Control of linear systems subject to time-domain constraints with polynomial pole placement and LMIs

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Abstract: The paper focuses on the control of continuous-time linear systems subject to time-domain constraints (input amplitude limitation, output overshoot) on closed-loop signals. Using recent results on positive polynomials, it is shown that finding a Youla-Kučera polynomial parameter of fixed degree (hence a controller of fixed order) such that time-domain constraints are satisfied amounts to solving a convex linear matrix inequality (LMI) optimization problem as soon as distinct strictly negative poles are assigned by pole placement.

Keywords: Linear systems, time-domain constraints, polynomials, pole placement, LMI

1 Introduction

Time-domain specifications on closed-loop system signals are ubiquitous (actuator amplitude or rate limits, no output signal overshoot or undershoot, trajectory planning constraints, among others), yet a very few computer-aided control system design techniques can handle them. Usually these time-domain constraints are met indirectly via frequency-domain constraints on weighted transfer functions [4].

Since physical actuators cannot deliver unlimited signals to controlled plants, the problem

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of amplitude limitation control and its impact on both closed-loop stability and performance has received a lot of attention in the last decade. The stability analysis and controller design problems have been addressed through two main approaches. The first approach consists in taking into account the saturation effects in the controller design phase. Different methods dealing with the global, semi-global and local stabilization of the closed-loop system can be identified, see, among others [15, 10, 5]. The second approach assumes that a controller has been previously designed for the linear system, and that the saturation effects on stability and performance are then considered a posteriori. In this paper, the technique developed fits the first approach: control saturation is avoided to ensure linear system behavior, control constraints being linearly satisfied. The closed-loop system must stay in a region of linear behavior, see, among others [2, 6].

In our previous work [6] we focused on discrete-time systems described by polynomials in the time-shift variable z, and we showed that by truncating the rational Youla-Kučera parameter to a finite degree polynomial in z^{-1}, time-domain constraints can be readily handled by standard linear matrix inequality (LMI) convex optimization techniques. Truncating the Youla-Kučera parameter amounts to placing all the system poles to the origin, i.e. enforcing a closed-loop characteristic polynomial to z^k for a sufficiently large positive integer power k. This approach is generally referred to as the finite settling time control.

In this paper we try to extend the results of [6] to continuous-time systems. As a first attempt, one can think of

1. applying a one-to-one bilinear mapping in order to transform a continuous-time system (described by polynomials in the Laplace variable s) into a discrete-time system (described by polynomials in the time-shift variable z);
2. solving the time-domain constrained control problem in the discrete-time setting, following [6];
3. converting back the discrete-time system to a continuous-time system using the inverse bilinear mapping.

Unfortunately, this approach turns out to be inappropriate because there is no direct correspondence between time-domain constraints on discrete-time signals and time-domain constraints on the corresponding continuous-time signals.

In the sequel we show that recent results on positive polynomials [12, 7] can be used to cope with continuous time-domain constraints as soon as distinct negative real closed-loop poles are assigned. This establishes a parallel with our previous work on discrete-time systems [6], where all the closed-loop poles were assigned to the origin. More importantly, we show that under this pole placement restriction, the problem of finding a polynomial Youla-Kučera parameter of given degree (hence a controller of given, presumably low order) such that time-domain constraints are satisfied can be formulated equivalently (necessary and sufficient conditions) as a convex LMI problem for which off-the-shelf solvers are available. For simplicity of exposition, we limit our study to scalar (single input single output) systems.

The outline of the paper is as follows. In section 2 we state the control problem to be solved. In section 3 we recall standard results on the polynomial Youla-Kučera parametrization of all the controllers assigning given poles in closed-loop. Then in section 4 we use results on positive polynomials to derive an LMI formulation of our control problem. The approach is illustrated in section 5 with a numerical example, and the final section 6 contains important concluding remarks and directions for further research.

