Semidefinite programming for optimizing convex bodies under width constraints

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Abstract

We consider the problem of minimizing a functional (like the area, perimeter, surface) within the class of convex bodies whose support functions are trigonometric polynomials. The convexity constraint is transformed via the Fejér-Riesz theorem on positive trigonometric polynomials into a semidefinite programming problem. Several problems such as the minimization of the area in the class of constant width planar bodies, rotors and space bodies of revolution are revisited. The approach seems promising to investigate more difficult optimization problems in the class of three-dimensional convex bodies.

Keywords: convexity, optimization, semidefinite programming, support function.

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1 Introduction

Many optimization problems involve convexity constraints. Such constraints naturally appear in the framework of geometric optimization problems, where the underlying admissible class is the set of convex bodies in Euclidean space. This paper aims to tackle the issue of a certain class of planar geometric optimization problems via semidefinite programming.

A convex body can be parametrized by its support function defined by homogeneity on the unit hypersphere. The convexity constraint amounts to the non-negativity of the principal radii of curvature of the boundary of the body (if they are defined). For the planar case, the convexity constraint is equivalent to the non-negativity of the radius of curvature of the boundary of the body, easily expressed via the support function. A reformulation of the convexity constraint allows the use of the calculus of variation and optimal control theory to treat

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several optimization problems for two- and three-dimensional bodies, such as the minimization of the area within the class of constant width bodies and rotors, see [3, 4, 5, 6, 10].
The approach which is chosen in this paper significantly differs from the previous ones. We consider planar convex bodies which are parametrized by their support function modeled by a truncated Fourier series. The convexity constraint becomes a trigonometric positivity constraint, and it turns out that it can be formulated as a linear decision problem in the convex cone of positive semidefinite matrices, with decision variables the Fourier coefficients of the support function.
Several optimization problems involving convex bodies can be formulated via this method. For example, the problem of minimizing the area within the class of constant width bodies and rotors [4] can be gathered into this form. The area is a concave quadratic functional with respect to Fourier coefficients, and the problem of minimizing the area under additional width constraints amounts to solving a nonconvex semidefinite programming problem (concave quadratic objective function, convex semidefinite constraints), for which we can use a public-domain Matlab implementation of a penalty augmented Lagrangian method. In this paper, we mainly deal with convex bodies satisfying width constraints, and the functional to be minimized is quadratic concave (area) or linear (perimeter) with respect to the decision variables.

The paper is organized as follows. In section 2, well-known results on the parametrization of a planar convex body by its support function are recalled. Then, the reformulation of the convexity constraint into a semidefinite programming (SDP) problem is presented. In the third section, we solve numerically (using YALMIP and PENBMI, see [16]) several optimization problems:

- minimization of the area within the planar constant width bodies (section 3.1),
- minimization of the area within the class of rotors (section 3.2),
- minimization of the area within three dimensional constant width bodies of revolution (section 3.3),
- minimization of the area and the perimeter within the class of convex bodies with given minimal and maximal width (section 3.4).

The numerical experiments allow to retrieve known results with closed-form solutions and to validate the method (problems of sections 3.1, 3.2 and 3.3). In section 3.4, the method is used to derive new results and to find numerical candidates for optimality (extrema of the perimeter or area) in the class of planar convex bodies for which the minimal width is 1 and the maximal width is 2. The last section 4 is devoted to the proof of the convergence of the method (for problem of sections 3.1, 3.2) when the number of Fourier coefficients of the truncated Fourier series goes to infinity. More precisely, we prove that the sequence of minimizers of the truncated problem converges in H^1 to the minimizer of the global optimization problem (Theorem 4.1). In particular, the value of the functional evaluated for a minimizer of the finite-dimensional problem converges to the value of the functional for the global optimization problem.

