = H^\star$ has one positive eigenvalue and one negative eigenvalue. Standard choices for $\mathcal{D}$ are the left half-plane ($a = 0, b = 1, c = 0$) and the unit disk ($a = -1, b = 0, c = 1$). Other choices of scalars a, b and c correspond to arbitrary half-planes and disks. Let $\partial\mathcal{D}$ denote the boundary of $\mathcal{D}$, i.e. the set $\{s \in \mathbb{C} : a + bs + b^\star s^\star + css^\star = 0\}$. In the sequel we say that a polynomial matrix is stable when all the roots of its determinant belong to $\mathcal{D}$. Similarly, we say that a rational matrix $G(s)$ is strictly positive real when its real part $(G(s)+G^\star(s))/2$ is positive definite when evaluated along $\partial\mathcal{D}$.

In the paper we consider real square polynomial matrices

$$N(s) = N_0 + N_1 s + \cdots + N_d s^d, \quad D(s) = D_0 + D_1 s + \cdots + D_d s^d$$

of size n and degree d. Throughout the paper we assume that both leading coefficient matrices N_d and D_d are non-singular, so that polynomial matrices $N(s)$ and $D(s)$ do not feature infinite zeros. The reader interested in knowing how to handle infinite zeros of polynomial matrices is advised to consult [Henrion, 2001b].

Lemma 1 *Polynomial matrix $N(s)$ is stable if and only if there exists a stable polynomial matrix $D(s)$ such that rational matrix $N(s)D^{-1}(s)$ is strictly positive real.*

Proof: If $D(s)$ is stable, then it is non-singular along $\partial\mathcal{D}$. By definition, $N(s)D^{-1}(s)$ strictly positive real with $D(s)$ stable implies $N(s)$ stable, see [Šiljak, 1971, Šiljak, 1998]. Conversely, if $N(s)$ is stable then the choice $D(s) = N(s)$ makes the rational matrix $N(s)D^{-1}(s) = I$ obviously strictly positive real. $\square$

When $D(s)$ is a given stable polynomial matrix, Lemma 1 is a sufficient condition for stability of matrix polynomial $N(s)$. Let $\mathcal{N}$ denote the set of coefficients of all polynomial matrices $N(s)$ of degree d, let $\mathcal{S}$ denote the subset of $\mathcal{N}$ corresponding to stable polynomial matrices only, and let $\mathcal{S}_D$ denote the subset of $\mathcal{N}$ such that $N(s)D^{-1}(s)$ is strictly positive real. Then it holds $\mathcal{S}_D \subset \mathcal{S} \subset \mathcal{N}$. It is well-known that generally $\mathcal{S}$ is not a convex set. In contrast, it turns out that $\mathcal{S}_D$ is a convex set (an inner approximation of the actual stability domain) that can be described by a linear matrix inequality (LMI), as captured by the following lemma.

Lemma 2 *Let*

$$N = [N_0 \quad N_1 \ \cdots \ N_d], \quad D = [D_0 \quad D_1 \ \cdots \ D_d]$$

and

$$\Pi = \begin{bmatrix} I_n & & & 0 \\ & \ddots & & \vdots \\ & & I_n & 0 \\ 0 & I_n & & \\ \vdots & & \ddots & \\ 0 & & & I_n \end{bmatrix}$$

be a matrix of size $2dn \times (d+1)n$. Given a stable polynomial matrix $D(s)$, polynomial matrix $N(s)$ ensures strict positive realness of rational matrix $N(s)D^{-1}(s)$ if and only if there exists a matrix $P = P^\star$ of dimension dn such that

$$D^\star N + N^\star D - H(P) \succ 0 \tag{1}$$

where

$$H(P) = \Pi^\star \begin{bmatrix} aP & bP \\ b^\star P & cP \end{bmatrix} \Pi.$$

Proof of sufficiency: Let

$$B_d(s) = \begin{bmatrix} I_n \\ sI_n \\ \vdots \\ s^d I_n \end{bmatrix}$$

be a basis for polynomial matrices of size n and degree d. Pre- and post-multiply LMI (1) by $B_d(s)$ to obtain

$$B_d^\star(s)(D^\star N + N^\star D - H(P))B_d(s) = D^\star(s)N(s) + N^\star(s)D(s) - B_d^\star(s)H(P)B_d(s) \succ 0, \quad \forall s \in \partial\mathcal{D}.$$