2 Problem setting

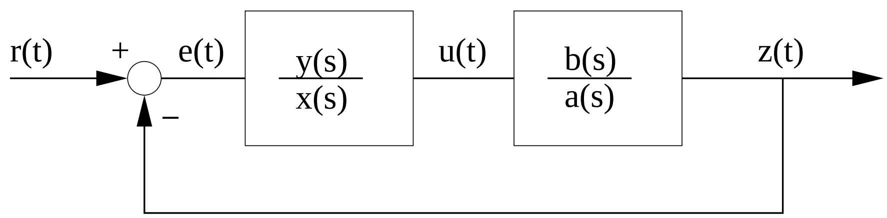

Figure 1: Standard negative feedback configuration.

Given a linear scalar continuous-time plant with input $u(t)$, output $z(t)$ and strictly proper transfer function

$$\frac{b(s)}{a(s)}$$

we are seeking a proper dynamic output feedback controller

$$\frac{y(s)}{x(s)}$$

in the standard negative feedback configuration of Figure 1 with reference input $r(t)$, such that the error signal $e(t)$ vanishes asymptotically and the time-domain constraints

$$u_{\min} \leq u(t) \leq u_{\max}, \quad z_{\min} \leq z(t) \leq z_{\max} \tag{1}$$

are satisfied for all $t \geq 0$, where $u_{\min}$, $u_{\max}$, $z_{\min}$ and $z_{\max}$ are given real numbers. Applying the Laplace transform, transfer functions between reference input $r(s)$ and input and output signals $u(s)$ and $z(s)$ are given by

$$\frac{u(s)}{r(s)} = \frac{a(s)y(s)}{a(s)x(s)+b(s)y(s)}, \quad \frac{y(s)}{r(s)} = \frac{b(s)y(s)}{a(s)x(s)+b(s)y(s)}$$

respectively.

We suppose that the above transfer functions are given in a polynomial setting, i.e. $a(s)$, $b(s)$ are given polynomials, and $x(s)$, $y(s)$ are polynomials to be found such that time-domain constraints (1) are met.

3 Pole placement design

Controller design is carried out by solving the following polynomial Diophantine equation

$$a(s)x(s) + b(s)y(s) = c(s) \tag{2}$$

where $c(s)$ is the pole polynomial of the closed-loop system. This is the so-called pole placement strategy, where polynomial $c(s)$ specifies the poles desired in closed-loop [8].

Recall that equation (2) is solvable if and only if every common divisor of $a(s)$ and $b(s)$ is a divisor of $c(s)$. Suppose that $a(s)$ and $b(s)$ are coprime, so that (2) is solvable for any right hand-side polynomial $c(s)$, and denote $x_0(s)$, $y_0(s)$ the unique solution pair such that $\deg y_0(s) < \deg a(s)$, the so-called y-minimal solution pair. Since the Diophantine equation is linear, any and all solutions pairs of (2) are given by

$$x(s) = x_0(s) + b(s)q(s), \quad y(s) = y_0(s) - a(s)q(s) \tag{3}$$

where

$$q(s) = q_0 + q_1 s + q_2 s^2 + \cdots$$

is an arbitrary polynomial such that $x_0(s)+b(s)q(s)$ is non-zero. Polynomial $q(s)$ is called the Youla-Kučera parameter [8].

We are interested only in polynomials $q(s)$ yielding a proper controller, and this is captured by the following result [9, Theorem 1], see also [3].