2 General results

2.1 Parametrization of a convex body by its support function

In this section we recall classical results on the parametrization of a convex set by its support function. Let K be a body, that is a non-empty compact set of $\mathbb{R}^n$, and let K be convex. The support function σ of K is the map defined by

$$u \in \mathbb{R}^n \backslash \{0\} \mapsto \sigma(u) = \max_{x \in K} x \cdot u \tag{1}$$

where the dot denotes the scalar product. The support function can be equivalently defined on the unit hypersphere $\mathbb{S}^{n-1}$ by homogeneity. It is standard that σ is of class C^1 if and only if K is strictly convex, see [22]. In this paper we consider convex bodies of $\mathbb{R}^2$, that is $n = 2$. The parametrized support function p of K is then defined by

$$\theta \in \mathbb{R} \mapsto p(\theta) = \sigma(\cos\theta, \sin\theta).$$

By extension, function p will be called support function of K. If K is strictly convex, its boundary ∂K can be described by

$$\begin{cases} x(\theta) = p(\theta)\cos\theta - p'(\theta)\sin\theta, \\ y(\theta) = p(\theta)\sin\theta + p'(\theta)\cos\theta \end{cases} \tag{2}$$

where the prime denotes differentiation.

Given a convex body K with support function p, there exists a non-negative Radon measure $\tilde{\rho}$ which is called *radial distribution of* K (see *e.g.* [13]) and such that:

$$p(\theta) = \int_0^\theta \sin(\theta - t)\tilde{\rho}(dt) + p(0)\cos\theta + p'(0)\sin\theta.$$

If the radial distribution $\tilde{\rho}(dt)$ is absolutely continuous with respect to the Lebesgue measure, we can write $\tilde{\rho}(dt) = \rho(t)dt$ where ρ is the *radius of curvature* of the boundary of K. Let $C^{1,1}$ denotes the space of continuously differentiable functions p such that p' is Lipschitz. If the support function p of a convex body K is of class $C^{1,1}$, p'' exists a.e. (Rademacher's Theorem) and coincides with p'' computed in the distribution sense. Thus p'' can be well represented in L^∞, and a computation shows that $\rho = p + p''$ almost everywhere (a.e.). By convexity of K, we must have a.e.:

$$\rho = p + p'' \geq 0. \tag{3}$$

By differentiating, one has a.e.:

$$\begin{cases} x'(\theta) = -\rho(\theta)\sin\theta, \\ y'(\theta) = \rho(\theta)\cos\theta. \end{cases}$$

If p is of class C^1, inequality (3) must be understood in the distribution sense: $\int_0^{2\pi}(p+p'')\phi \geq 0$ for all functions $\phi \geq 0$ in $C^\infty_{2\pi}(\mathbb{R})$, the space of regular 2π-periodic functions. Conversely, given a function p of class C^1, 2π-periodic and satisfying $p + p'' \geq 0$ in the distribution sense, there exists a unique strictly convex body K whose support function is p. In the present work, we mainly deal with strictly convex bodies whose support functions p are at least of class $C^{1,1}$

(except for problems of section 3.4). For $C^{1,1}$ functions p, the radius of curvature $\rho = p + p''$ is well represented in L^∞ and inequality (3) is satisfied a.e. This is obviously the case if p is a trigonometric polynomial, and also for constant width bodies and rotors whose support functions are of class $C^{1,1}$ (see [4, 5]). Therefore, to ensure convexity of a body K represented by a support function p of class $C^{1,1}$, we assume that (3) holds in the classical sense.

Remark 2.1. *The convexity constraint of K can be equivalently expressed in polar coordinates, see [19]. Consider a body K of $\mathbb{R}^2$ whose boundary is described by*

$$\begin{cases} x(\theta) = f(\theta)\cos\theta, \\ y(\theta) = f(\theta)\sin\theta, \end{cases} \tag{4}$$

where f is a periodic continuous function. By standard computations, convexity of K amounts to saying that $u + u'' \geq 0$, where $u = 1/f$. Notice that the convexity constraint is similar to (3). The parametrization by support function takes easily into account width constraints (see problems of section 3) whereas the polar parametrization handles other constraints.

