Reliable Algorithms for Polynomial Matrices

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December 1998

OUTLINE

1. Introduction

2. Polynomial matrices in control

3. Existing numerical methods

4. New numerical methods

5. Conclusion

1. INTRODUCTION

Main approaches to linear system control

- state-space methods, for which numerically reliable and efficient algorithms are available: ARE, LMIs

- polynomial methods, more in favor for educational purposes: polynomial matrix equations

Objective – Design of numerically **reliable** and **efficient** algorithms for polynomial matrices

Contribution – **Unified** approach based on Sylvester matrices and interpolation

2. POLYNOMIAL MATRICES IN CONTROL

Polynomial matrices are ubiquitous in control: dead-beat control, LQG regulation, $\mathcal{H}_\infty$ control, MIMO model matching, non-interacting control..

Operations on polynomial matrices

- **Transformations**

- **Linear equations**

- Quadratic equations

Most general formulation for transformations and linear equations: resolution of polynomial matrix equation

$$A(z)X(z) = B(z)$$

where A, B are given and X is to be found

Transformation of polynomial matrices

Preliminary step in control problems: reduction into canonical forms such as

- State-space realization to coprime MFD
- Non-coprime MFD to coprime MFD
- Reduction into row- or column-reduced form
- Extraction of GCD or LCM
- Triangular or Hermite form
- Diagonal or Smith form

Unilateral equations

Diophantine equation

$$\hat{A}(z)\hat{X}(z) + \hat{B}(z)\hat{Y}(z) = \hat{C}(z)$$

- State or output feedback dynamics assignment

- Dead-beat regulation with $\hat{C} = I$

- Factor extraction by polynomial division

Additional degree requirements on solution

Denoting

$$A = [\hat{A} \quad \hat{B}], \qquad X = \begin{bmatrix} \hat{X} \\ \hat{Y} \end{bmatrix}$$

and $B = \hat{C}$, Diophantine equation reads

$$A(z)X(z) = B(z)$$

Bilateral equations

Coupled equations

$$\begin{array}{rcl}\widehat{A}(z)\widehat{X}(z)+\widehat{Z}(z)\widehat{B}(z) & = & \widehat{C}(z) \\ \widehat{A}(z)\widehat{Y}(z)+\widehat{Z}(z)\widehat{D}(z) & = & \widehat{E}(z)\end{array}$$

- LQG regulation or prediction problems

- $\mathcal{H}_2$ control

Additional degree requirements on solution

Using Kronecker product, coupled equations can also be written as

$$A(z)X(z) = B(z)$$

Symmetric equations

Bilateral linear equations

$$\widehat{A}^\star(z)\widehat{X}(z) + \widehat{X}^\star(z)\widehat{A}(z) = \widehat{B}(z)$$

- Linearization scheme for solving spectral factorization

- ARMA covariance function

Using Kronecker product and projection operators, symmetric equation can be written as

$$A(z)X = B(z)$$

where X constant vector to be found

3. EXISTING NUMERICAL METHODS FOR POLYNOMIAL MATRICES

Different categories – pros and cons

- Elementary polynomial operations: Euclidean division for polynomials

  $\oplus$ accuracy, low memory requirements

  $\ominus$ potential instability

- State-space realizations

  $\oplus$ efficiency and reliability of state-space methods

  $\ominus$ numerical difficulties when passing from one representation to the other

- Matrix pencil algorithms: linear algebra of pencils of matrices

  $\oplus$ efficiency and reliability

  $\ominus$ not yet unified

- Symbolic methods: exact arithmetic

  $\oplus$ no roundoff errors

  $\ominus$ small size problems only

- Sylvester matrices and interpolation: linear system of equations

  $\oplus$ efficiency and reliability

  $\ominus$ memory requirements

Unsatisfactory status entails need for **unified**, **efficient** and **reliable** approach

4. NEW NUMERICAL METHODS FOR POLYNOMIAL MATRICES

Main contribution of this work is use of **Sylvester matrices** and **interpolation** techniques

- unified approach
- resolution of linear system of equations
- efficient and reliable algorithms (QR, SVD)