Lemma 1 *In Diophantine equation (2) suppose that* $\deg a(s) > \deg b(s)$ *and* $\deg c(s) \geq 2\deg a(s) - 1$. *Then polynomials* $x(s)$, $y(s)$ *in (3) are such that* $\deg x(s) \geq \deg y(s)$ *if and only if*

$$\deg q(s) \leq \deg c(s) - 2\deg a(s). \tag{4}$$

4 LMI formulation of time-domain constraints

Suppose that the system is excited with a step reference input $r(t)$. The Laplace transforms of input signal $u(t)$ and output signal $z(t)$ are

$$u(s) = \frac{1}{s}\frac{a(s)y(s)}{c(s)} = \frac{1}{s}\frac{a(s)y_0(s)}{c(s)} - \frac{1}{s}\frac{a^2(s)}{c(s)}q(s)$$

and

$$z(s) = \frac{1}{s}\frac{b(s)y(s)}{c(s)} = \frac{1}{s}\frac{b(s)y_0(s)}{c(s)} - \frac{1}{s}\frac{a(s)b(s)}{c(s)}q(s)$$

respectively. Note that these closed-loop transfer functions are affine in $q(s)$, which is a fundamental property of the Youla-Kučera parametrization.

Now denoting p_i, $i = 1, \ldots, n$ the assigned closed-loop poles, i.e. the roots of pole polynomial $c(s) = \prod_{i=1}^{n}(s - p_i)$, we make the following fundamental assumption.

Assumption 1 *Assigned poles p_i are distinct strictly negative rational numbers.*

Under this assumption, both input and output signal Laplace transforms can be decomposed as

$$u(s) = \sum_{i=0}^{n} \frac{u_i}{s - p_i}, \quad z(s) = \sum_{i=0}^{n} \frac{z_i}{s - p_i}$$

where $p_0 = 0$ corresponds to the Laplace transform of the step reference input, and u_i, z_i are constant real residues of the fraction decompositions. Corresponding time-domain signals, obtained by the inverse Laplace transform, are sums of decaying exponential modes

$$u(t) = \sum_{i=0}^{n} u_i e^{-p_i t}, \quad z(t) = \sum_{i=0}^{n} z_i e^{-p_i t}.$$

Let $p_i = n_i/d_i$ be ratios of integers, and let m denote the least common multiple of the denominators, such that $p_i = k_i/m$ for some integers k_i. Input and output signals can now be expressed as polynomials in the indeterminate

$$\lambda = e^{-t/m}$$

namely

$$u(\lambda) = \sum_{i=0}^{n} u_i \lambda^{k_i}, \quad z(\lambda) = \sum_{i=0}^{n} z_i \lambda^{k_i}.$$

When time t increases from 0 to $+\infty$, indeterminate λ decreases from 1 to 0, so that polynomials $u(\lambda)$ and $z(\lambda)$ are defined only on the interval $[0,\ 1]$. Enforcing constraints (1) then amounts to enforcing polynomial bound constraints

$$u_{\min} \leq u(\lambda) \leq u_{\max}, \quad z_{\min} \leq z(\lambda) \leq z_{\max}, \quad \forall \lambda \in [0,\ 1]. \tag{5}$$

With the help of the following result, we can derive an LMI formulation of these constraints [7, Lemma 1].

Lemma 2 *Polynomial non-negativity constraint*

$$p(\lambda) = \sum_{i=0}^{2n} p_i \lambda^i \geq 0, \quad \forall \lambda \in [\lambda_{\min},\ \lambda_{\max}]$$

is equivalent to the existence of symmetric matrices $P_{\min}$, $P_{\max}$ of size $n+1$ satisfying the convex LMI constraints

$$\begin{array}{l} p_i = \mathrm{trace}\left[P_{\min}(H_{i-1} - \lambda_{\min} H_i)\right] + \mathrm{trace}\left[P_{\max}(\lambda_{\max} H_i - H_{i-1})\right], \quad i = 0, 1, \ldots, 2n \\ P_{\min} \succeq 0, \quad P_{\max} \succeq 0 \end{array} \tag{6}$$

where H_i is the Hankel matrix with ones along the $(i+1)$-th anti-diagonal, and $\succeq 0$ means positive semidefinite.