By standard computations, the area of a strictly convex body K can be expressed by:

$$A(p) = \frac{1}{2}\int_0^{2\pi} (p^2(\theta) - p'^2(\theta))d\theta = \frac{1}{2}\int_0^{2\pi} p(\theta)(p(\theta) + p''(\theta))d\theta. \tag{5}$$

The previous expression is well-defined for a strictly convex body, since then p is C^1. For a general convex body, the previous expression must be understood as the product of a continuous function and a Radon measure (for example if K is a polygon). The perimeter of K is linear with respect to p:

$$P(p) = \int_0^{2\pi} p(\theta)d\theta. \tag{6}$$

The Fourier series of p can be written

$$p(\theta) = \sum_{k\in\mathbb{Z}} p_k e^{ik\theta}, \tag{7}$$

with complex Fourier coefficients

$$p_k = \frac{1}{2\pi}\int_0^{2\pi} p(\theta)e^{-ik\theta}d\theta = a_k - ib_k,\ a_k = \operatorname{Re} p_k,\ b_k = \operatorname{Im} p_k,$$

satisfying $\overline{p}_k = p_{-k}$, so that we can alternatively write

$$p(\theta) = p_0 + \sum_{k\geq 1} 2\operatorname{Re}(p_k e^{ik\theta}) = a_0 + 2\sum_{k\geq 1}(a_k\cos k\theta + b_k \sin k\theta).$$

Notice that $p_0 = a_0$. By Parseval's Theorem, the area of K can be expressed as follows:

$$A(p) = \pi\sum_{k\in\mathbb{Z}}(1-k^2)|p_k|^2 = \pi a_0^2 + 2\pi\sum_{k\geq 2}(1-k^2)(a_k^2 + b_k^2) \tag{8}$$

and the perimeter of K is obviously $P(p) = 2\pi a_0$. Naturally, $A(p)$ and $P(p)$ do not depend on p_1 as this coefficient can be chosen zero by translating K along vector $(-2a_1, 2b_1)$.

2.2 Convexity constraint as a semidefinite programming problem

This section is devoted to the reformulation of the convexity constraint into a semidefinite programming (SDP) problem. We first set some notations:

- Let $z(\theta) = (1, e^{i\theta}, ..., e^{im\theta}) \in \mathbb{C}^{m+1}$, $\theta \in \mathbb{R}$.
- Let $0 \leq k \leq m$ and $T_k \in \mathbb{R}^{(m+1)\times(m+1)}$ denote the Toeplitz matrix with ones on the k-th subdiagonal and zeros elsewhere (lower shift matrix). Similarly define T_{-k} as the transpose of T_k. Note that T_0 is the identity matrix.
- A^* denotes the transpose conjugate of a matrix A.
- The notation $A \succeq 0$ means that matrix A is hermitian positive semidefinite, i.e. with nonnegative real eigenvalues.
- We denote by $A \cdot B$ the inner product of two matrices A and B of the same size, that is the trace of the product of A^* by B.

Given a hermitian matrix $Q \in \mathbb{C}^{(m+1)\times(m+1)}$, let

$$q(\theta) = z^*(\theta)Qz(\theta) = \sum_{k=-m}^{m} q_k e^{ik\theta} \tag{9}$$

be the value of the quadratic form defined by Q and evaluated at $z(\theta)$, where the star denotes transpose conjugate. Function $q(\theta)$ is a trigonometric polynomial of degree m. We next recall a classical lemma showing the linear dependence between entries of matrix Q and Fourier coefficients q_k (see [7]).

Lemma 2.1. *Hermitian matrix Q and trigonometric polynomial $q(\theta)$ sastify relation (9) if and only if*

$$q_k = T_k \cdot Q, \quad k = 0, 1, \dots, m. \tag{10}$$

Proof. Denoting by $q_{j,k}$ the (j,k)-th entry of matrix Q, we write:

$$q(\theta) = \sum_{j=1}^{m+1} q_{j,j} + \sum_{k=1}^{m} \sum_{j=1}^{m+1-k} q_{j,j+k} e^{ik\theta} + \sum_{k=1}^{m} \sum_{j=k+1}^{m+1} q_{j,j-k} e^{-ik\theta},$$

and as Q is hermitian, we have:

$$q(\theta) = \sum_{j=1}^{m+1} q_{j,j} + \sum_{k=1}^{m} \left(e^{ik\theta} \sum_{j=1}^{m+1-k} q_{j,j+k} + e^{-ik\theta} \sum_{j=1}^{m+1-k} \overline{q}_{j,j+k} \right).$$

We obtain relation (10) by using $\sum_{j=1}^{m+1-k} q_{j,j+k} = \sum_{j=1}^{m+1-k} \overline{q}_{j,j+k} = T_k \cdot Q$. □

Notice that if Q is real, then $q(\theta) = T_0 \cdot Q + \sum_{k=1}^{m} 2\cos(k\theta) T_k \cdot Q$ becomes a real trigonometric polynomial.