From the definition of region $\mathcal{D}$, it holds

$$B_d^\star(s)H(P)B_d(s) = (a + bs + b^\star s^\star + css^\star)B_{d-1}^\star(s)PB_{d-1}(s) = 0, \quad \forall s \in \partial\mathcal{D} \tag{2}$$

and then

$$D^\star(s)N(s) + N^\star(s)D(s) \succ 0, \quad \forall s \in \partial\mathcal{D}. \tag{3}$$

Polynomial matrix $D(s)$ being stable, it is non-singular along $\partial\mathcal{D}$, and we can equivalently write

$$N(s)D^{-1}(s) + D^{-\star}(s)N^\star(s) \succ 0, \quad \forall s \in \partial\mathcal{D}$$

which is the strict positive realness condition for rational matrix $N(s)D^{-1}(s)$. □

Proof of necessity: Now assuming that rational matrix $N(s)D^{-1}(s)$ is strictly positive real, i.e. that matrix inequality (3) holds, there always exists a strictly positive scalar γ such that

$$R(s) = D^\star(s)N(s) + N^\star(s)D(s) - \gamma B_d^\star(s)B_d(s) \succeq 0, \quad \forall s \in \partial\mathcal{D}.$$

Relating strict positive realness of rational matrix $N(s)D^{-1}(s)$ and positivity of polynomial matrix $R(s)$ along the stability boundary is an idea that can be found e.g. in [Šiljak, 1971, Šiljak, 1998].

Now we show that the set of all constant matrices R such that

$$B_d^\star(s)RB_d(s) = R(s), \quad \forall s \in \partial\mathcal{D}$$

can be described as

$$R = D^\star N + N^\star D - H(P) - \gamma I_{(d+1)n}. \tag{4}$$

where $P = P^\star$ is an arbitrary matrix. Indeed, the whole set of polynomial matrices of degree d or less in s and $s^\star$ vanishing along $s \in \partial\mathcal{D}$ is described by the term $B_d^\star(s)H(P)B_d(s)$. To prove this, note that in equation (2) the vanishing scalar term $a + bs + b^\star s^\star + css^\star$ necessarily appears as a common factor in matrix $B_d^\star(s)H(P)B_d(s)$, while the remaining matrix factor $B_{d-1}^\star(s)PB_{d-1}(s)$ describes the whole set of polynomial matrices of degree $d-1$ or less.

The last step in the proof consists in showing that there always exists a matrix $P = P^\star$ such that $R \succeq 0$ as soon as a $R(s) \succeq 0$ along $s \in \partial\mathcal{D}$. This follows from the existence of a polynomial matrix spectral factor, i.e. there exists a matrix $Q(s) = Q_0 + Q_1 s + \cdots + Q_d s^d$ such that $R(s) = Q^\star(s)Q(s)$, see e.g. [Callier, 1985] or [Genin, 2000]. In other words, matrix R can be written as

$$R = \left[\begin{array}{c} Q_0 \\ Q_1 \\ \vdots \\ Q_d \end{array}\right] \left[\begin{array}{c} Q_0 \\ Q_1 \\ \vdots \\ Q_d \end{array}\right]^\star$$

which is a positive semidefinite matrix. Since matrix $P = P^\star$ describes in (4) the whole set of matrices $R(s)$ in basis $B_d(s)$, there necessarily exists a solution to LMI (1). $\square$

Combining Lemma 1 with Lemma 2, we deduce the following result which will be instrumental to the derivation of the LMI stability conditions in the sequel of the paper.

Lemma 3 *Polynomial matrix $N(s)$ is stable if and only if there exists a stable polynomial matrix $D(s)$ and a matrix $P = P^\star$ satisfying LMI (1).*

3 Connection with Existing LMI Stability Conditions

In this section, we show that we can easily recover the improved LMI stability conditions of [de Oliveira, 1999, Peaucelle, 2000] with the help of Lemma 3.

3.1 State-space Framework

Stability of the pencil matrix $N(s) = sI - A$ is equivalent to stability of the constant matrix A, and the LMI condition of Lemma 3 yields the following result.

