Semidefinite representation of convex hulls of rational varieties*

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Abstract

Using elementary duality properties of positive semidefinite moment matrices and polynomial sum-of-squares decompositions, we prove that the convex hull of rationally parameterized algebraic varieties is semidefinite representable (that is, it can be represented as a projection of an affine section of the cone of positive semidefinite matrices) in the case of (a) curves; (b) hypersurfaces parameterized by quadratics; and (c) hypersurfaces parameterized by bivariate quartics; all in an ambient space of arbitrary dimension.

Keywords: semidefinite programming, LMI, convex geometry, real algebraic geometry

Introduction

Semidefinite programming, a versatile extension of linear programming to the convex cone of positive semidefinite matrices (semidefinite cone for short), has found many applications in various areas of applied mathematics and engineering, especially in combinatorial optimization, structural mechanics and systems control. For example, semidefinite programming was used in [3] to derive a hierarchy of embedded convex linear matrix inequality (LMI) outer approximations of non-convex semi-algebraic sets arising in control problems.

It is easy to prove that affine sections and projections of the semidefinite cone are convex semi-algebraic sets, but it is still unknown whether all convex semi-algebraic sets can be modeled like this, or in other words, whether all convex semi-algebraic sets are semidefinite representable. Following the development of polynomial-time interior-point algorithms to solve semidefinite programs, a long list of semidefinite representable semi-algebraic sets and convex hulls is provided in [1]. Latest achievements in the field are reported in [2] and [4].

We aim at enlarging the class of semi-algebraic sets whose convex hulls are explicitly semidefinite representable. Using elementary duality properties of positive semidefinite moment matrices and polynomial sum-of-squares decompositions – nicely recently surveyed in [5] – we prove that the convex hull of rationally parameterized algebraic varieties is explicitly semidefinite representable in the case of (a) curves; (b) hypersurfaces parameterized by quadratics; and (c) hypersurfaces parameterized by bivariate quartics; all in an ambient space of arbitrary dimension. These convex hulls can be expressed as projections of affine sections of the semidefinite cone, by introducing a sufficient number of lifting variables.

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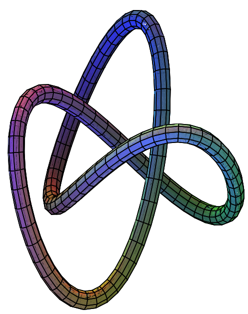
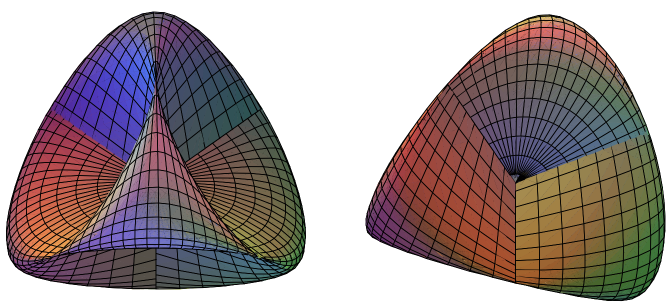
Rationally parameterized surfaces arise often in engineering, and especially in computer-aided design (CAD). For example, the CATIA (Computer Aided Three-dimensional Interactive Application) software, developed since 1981 by the French company Dassault Systèmes, uses rationally parameterized surfaces as its core 3D surface representation. CATIA was originally used to develop Dassault's Mirage fighter jet for the French airforce, and then it was adopted in aerospace, automotive, shipbuilding, and other industries. Airbus aircrafts are designed in Toulouse with the help of CATIA, and architect Frank Gehry has used the software to design his curvilinear buildings, like the Guggenheim Museum in Bilbao or the Dancing House in Prague, near the Charles Square buildings of the Czech Technical University.

Examples



Tube plot of the trefoil knot curve, whose convex hull is exactly semidefinite representable as a 4-by-4 LMI with 3 liftings.



Two views of Steiner's Roman surface, whose convex hull is semidefinite representable as a 3-by-3 LMI with 2 liftings.

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