Let us now consider the set of convex bodies whose support functions are trigonometric polynomials of degree m:

$$p(\theta) = \sum_{k=-m}^{m} p_k e^{ik\theta}. \tag{11}$$

Theorem 2.1. *Let K be a convex body with polynomial support function (11). Then there exists a matrix $Q \in \mathbb{C}^{(m+1)\times(m+1)}$ such that*

$$Q \succeq 0, \quad (1-k^2)p_k = T_k \cdot Q, \quad k = 0, 1, \ldots, m. \tag{12}$$

Conversely, any polynomial (11) satisfying relations (12) for some matrix Q is the support function of a convex body K.

Proof. Differentiate twice the support function of K to obtain $p(\theta) + p''(\theta) = \sum_{k=-m}^{m}(1 - k^2)p_k e^{ik\theta}$. Recalling relation (3), function $p + p''$ is a non-negative trigonometric polynomial, and the Fejér-Riesz theorem ensures that there exists a trigonometric polynomial $r(\theta)$ such that $p(\theta)+p''(\theta) = |r(\theta)|^2$ with $r(\theta) = \sum_{k=0}^{m} r_k e^{ik\theta}$. It follows that $|r(\theta)|^2 = \sum_{0\leq j,k\leq m} r_j \overline{r}_k e^{i(j-k)\theta}$. Denote by Q the matrix with entries $q_{j,k} = r_j \overline{r}_k$, $0 \leq j, k \leq m$. By construction, Q is positive semidefinite hermitian of size $m + 1$. Indeed, if $q = (r_j)_{0\leq j\leq m} \in \mathbb{C}^{m+1}$, then one has $Q = q^* q$ which shows that Q is positive semidefinite. We can then apply Lemma 2.1 on $q(\theta) = |r(\theta)|^2$. □

Given p in (11), finding a matrix Q satisfying (12) amounts geometrically to finding a point in the intersection of an affine subspace and the convex cone of positive semidefinite matrices. Finding jointly Q and p (presumably satisfying further linear constraints) is equivalent to solving a semidefinite programming (SDP) problem. For more information on the SDP representation of the convex cone of nonnegative trigonometric polynomials, see [15] or [7].

Remark 2.2. *Geometrical constraints expressed in terms of linear equations with respect to support function (width, perimeter, Steiner point..) can be transformed via the correspondence given by Theorem 2.1. Any optimization problem involving such constraints and formulated in the class of support functions which are trigonometric polynomials can be then formulated into an SDP problem under additional linear constraints.*

3 Constant width planar bodies

In this section we apply the procedure described in the previous section to the minimization of the area for the four problems mentioned in the introduction.

3.1 Concave SDP problem

We start with convex planar bodies of constant width and the Blaschke-Lebesgue Theorem [10] to test the efficiency of the method on an optimization problem for which we know the solution.

If $K \subset \mathbb{R}^2$ is a strictly convex body of support function p, then the width function is

$$w(\theta) = p(\theta) + p(\theta + \pi),$$

and it represents the distance between two different parallel support lines to K. As recalled in [4], any planar constant width body K can be represented by a 2π-periodic function $p \in C^{1,1}$ which satisfies

$$\begin{cases} p(\theta) + p(\theta + \pi) = 1, \ \forall \, \theta \in \mathbb{R}, \\ p + p'' \geq 0 \text{ a.e.} \end{cases} \tag{13}$$