Lemma 4 *Matrix A is stable if and only if there exists a non-singular matrix G, a matrix F and a matrix $P = P^\star \succ 0$ satisfying LMI*

$$\begin{array}{c} \left[\begin{array}{cc} F^\star A + A^\star F - aP & -A^\star G - F^\star - bP \\ -G^\star A - F - b^\star P & G + G^\star - cP \end{array}\right] = \\ \left[\begin{array}{c} -F^\star \\ G^\star \end{array}\right] \left[\begin{array}{cc} -A & I \end{array}\right] + \left[\begin{array}{c} -A^\star \\ I \end{array}\right] \left[\begin{array}{cc} -F & G \end{array}\right] - \left[\begin{array}{cc} aP & bP \\ b^\star P & cP \end{array}\right] \succ 0. \end{array} \quad (5)$$

Proof: Assume that LMI (5) holds for some $P = P^\star \succ 0$. Then

$$\begin{array}{l} -\left[\begin{array}{c} I \\ A \end{array}\right]^\star \left(\left[\begin{array}{c} -F^\star \\ G^\star \end{array}\right] \left[\begin{array}{cc} -A & I \end{array}\right] + \left[\begin{array}{c} -A^\star \\ I \end{array}\right] \left[\begin{array}{cc} -F & G \end{array}\right] - \left[\begin{array}{cc} aP & bP \\ b^\star P & cP \end{array}\right] \right) \left[\begin{array}{c} I \\ A \end{array}\right] \\ = aP + bPA + b^\star A^\star P + cA^\star PA \prec 0. \end{array}$$

In virtue of Lyapunov's Theorem (see e.g. [Chilali, 1996] or [Henrion, 2001c]), this implies that A is stable. Conversely, if A is stable then choose $N(s) = sI - A$ and $D(s) = sG - F$ as stable pencil matrices, and in virtue of Lemma 3 there exists a stable matrix $G^{-1}F$ and a matrix $P = P^\star$ satisfying LMI (5). Then

$$-\left[\begin{array}{c} I \\ G^{-1}F \end{array}\right]^\star \left(\left[\begin{array}{c} -F^\star \\ G^\star \end{array}\right]\left[\begin{array}{cc} -A & I \end{array}\right] + \left[\begin{array}{c} -A^\star \\ I \end{array}\right]\left[\begin{array}{cc} -F & G \end{array}\right] - \left[\begin{array}{cc} aP & bP \\ b^\star P & cP \end{array}\right]\right)\left[\begin{array}{c} I \\ G^{-1}F \end{array}\right]$$
$$= aP + bP(G^{-1}F) + b^\star(G^{-1}F)^\star P + c(G^{-1}F)^\star P(G^{-1}F) \prec 0.$$

Since matrix $G^{-1}F$ is stable, we can invoke Theorem 2.2.3 in [Horn, 1991] to conclude that $P \succ 0$. □

From Lemma 4 we recover discrete-time Theorem 1 of [de Oliveira, 1999] as a special case by letting $a = -1$, $b = 0$, $c = 1$ and $F = 0$. We also recover Theorem 3 of [Peaucelle, 2000] (for stability regions defined by a scalar inequality) by letting $a = R_{11}$, $b = R_{12}$ and $c = R_{22}$. We see here clearly the role played by additional variables F and G: they are coefficients of a polynomial matrix ensuring stability via a strict positive realness condition.

The nice feature in Lemma 4 is that system matrix A and Lyapunov matrix P are decoupled in the sense that, contrary to the standard Lyapunov inequality, there is no cross-product between A and P. As shown in [de Oliveira, 1999] and [Peaucelle, 2000], when matrix A is affected by parametric uncertainty, this allows for the construction of a parameter-dependent Lyapunov matrix significantly narrowing the gap between conservative standard quadratic stability results (corresponding to a parameter-independent Lyapunov matrix) and intractable robust stability results.

Note that the nice decoupling between system matrix and Lyapunov matrix follows from the use of strict positive realness conditions to ensure stability. The decoupling disappears when formulating stability using the standard Lyapunov conditions.

3.2 Polynomial Framework

Now if we assume that $N(s)$ is an arbitrary polynomial matrix, Lemma 3 can be reformulated as follows.