Polynomial bound constraints (5) can be written as positivity constraint for polynomials $p_1(\lambda) = u(\lambda) - u_{\min}$, $p_2(\lambda) = -u(\lambda) + u_{\max}$, $p_3(\lambda) = z(\lambda) - z_{\min}$ and $p_4(\lambda) = -z(\lambda) + z_{\max}$, respectively. In turn, these positivity constraints can be expressed and combined as LMI constraints using Lemma 2. Since coefficients of polynomials $u(\lambda)$ and $z(\lambda)$ are affine in unknown design parameters q_i, the coefficients of Youla-Kučera polynomial parameter $q(s)$, we come up with our main result.

Theorem 1 *Under pole placement Assumption 1, closed-loop time-domain constraints (1) are satisfied by controller polynomials (3) if and only if there exists a polynomial Youla-Kučera parameter $q(s)$ with degree bound (4) whose coefficients satisfy the LMIs (6) corresponding to polynomial positivity constraints (5).*

5 Example

In the following numerical example, polynomial equations were solved with the Polynomial Toolbox for Matlab [13], and LMI problems were solved with the help of the YALMIP interface [11] and the SeDuMi solver [14].

Consider the continuous-time plant

$$\frac{b(s)}{a(s)} = \frac{0.5 + s}{s(s-2)}$$

for which we want to design a controller

$$\frac{y(s)}{x(s)}$$

ensuring asymptotic step reference tracking when placed in a standard negative feedback configuration of Figure 1. The open-loop plant integrator ensures asymptotic zero error position, and stabilization is ensured by pole placement.

Assume we want to assign closed-loop poles at -1, -2, -3, -4 and -5. This can be done by solving the polynomial Diophantine equation (2) with right hand-side

$$c(s) = (s+1)(s+2)(s+3)(s+4)(s+5).$$

We obtain the third-order controller

$$\frac{y_0(s)}{x_0(s)} = \frac{240 + 384s}{79 + 119s + 17s^2 + s^3}$$

corresponding to the y-minimal solution pair $x_0(s)$, $y_0(s)$.

The step-response of the corresponding closed-loop transfer function

$$\frac{b(s)y_0(s)}{c(s)} = \frac{384(s+0.5)(s+0.625)}{(s+1)(s+2)(s+3)(s+4)(s+5)} = \frac{120 + 432s + 384s^2}{120 + 274s + 225s^2 + 85s^3 + 15s^4 + s^5}$$

is represented in Figure 2. We can see that, although all the assigned closed-loop poles are negative real, the step reference input is tracked with an unacceptable overshoot of 141%. Inspection of the Bode magnitude plot reveals that the overshoot is due to the presence of two system zeros at -0.5 and -0.625, at the left of the first system pole -1.