By taking into account this width constraint, the Fourier series of p becomes

$$p(\theta) = \frac{1}{2} + 2\text{Re}(p_1 e^{i\theta}) + 2 \sum_{k \geq 1} \left(a_{2k+1} \cos(2k+1)\theta + b_{2k+1} \sin(2k+1)\theta \right),$$

and the area becomes:

$$A(p) = \frac{\pi}{4} + 2\pi \sum_{k \geq 1} (1 - (2k+1)^2)(a_{2k+1}^2 + b_{2k+1}^2).$$

A standard optimization problem consists in determining the constant width convex body of maximal and minimal area. Since any constant width body has the same perimeter as the ball of same diameter (Barbier's theorem), it follows by the isoperimetric inequality that the maximizer is the ball, with area $\frac{\pi}{4} \simeq 0.78540$. It has been proved by Blaschke and Lebesgue in 1920 that the minimizer is the Reuleaux triangle which consists of the intersection of three discs of unit radius centered at the three vertices of an equilateral triangle of unit side, see figure 1. The area of the Reuleaux triangle is equal to $\frac{\pi}{2} - \frac{\sqrt{3}}{2} \simeq 0.70477$. This result has been revisited recently by optimal control theory and calculus of variation [4, 10].

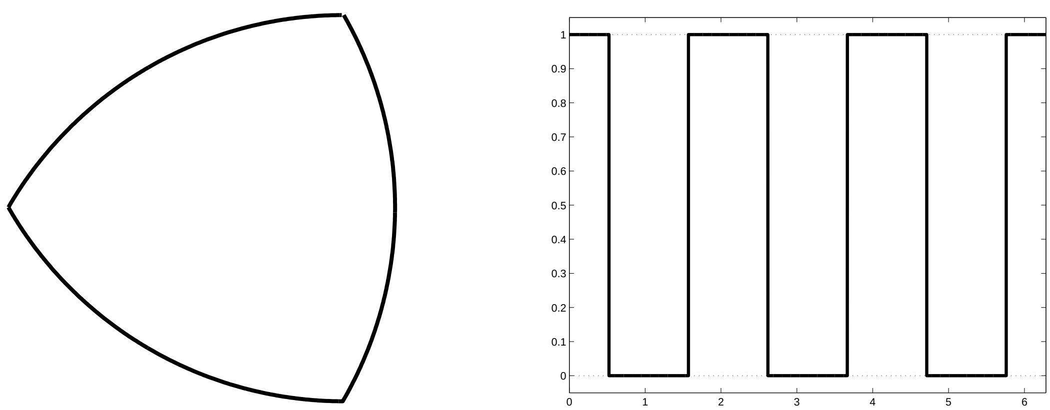

Figure 1: Reuleaux triangle (left) and its radius of curvature (right).

If p_R is the Fourier series of the support function of the Reuleaux triangle, the radius of curvature ρ_R satisfies for a.e. $\theta \in \mathbb{R}$:

$$\rho_R(\theta) = p_R(\theta) + p_R''(\theta) = \mathbb{I}_{[-\frac{\pi}{6}, \frac{\pi}{6}[} + \mathbb{I}_{[\frac{\pi}{2}, \frac{5\pi}{6}[} + \mathbb{I}_{[\frac{7\pi}{6}, \frac{3\pi}{2}[} = \frac{1}{2} + \frac{2}{\pi} \sum_{k \geq 0} \frac{(-1)^k}{2k+1} \cos(6k+3)\theta, \tag{14}$$

and it is therefore lacunary. In the above expression, $\mathbb{I}_J$ denotes the indicator function of a set J.

Let us now consider the set of trigonometric polynomials p of degree $2N+1$ of the form:

$$p(\theta) = \frac{1}{2} + 2\sum_{k=0}^{N}(a_{2k+1}\cos(2k+1)\theta + b_{2k+1}\sin(2k+1)\theta) \tag{15}$$

and satisfying (13). The area of the body with support function p is

$$A(p) = \frac{\pi}{4} + 2\pi\sum_{k=1}^{N}(1-(2k+1)^2)(a_{2k+1}^2 + b_{2k+1}^2)$$

and hence the minimization problem we consider is

$$\begin{array}{ll} \min_p & A(p) \\ \text{s.t.} & (13) \text{ and } (15). \end{array} \tag{16}$$