Lemma 5 *Polynomial matrix $N(s)$ is stable if and only if there exists a polynomial matrix $D(s)$ and a matrix $P = P^\star \succ 0$ satisfying*

$$D^\star N + N^\star D - H(P) \succ 0. \tag{6}$$

Proof: We proceed as in the proof of Lemma 4, showing that the requirement that $D(s)$ be stable in Lemma 3 amounts to enforcing positivity of P. We pre- and post-multiply

LMI (6) by matrix

$$\begin{bmatrix} I_n & & \\ & \ddots & \\ & & I_n \\ -D_d^{-1}D_0 & \cdots & -D_d^{-1}D_{d-1} \end{bmatrix}$$

and stability is proved with Lyapunov's Theorem applied to the companion matrix associated with polynomial matrix $D(s)$. □

A slightly different version of Lemma 5 appeared in [Henrion, 2001b], with a different proof based on quadratic programming and semidefinite programming duality.

Lemma 6 *[Henrion, 2001b] Polynomial matrix $N(s)$ is stable if and only if there exists a matrix $P = P^\star \succ 0$ satisfying*

$$N^\star N - H(P) \succ 0.$$

Proof: It is an application of the projection lemma, an algebraic result well-known to the control community, see e.g. [Skelton, 1998, Theorem 2.3.12]. □

4 Application to Robust Stability Analysis

In all the following numerical examples we solved the LMI problems with SeDuMi 1.04 [Sturm, 1999] running under Matlab 6.1 on a Sun Sparc Workstation Ultra 5.

We notice that the LMI condition of Lemma 5 is simultaneously linear in coefficients of $N(s)$ and in Lyapunov matrix P. This property can be exploited to provide less conservative convex conditions than the well-known quadratic stability for assessing robust stability of polynomial matrices, as shown in the following.

Assume that $N(s,\lambda)$ is a polynomial matrix with multi-linear dependence in a parameter vector λ that lives in an m-dimensional hyper-rectangle. When λ describes the vertices of Λ we denote the vertices of $N(s,\lambda)$ by $N_i(s)$ for $i = 1,\ldots,2^m$. With these notations, we can formulate the following lemma.

Lemma 7 *[Henrion, 2001a] Multi-linear polynomial matrix $N(s,\lambda)$ with vertices $N_i(s)$ is robustly stable if there exists a matrix D and 2^m matrices $P_i = P_i^\star \succ 0$ satisfying the LMI*

$$D^\star N_i + N_i^\star D - H(P_i) \succ 0, \quad i = 1,\ldots,2^m. \tag{7}$$

Proof: Since LMI (7) is simultaneously linear in N_i and in P_i, parameter λ enters multi-linearly in $N(s,\lambda)$, and Λ is a hyper-rectangle, it follows that

$$\sum_{i=1}^{2^m} \lambda_i (D^\star N_i + N_i^\star D - H(P_i)) = D^\star N(\lambda) + N(\lambda) D - H(P(\lambda))\succ 0$$

where $P(\lambda) \succ 0$ is a parameter-dependent multi-linear Lyapunov matrix proving robust stability of polynomial matrix $N(s, \lambda)$ in virtue of Lemma 5. $\square$

Note that when all system vertices share the same Lyapunov matrix $P_i = P$ we retrieve the quadratic stability results of [Henrion, 2001a] as a special case. As a consequence, LMI stability condition (7) is always less conservative than the standard LMI quadratic stability condition of [Henrion, 2001a] when assessing robust stability of a polytope of polynomial matrices.

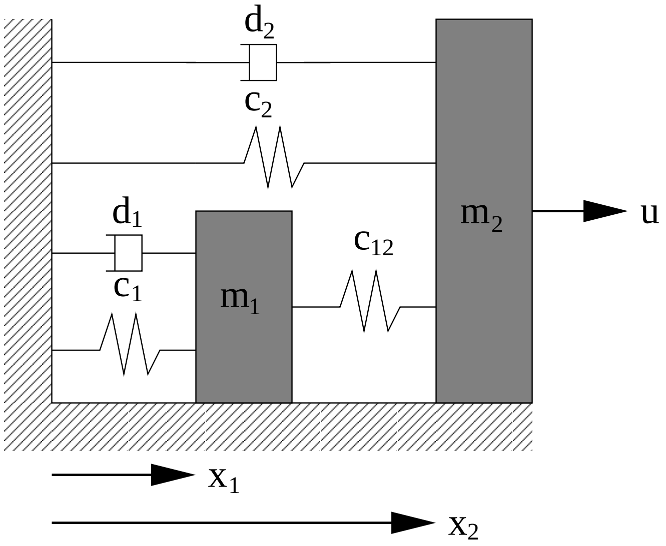

Figure 1: Mechanical system.