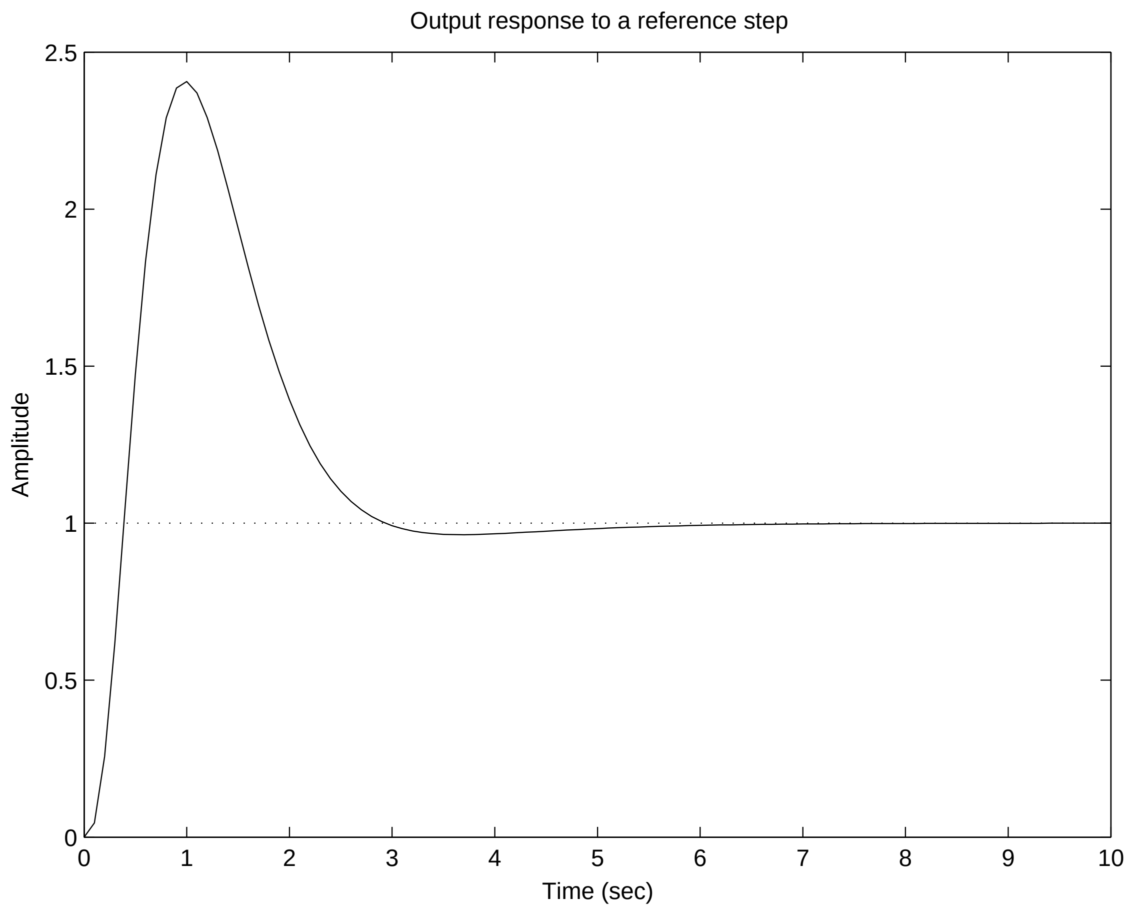

Figure 2: Closed-loop time response with unacceptable overshoot.

In virtue of the Youla-Kučera parameterization, the whole set of controllers assigning the same pole polynomial is parameterized as in (3) as

$$\frac{y(s)}{x(s)} = \frac{y_0(s) - a(s)q(s)}{x_0(s) + b(s)q(s)} = \frac{240 + 384s - s(s-2)q(s)}{79 + 119s + 17s^2 + s^3 + (0.5+s)q(s)}$$

where $q(s)$ is an arbitrary polynomial. So, at the price of increasing the order of the controller, there should be some freedom to reduce the overshoot.

The whole set of closed-loop transfer functions is parameterized affinely in $q(s)$ as

$$\frac{b(s)y(s)}{c(s)} = \frac{b(s)y_0(s) - a(s)b(s)q(s)}{c(s)} = \frac{120 + 432s + 384s^2 - (-s - 1.5s^2 + s^3)q(s)}{(s+1)(s+2)(s+3)(s+4)(s+5)}.$$

The Laplace transform of the closed-loop time response to a step input is given by

$$z(s) = \frac{1}{s}\frac{b(s)x(s)}{c(s)} = \frac{z_0}{s} + \frac{z_1}{s+1} + \frac{z_2}{s+2} + \frac{z_3}{s+3} + \frac{z_4}{s+4} + \frac{z_5}{s+5}$$

where the z_i are constant real residues of the partial fraction decomposition.