By using the reformulation of the convexity constraint into an SDP problem, this optimization problem can be transformed as follows. Consider the set of hermitian matrices Q of size $m = 2N+2$ satisfying the affine equations:

$$T_0 \cdot Q = \frac{1}{2},\ T_1 \cdot Q = 0,\ T_{2k} \cdot Q = 0,\ k = 1, 2, \ldots, N \tag{17}$$

and let

$$\mathcal{A}(Q) = \frac{\pi}{4} + 2\pi\sum_{k=1}^{N}(1-(2k+1)^2)(((T_{2k+1} + T_{2k+1}^*) \cdot Q)^2 + ((T_{2k+1} - T_{2k+1}^*) \cdot Q)^2). \tag{18}$$

Proposition 3.1. *Problem (16) is equivalent to:*

$$\begin{array}{ll} \min_Q & \mathcal{A}(Q) \\ \text{s.t.} & Q \succeq 0 \text{ and } (17) \end{array} \tag{19}$$

which is a semidefinite programming problem with a concave quadratic objective function and a convex feasible set.

Proof. The affine constraint (17) takes into account the expression of p given by (15). The objective function (18) is a reformulation of the expression (8) of the area with Fourier coefficients $p_k = a_k - ib_k$ such that $a_k = (T_{2k+1} + T_{2k+1}^*) \cdot Q$ and $b_k = (T_{2k+1} - T_{2k+1}^*) \cdot Q$. The convexity constraint is then equivalent to the semidefinite constraint $Q \succeq 0$. Finally, concavity of the objective function follows from the negative signs of the quadratic terms in (8). $\square$

3.1.1 Concave semidefinite programming

To solve problem (19) numerically, we use PENBMI, an implementation of a penalty-augmented Lagrangian method handling nonconvex quadratic semidefinite programming problems [16]. As far as we know, this is the only publicly available solver that can deal with such optimization problems. We use YALMIP [20] as a Matlab interface to PENBMI.

PENBMI returns local optima (satisfying first-order optimality conditions) with no guarantee of global optimality, unless the optimization problem is convex. Problem (19) is not convex, and since the objective function is concave we may suspect that there are several local (and maybe global) optima along the boundary of the convex feasible set. To avoid getting trapped into a critical point which is not a global optimum, we run PENBMI with several (typically 10) randomly generated initial points.

PENBMI does not handle complex-valued data, so we rewrite the complex constraint $Q = Q_R + iQ_I \succeq 0$ as the real constraint

$$\begin{bmatrix} Q_R & Q_I \\ -Q_I & Q_R \end{bmatrix} \succeq 0$$

where $Q_R = Q_R^*$ is the symmetric real part of Q and $Q_I = -Q_I^*$ is the skew-symmetric imaginary part of Q. Then it follows that $a_k = (T_{2k+1} + T_{2k+1}^*) \cdot Q = (T_{2k+1} + T_{2k+1}^*) \cdot Q_R$ and $b_k = (T_{2k+1} - T_{2k+1}^*) \cdot Q = (T_{2k+1} - T_{2k+1}^*) \cdot Q_I$.

3.1.2 Real Fourier coefficients

In this section, we make some comments on the problem to improve the efficiency of the method.

We first use PENBMI to solve problem (19) with complex Fourier coefficients (both a_k and b_k are free, i.e. Q is a complex hermitian matrix) and real Fourier coefficients ($b_k = 0$, i.e. Q is a real symmetric matrix) for increasing values of the Fourier series truncation degree m. Our results are reported in table 1, to 5 significant digits.

degree	3	5	7	9	11	13
complex	0.73631	0.73631	0.73631	0.71976	0.71976	0.71976
real	0.73631	0.73631	0.73631	0.71963	0.71963	0.71963
degree	15	17	19	21	23	25
complex	0.71347	0.71347	0.71347	0.71045	0.71045	0.71045
real	0.71319	0.71319	0.71319	0.71003	0.71003	0.71003

Table 1: Minimum area vs. degree, for complex and real Fourier coefficients.