Developed, implemented and tested on three typical problems involving polynomial matrices

- Rank evaluation
- Triangularization
- Symmetric equation

RANK EVALUATION

Sylvester matrix approach

Basis K for polynomial null-space of A

$$A(z)K(z) = 0$$

Then

$$\text{rank}\, A = \text{col.dim.}\, A - \text{col.dim.}\, K$$

Upper bound on degree of K used to assess column dimension of K

Interpolation approach

$z_0, z_1, \ldots, z_d$ distinct complex numbers, where d number of zeros of A

$$\text{rank}\, A = \max_{i=0,1,\ldots,d} \text{rank}\, A(z_i)$$

TRIANGULARIZATION

Find unimodular U and triangular T such that

$$A(z)U(z) = T(z)$$

Sylvester matrix approach

- Upper bound d on degree of T
- QR decomposition of Sylvester matrix of A of order d

Interpolation approach

- Upper bound d on degree of T
- QR decomposition of interpolated matrix built from values $A(z_0)$, $A(z_1) \ldots$, $A(z_d)$

TRIANGULARIZATION NUMERICAL EXPERIMENTS

Five algorithms were tested

A – Sylvester matrix

B – Interpolation

C – Gaussian elimination

D – Elementary polynomial operations

E – Symbolic computation

Algorithms A,B,C,D implemented in MATLAB

Algorithm E implemented in MAPLE

Comparative execution times in seconds

d	n	**A**	**B**	C	D	E
		Algorithm				
3	3	**0.21**	**1.3**	0.62	16	0.65
5	3	**0.51**	**2.4**	1.1	26	2.7
7	3	**0.87**	**4.0**	1.6	39	12
9	3	**1.5**	**6.1**	2.3	54	30
3	4	**0.46**	**2.4**	1.1	29	8.8
5	4	**2.0**	**5.6**	2.5	59	111
7	4	**4.1**	**10**	3.7	$\star$	603
9	4	**7.8**	$\star$	5.5	$\star$	$\star\star$
3	5	**2.1**	**4.6**	2.3	$\star$	109
5	5	**7.3**	**12**	4.6	$\star$	$\star\star$
7	5	**14.4**	$\star$	8.0	$\star$	$\star\star$
9	5	**29**	$\star$	12	$\star$	$\star\star$
3	9	**48**	**59**	17	$\star$	$\star\star$
5	9	**220**	$\star$	$\star$	$\star$	$\star\star$
7	9	**580**	$\star$	$\star$	$\star$	$\star\star$
9	9	**1200**	$\star$	$\star$	$\star$	$\star\star$

$\star$ triangularization failed

$\star\star$ execution time $>$ 1500 seconds

SYMMETRIC EQUATION

Using Kronecker product, equation

$$\widehat{A}^{\star}(z)\widehat{X}(z) + \widehat{X}^{\star}(z)\widehat{A}(z) = \widehat{B}(z)$$

becomes

$$A(z)X = B(z)$$

Sylvester matrix approach

Linear system of equations with Sylvester matrix of A of order $d = \deg A = \deg X$

Interpolation approach

Linear system of equations built from values $A(z_0)$, $A(z_1), \ldots,$ $A(z_d)$

Links with stability theory, Hurwitz matrices

5. CONCLUSION

Unsatisfactory status

- Polynomial matrix equations frequently occur in linear system control, yet

- Numerical methods to date feature bad numerical properties

Contribution

- Unified approach to deal with polynomial matrix equations

- Efficient and reliable resolution of linear systems of equations

Sylvester matrix and interpolation methods also apply to several other problems

- Polynomial matrix division
- GCD or LCM extraction
- Polynomial matrix inverse computation
- Determinant computation (through FFT)

Implemented in the Polynomial Toolbox Version 2.0 for MATLAB

Current research directions

- Tighter bounds on polynomial null-space degree

- Efficient resolution of Toeplitz-structured linear system of equations

- Numerical comparison of state-space and polynomial methods

- Numerical instability of elementary polynomial operations

- Reduction into Smith or diagonal form with Sylvester matrices or interpolation