In order to ensure properness of the controller, in virtue of Lemma 1 we must have $\deg q(s) \leq \deg c(s) - 2\deg a(s) = 1$, so we restrict our search to Youla-Kučera parameters of first degree, i.e.

$$q(s) = q_0 + q_1 s.$$

Coefficients z_i are obtained by simple identification of like powers of s in the numerator of the partial fraction decomposition:

$$\begin{aligned} & z_0(s+1)(s+2)(s+3)(s+4)(s+5) + z_1 s(s+2)(s+3)(s+4)(s+5) + \\ & \quad z_2 s(s+1)(s+3)(s+4)(s+5) + z_3 s(s+1)(s+2)(s+4)(s+5) + \\ & \quad z_4 s(s+1)(s+2)(s+3)(s+5) + z_5 s(s+1)(s+2)(s+3)(s+4) = \\ & \quad = 120 + 432s + 384s^2 - (-s - 1.5s^2 + s^3)(q_0 + q_1 s) \end{aligned}$$

yielding the linear system of equations

$$\begin{bmatrix} 120 & 0 & 0 & 0 & 0 & 0 \\ 274 & 120 & 60 & 40 & 30 & 24 \\ 225 & 154 & 107 & 78 & 61 & 50 \\ 85 & 71 & 59 & 49 & 41 & 35 \\ 15 & 14 & 13 & 12 & 11 & 10 \\ 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \\ z_4 \\ z_5 \end{bmatrix} = \begin{bmatrix} 120 \\ 432 \\ 384 \\ 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1.5 & 1 \\ -1 & 1.5 \\ 0 & -1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} q_0 \\ q_1 \end{bmatrix}.$$

The closed-loop time response, obtained by inverse Laplace transform of $z(s)$, is a sum of decaying exponential modes

$$z(t) = z_0 + z_1 e^{-t} + z_2 e^{-2t} + z_3 e^{-3t} + z_4 e^{-4t} + z_5 e^{-5t}$$

or a polynomial in the indeterminate $\lambda = e^{-t}$, i.e.

$$z(\lambda) = z_0 + z_1\lambda + z_2\lambda^2 + z_3\lambda^3 + z_4\lambda^4 + z_5\lambda^5.$$

Enforcing a maximum overshoot of γ (in percent) on the closed-loop time response is equivalent to enforcing the time-domain constraint

$$z(t) \leq (1+\gamma)z_0$$

for all time $t \geq 0$, or equivalently to enforcing the polynomial positivity condition

$$p(\lambda) = \sum_{i=0}^{5} p_i \lambda^i = \gamma z_0 - z_1\lambda - z_2\lambda^2 - z_3\lambda^3 - z_4\lambda^4 - z_5\lambda^5 \geq 0$$

along the interval $\lambda \in [0,\ 1]$.

As shown in Lemma 2, enforcing positivity of polynomial along this interval can be formulated as the LMI feasibility problem (6):

$$\begin{array}{l} p_i = \mathrm{trace}\,[P_0 H_{i-1}] + \mathrm{trace}\,[P_1(H_i - H_{i-1})]\,, \quad i = 0, 1 \ldots, 6 \\ P_0 \succeq 0, \quad P_1 \succeq 0 \end{array}$$

where P_0 and P_1 are unknown symmetric matrices of dimension 4. Since coefficients p_i are linear in parameters z_i, which are in turn linear in design parameters q_i, enforcing the time-domain overshoot constraint is an LMI optimization problem in the Youla-Kučera parameter coefficients.