First, we see from table 1 that the area does not decrease strictly monotonically: only the harmonics of orders $6k + 3$, with k integer, contribute, which is consistent with the lacunary Fourier series (14).

Second, repeated experiments reveal that the behavior of PENBMI is much more consistent on different runs with random initial conditions when restricting the search to real Fourier coefficients. The algorithm converges almost everytime to the same local minimum, which leads us to conclude that it is actually a global minimum (even though we cannot prove this rigorously). In the case of complex Fourier coefficients, the objective function is invariant upon multiplication of $p_k = a_k - ib_k$ by any complex number of unit magnitude, and hence there are more degrees of freedom. As a result, the algorithm gets more often trapped around suboptimal critical points.

Third, and most importantly, there is no significant contribution provided by the complex parts b_k when the degree increases. Notice that in view of (14), it is natural to force this constraint: it is always possible to assume that the radius of curvature ρ_R of the Reuleaux triangle is even (by doing a translation). By integration, its support function p_R is also even and we can restrict the optimization problem to coefficients a_k.

For these reasons, we decide to enforce $b_k = 0$ and to search for a real symmetric matrix Q for orders of the form $6k + 3$, in the experiments below.

3.1.3 Approximations

On figure 2 we represent the minimizers and corresponding radii of curvature obtained for various values of the truncation degree m, and on table 2 we report the corresponding Fourier coefficients, to 3 significant digits.

For $N = 33$, a typical computational time for one run of PENBMI is less than 5 seconds on a standard PC.

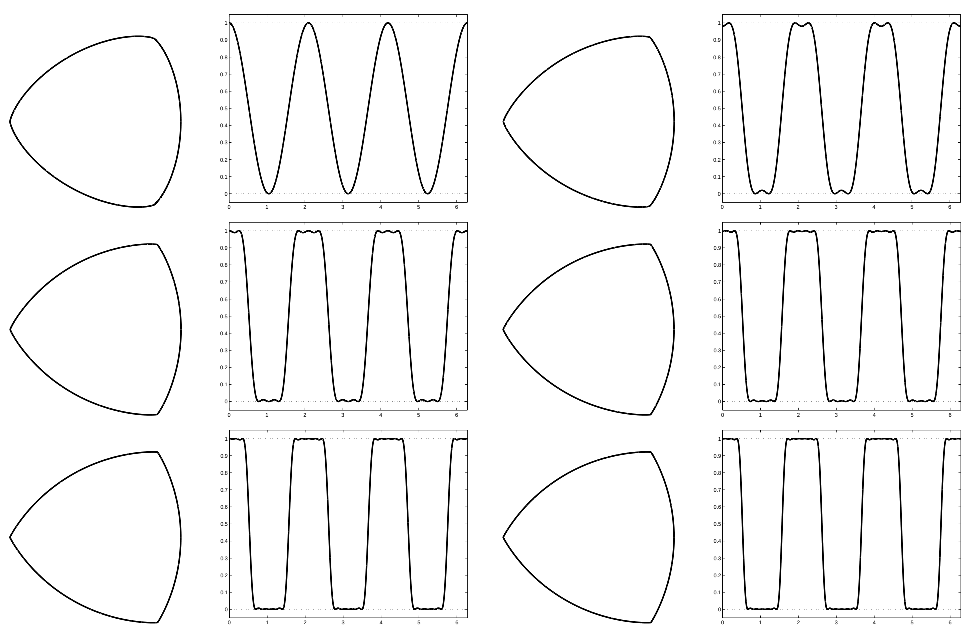

Figure 2: Minimizers and radii of curvature obtained for degrees $3, 9, 15, 21, 27, 33$.