As a numerical example, we consider the mechanical system studied in [Ackermann, 1993, Example 6.10], represented in Figure 1, whose differential equations after application of the Laplace transform are given by

$$\begin{bmatrix} m_1 s^2 + d_1 s + c_1 + c_{12} & -c_{12} \\ -c_{12} & m_2 s^2 + d_2 s + c_2 + c_{12} \end{bmatrix} \begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} 0 \\ u(s) \end{bmatrix}. \tag{8}$$

We assume that system parameters $\lambda = [\, m_1 \quad d_1 \quad c_1 \quad m_2 \quad d_2 \quad c_2 \,]$ belong to the uncertainty hyper-rectangle $\Lambda = [1,\ 3] \times [0.5,\ 2] \times [1,\ 2] \times [2,\ 5] \times [0.5,\ 2] \times [2,\ 4]$ and we set $c_{12} = 1$. As pointed out in [Ackermann, 1993], this mechanical system is passive so it must be open-loop stable (when $u(s) = 0$) independently of the values of the masses, springs, and dampers. However, it is a non-trivial task to know whether the open-loop system is robustly stable in some stability region $\mathcal{D}$ ensuring a certain damping. Here we choose the disk of radius 12 centered at -12, i.e.

$$\mathcal{D} = \{s \in \mathbb{C} : (s+12)^2 < 12^2\} = \{s \in \mathbb{C} : \begin{bmatrix} 1 \\ s \end{bmatrix}^\star \begin{bmatrix} 0 & 12 \\ 12 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0\}.$$

The robust stability analysis problem amounts then to assessing whether the second degree polynomial matrix in the left hand-side of equation (8) is robustly stable in $\mathcal{D}$ for all admissible uncertainty. This is an interval polynomial matrix with $2^6 = 64$ vertices.

We set $a = 0$, $b = 12$ and $c = 1$ in LMI (7) and with the help of SeDuMi we find this problem feasible. Therefore, the root-locus of the polynomial matrix remains in disk $\mathcal{D}$ for all admissible uncertainty. In Figure 2 we represented the roots of the 64 polynomial matrix vertices. Note however that an accurate 6-dimensional brute force gridding of the root-locus would be required to check robust stability graphically, since there is no vertex or edge result available for interval polynomial matrices.