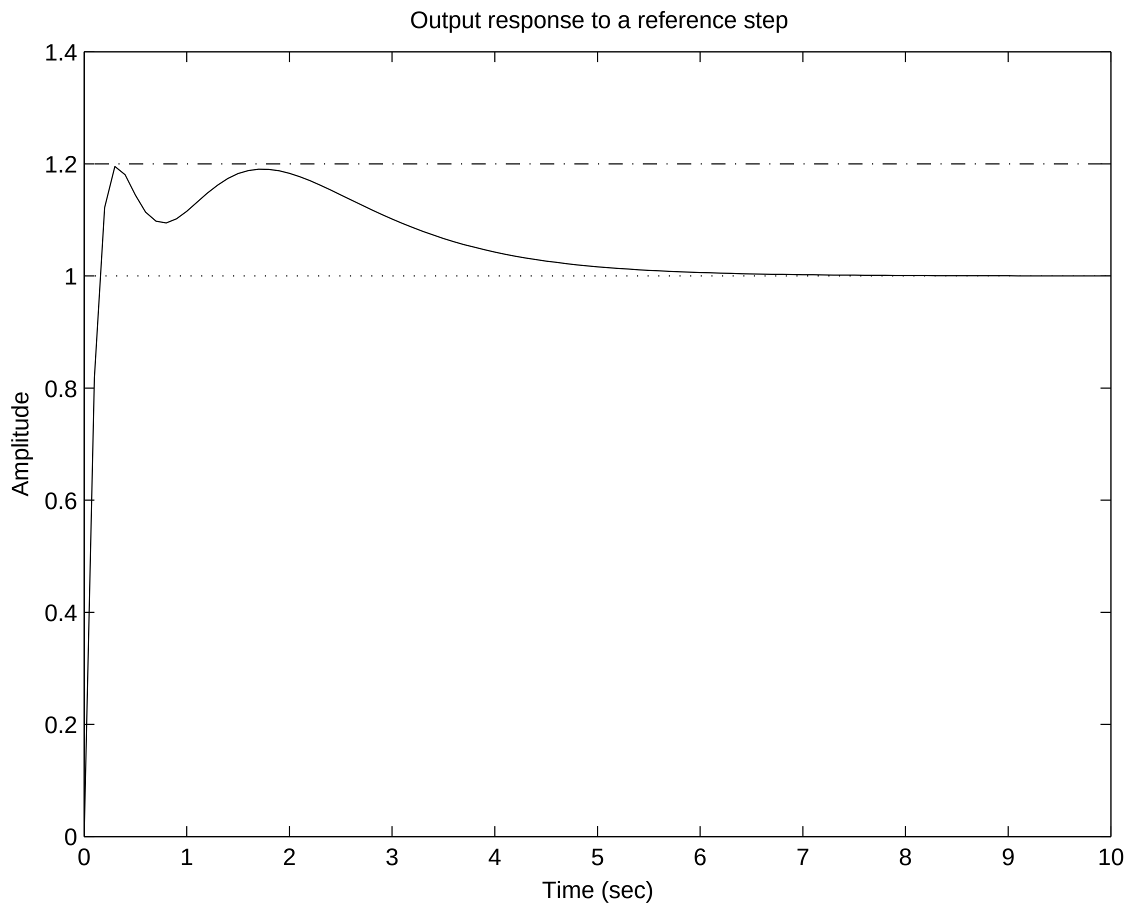

Figure 3: Closed-loop time response with reduced overshoot.

Now solving the LMI problem for a desired overshoot of $\gamma = 20\% = 0.2$, we obtain the Youla-Kučera parameter

$$q(s) = -100.3641 - 12.2700s$$

and the resulting controller

$$\frac{y(s)}{x(s)} = \frac{240.0000 + 183.2719s + 75.8240s^2 + 12.2700s^3}{28.8180 + 12.5009s + 4.7300s^2 + s^3}$$

producing the proper closed-loop transfer function

$$
\begin{aligned}
\frac{b(s)y(s)}{c(s)} &= \frac{120.0000 + 331.6359s + 221.1839s^2 + 81.9590s^3 + 12.2700s^4}{120 + 274s + 225s^2 + 85s^3 + 15s^4 + s^5} \\
&= \frac{12.2700(s + 0.5000)(s + 3.5126)(s^2 + 2.6670s + 5.5684)}{(s + 1)(s + 2)(s + 3)(s + 4)(s + 5)}
\end{aligned}
$$

and the closed-loop time response

$$
z(t) = 1 + 2.5059e^{-t} - 9.8240e^{-2t} + 8.6053e^{-3t} + 9.5081e^{-4t} - 11.7953e^{-5t}
$$

shown in Figure 3. One can check that the overshoot is indeed less than 20%.