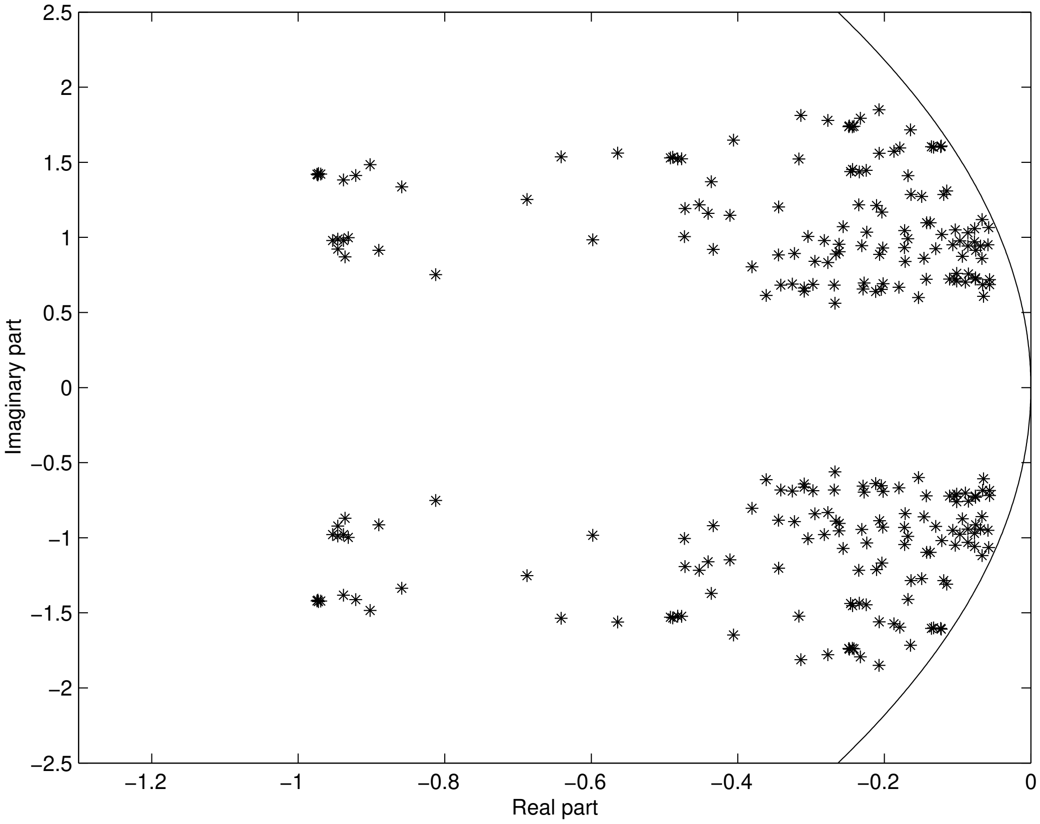

Figure 2: Disk of radius 12 and center -12 including the roots of the 64 polynomial matrix vertices modeling the uncertain mechanical system.

5 Application to Robust Design

Now we extend Lemma 3 to perform robust design of linear systems described by polynomial matrices. The transfer function of the uncertain linear system is $B(s,\lambda)A^{-1}(s,\lambda)$ where both $A(s,\lambda)$ and $B(s,\lambda)$ are polynomial matrices that are multi-linear in a parameter vector λ that lives in an m-dimensional hyper-rectangle Λ. The design problem amounts to finding a dynamical output-feedback controller with transfer function $X^{-1}(s)Y(s)$ such that the closed-loop denominator matrix

$$N(s,\lambda) = A(s,\lambda)X(s) + B(s,\lambda)Y(s)$$

is robustly stable for all admissible uncertainty vector λ. As usual, when λ describes the vertices of Λ we denote the vertices of $A(s,\lambda)$, $B(s,\lambda)$ and $N(s,\lambda)$ by A_i, B_i and N_i

respectively, for $i = 1, \ldots, 2^m$.

Moreover, we assume that the controller polynomial matrices $X(s) = X_0 + X_1 s + \cdots$ and $Y(s) = Y_0 + Y_1 s + \cdots$ entering linearly in polynomial matrix $N(s,\lambda)$ have prescribed structure, which we denote by the LMI

$$G(N) \succeq 0. \tag{9}$$

For example, we can assume that $X(s)$ and $Y(s)$ have given (presumably low) degree. Some coefficients in $X(s)$ and $Y(s)$ may be given, such as with a PID controller

$$\frac{Y(s)}{X(s)} = k_P + \frac{k_I}{s} + k_D s \tag{10}$$

where $X_0 = 0$, $X_1 = 1$, $X_2 = 0$ and $Y_0 = k_I$, $Y_1 = k_P$, $Y_2 = k_D$.

Under these assumptions, we can formulate the following lemma.

Lemma 8 *System $A(s,\lambda)$, $B(s,\lambda)$ with multi-linear uncertainty and vertices $A_i(s)$, $B_i(s)$ is robustly stabilizable by a constrained output feedback controller $X(s)$, $Y(s)$ if, given a stable polynomial matrix $D(s)$ of the same degree as polynomial matrices $N_i(s) = A_i(s)X(s) + B_i(s)Y(s)$, there exists some matrices $P_i = P_i^\star$ satisfying the LMI*

$$D^\star N_i + N_i^\star D - H(P_i) \succ 0, \quad i = 1, \ldots, 2^m. \tag{11}$$

with the additional LMI constraint (9).

To illustrate this result, we consider the scalar system studied in [Jaulin, 1996] described by the transfer function

$$\frac{B(s)}{A(s)} = \frac{K w_0^2}{(1+Ts)(s^2 + 2z w_0 s + w_0^2)}$$

with nominal parameters $z = 1$, $T = -1$, $w_0 = 1$, $K = 1$. It is assumed that parameters z, T and K are uncertain and may vary by $\pm 10\%$ around their nominal values. We seek a PID controller (10) that robustly stabilizes the system for all possible parametric uncertainty. Moreover, we would like to ensure a robust stability margin of 0.1, i.e. the stability region is the shifted left half-plane

$$\mathcal{D} = \{s \in \mathbb{C} : \mathrm{Re}\, s < -0.1\} = \{s \in \mathbb{C} : \begin{bmatrix} 1 \\ s \end{bmatrix}^\star \begin{bmatrix} 0.2 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ s \end{bmatrix} < 0\}.$$

The closed-loop characteristic polynomial can be written as

$$N(s,\lambda) = w_0^2 K k_I + w_0^2(1 + K k_P)s + w_0(2z + w_0 T + w_0 K k_D)s^2 + (2z w_0 T + 1)s^3 + T s^4$$

which is a multi-linear function of uncertain parameters z, T and K and a linear function of controller parameters k_I, k_P and k_D, so that we can apply Lemma 8.

First we stabilize the nominal system with an arbitrary stabilizing PID controller, say $-(4+1/s+8s)$ which corresponds to the stable nominal polynomial

$$D(s) = -(1+3s+7s^2+s^3+s^4).$$

Second, we set $a = 0.2$, $b = 1$ and $c = 0$ in LMI (11) and with the help of SeDuMi we find this problem feasible. We obtain the robustly stabilizing PID controller

$$-(2.839+0.8105/s+7.400s).$$

Finally, note that another choice of a nominally stabilizing PID controller, such as the one provided in [Jaulin, 1996], namely $-(3.750+0.6250/s+8.750s)$, leads to a different, still robustly stabilizing PID controller

$$-(3.222+0.5818/s+8.069s).$$

6 Conclusion

We have proposed a new approach to derive improved LMI robustness conditions, narrowing the gap between conservative convex quadratic stability conditions and intractable non-convex robust stability conditions. The approach is general enough to treat in a unifying way continuous-time and discrete-time systems in the state-space and polynomial frameworks. It is based on the theory of positive polynomial matrices: stability of a polynomial matrix $N(s)$ is ensured if and only if some bilinear polynomial matrix form in $N(s)$ and $D(s)$ is positive along the stability boundary for some stable polynomial matrix $D(s)$ to be found.

Usually, polynomial matrix $N(s)$ is unknown (it may depend linearly on design parameters) and polynomial matrix $D(s)$ is given (it plays the role of an additional slack variable decoupling $N(s)$ from the Lyapunov matrix). So when $N(s)$ is assumed to belong to a given set accounting for possible uncertainty affecting some dynamical system, one important step in solving the robustness problem amounts to finding a sensible choice of an additional variable $D(s)$. Based on the numerical examples described in this paper, it is tempting to set $D(s)$ to some nominal value of $N(s)$, that is to say the nominal closed-loop system denominator matrix obtained with a standard design algorithm in the absence of uncertainty. In particular, it is believed that such a choice can prove useful in heuristic iterative design schemes based on cone complementarity, where $N(s)$ and $D(s)$ are iteratively sought and an initial guess on $D(s)$ is generally required to start the overall algorithm. However, recent findings on robust strictly positive real design with l_2 uncertainty [Bianchini, 2001] tend to show that the choice of the nominal polynomial $N(s)$ as an additional variable $D(s)$ is not always appropriate. This point needs to be studied in deeper detail.

Note that we considered only very simple stability regions $\mathcal{D}$ such as half-planes or disks. More involved stability regions (sectors, ellipses, hyperbolas) can be handled similarly with the help of the Kronecker product, see [Peaucelle, 2000]. This is not detailed in this paper for notational simplicity.

We have studied robust stability matrices and polynomial matrices affected by (highly structured) multi-linear interval uncertainty. We are now applying the same ideas to linear systems affected by (unstructured) ellipsoidal uncertainty, also known as l_2, or rank-one uncertainty. We may also extend our approach to meet other requirements that mere stability, such as H_2 or H_∞ performance, tracking or disturbance rejection. The key point is in fixing some nominally stable denominator polynomial matrix $D(s)$ in order to formulate stability, H_2 or H_∞ conditions via strict positive realness conditions. Related results will be reported elsewhere.

Besides providing a clear, new interpretation of the additional variables featured in the improved LMI robustness conditions, the approach proposed in this paper is also interesting from the numerical point of view. Indeed, the Hankel or Toeplitz structure of the dual LMI problem associated with problems coming from positivity of polynomials can be exploited to reduce the computational cost of solving the original primal problem, as shown in [Alkire, 2001] in the scalar case. Extension of these algorithms to block-Hankel and block-Toeplitz matrices is a challeging research topic. Numerical properties of these kind of optimization problems must however be studied in further detail, since it is well-known that positive definite Hankel matrices are exponentially ill-conditionned.

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