6 Conclusion

Pole placement is a widely used design technique for linear systems control. The shape of the transient closed-loop time response however depends not only on the assigned poles, but also on the assigned zeros. For example, unwanted overshoot can occur even when all the assigned closed-loop poles are strictly negative real.

In this work, we investigated to what extend the shape of the time response can be altered by playing with the degrees of freedom of the Youla-Kučera parameterization. We focused on continuous-time systems, since time-domain constraints for discrete-time systems can be handled easily by placing all the poles at the origin, the so-called finite settling-time control [6]. We showed that, as soon as distinct negative real closed-loop poles are assigned, satisfying time-domain constraints on closed-loop system signals (such as input amplitude or rate limitations, output overshoot or undershoot) amounts to solving a convex LMI optimization problem when the degree of the polynomial Youla-Kučera parameter (hence the order of the controller) is fixed. The key point is in writing system signals as polynomials of the fastest decaying exponential term $\lambda = e^{-t/m}$, powers of λ corresponding then to the assigned closed-loop poles. In this framework, time-domain constraints can be written as positivity conditions for polynomials, which can be formulated as convex LMIs in the polynomial coefficients [12, 7].

It must be underlined that the time-domain constraints that can be handled by our approach are not restricted to upper or lower-bound limits as in (1). Actually, any constraint on a given time interval which can be formulated as a polynomial or rational function of the indeterminate $\lambda = e^{-t/m}$ can be incorporated into the framework.

Multivariable systems can be handled in a similar way, by replacing scalar polynomials by matrix polynomials, see [6].

Current limitations of the approach, and possible directions for further research, are as follows:

- Repeated poles are not allowed, otherwise system signals would feature terms $t^k e^{-pt}$ that cannot be formulated anymore as polynomials of the indeterminate $\lambda = e^{-t/m}$. We do not think however that it is a key limitation, since repeated poles with high multiplicity may correspond to a system canonical form with large Jordan blocks which are notoriously numerically sensitive;

- It is not clear how to handle assignment of complex conjugate poles, since terms $e^{-pt}\cos(\alpha+\beta t)$ would appear that are non-polynomial in $\lambda = e^{-t/m}$. This limitation is more severe and deserves further research, since it can be irrealistic to assign purely real poles in e.g. lightly damped systems such as flexible structures;

- Assigned poles should rather be integers, or ratios of small integers. If poles p_i are ratios of integers, and m denotes the least common multiple of the denominators, then system signals are polynomials of the indeterminate $\lambda = e^{-t/m}$. For example, if $p_1 = -1/2$, $p_2 = -2/3$ and $p_3 = -3/5$, then $m = 30$ and the 3 decaying exponential modes are respective powers λ^{15}, λ^{20} and λ^{18} of $\lambda = e^{-t/30}$. Since the number of decision variables in the design LMI is proportional to the square of the degree of λ in the polynomial expressions, integer m must be as small as possible. A poor choice of assigned poles would be for example $p_1 = -1$, $p_2 = -2.001$ for which a high-degree but sparse polynomial with powers λ^{1000} and λ^{2001} of the indeterminate $\lambda = e^{-t/1000}$ would be required;

- In relation with the latter point, the effect of numerical rounding errors on the poles, or more generally, uncertainty affecting the systems coefficients, must be studied in further detail. If the assigned poles are $p_1 = -1$ and $p_2 = -2$, system signals decompose as $z(t) = z_0 + z_1 e^{-p_1 t} + z_2 e^{-p_2 t}$. Suppose now that there is a small amount of uncertainty in p_2, which can vary between -1.999 and -2.001. In view of the above discussion, it is certainly preferable to model the uncertainty in p_2 as uncertainty in coefficient z_2 rather than in the integer power of the indeterminate $\lambda = e^{-t/1000}$. We believe that uncertainty of this kind can be handled via robust LMI optimization [1], i.e. solving time-constraint design LMIs for solutions that would be numerically insensitive to uncertainty and perturbations.

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