degree	Fourier coefficients
3	$a_0 = 0.500$, $a_3 = -0.0313$
9	$a_0 = 0.500$, $a_3 = -0.0361$, $a_5 = 5.52\ 10^{-14}$, $a_7 = -1.23\ 10^{-14}$, $a_9 = 6.06\ 10^{-4}$
15	$a_0 = 0.500$, $a_3 = -0.0377$, $a_5 = 9.15\ 10^{-12}$, $a_7 = 2.86\ 10^{-12}$, $a_9 = 8.81\ 10^{-4}$, $a_{11} = -2.33\ 10^{-12}$, $a_{13} = -1.13\ 10^{-12}$, $a_{15} = -8.34\ 10^{-5}$
21	$a_0 = 0.500$, $a_3 = -0.0385$, $a_5 = 1.02\ 10^{-13}$, $a_7 = 1.60\ 10^{-14}$, $a_9 = 1.02\ 10^{-3}$, $a_{11} = -2.70\ 10^{-14}$, $a_{13} = -9.51\ 10^{-15}$, $a_{15} = -1.39\ 10^{-4}$, $a_{17} = 7.08\ 10^{-15}$, $a_{19} = 3.70\ 10^{-15}$, $a_{21} = 2.11\ 10^{-5}$

Table 2: Fourier coefficients for degrees $3, 9, 15, 21$.

3.1.4 Low degree bodies

In the case $m = 3$, problem (16) restricted to real coefficients reads

$$\begin{array}{ll} \min_{a_3} & \frac{\pi}{4} - 16\pi a_3^2 \\ \text{s.t.} & \frac{1}{2} - 16 a_3 \cos 3\theta \geq 0, \ \forall \theta \in \mathbb{R} \end{array}$$

and from inspection we obtain optimal values $a_3 = \pm\frac{1}{32}$, support functions $p(\theta) = \frac{1}{2} \pm \frac{1}{16} \cos 3\theta$ and radii of curvature $\rho(\theta) = \frac{1}{2}(1 \pm \cos 3\theta)$. This should be compared with the results obtained in [21, Section 3], where the author observes graphically that the support function $p(\theta) = \frac{4}{9} + \frac{1}{9}\cos^2 \frac{3}{2}\theta = \frac{1}{2} + \frac{1}{18}\cos 3\theta$ shapes a smooth convex body of constant width whose boundary is an irreducible algebraic plane curve of degree 8.

In the case $m = 5$, problem (16) restricted to real coefficients reads

$$\begin{array}{ll} \min_{a_3, a_5} & \frac{\pi}{4} - 16\pi a_3^2 - 48\pi a_5^2 \\ \text{s.t.} & \frac{1}{2} - 16 a_3 \cos 3\theta - 48 a_5 \cos 5\theta \geq 0, \ \forall \theta \in \mathbb{R}. \end{array}$$

On figure 3 we represent the two-dimensional convex feasible set obtained by minimizing linear functions along a sufficiently dense numbers of angular directions, as implemented in YALMIP function `plot` [20]. Numerically, we observe that the minimum area is achieved at the boundary points $a_3 = \pm\frac{1}{32}$ on the axis $a_5 = 0$.

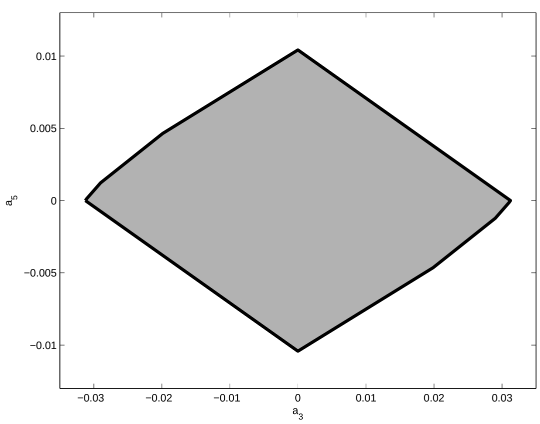

Figure 3: In shaded gray, admissible Fourier coefficients for constant width convex bodies with trigonometric polynomial support functions of fifth degree.

In the case $m = 7$, problem (16) restricted to real coefficients reads

$$\begin{array}{ll} \min_{a_3, a_5, a_7} & \frac{\pi}{4} - 16\pi a_3^2 - 48\pi a_5^2 - 96\pi a_7^2 \\ \text{s.t.} & \frac{1}{2} - 16 a_3 \cos 3\theta - 48 a_5 \cos 5\theta - 96 a_7 \cos 7\theta \geq 0, \ \forall \theta \in \mathbb{R}. \end{array}$$

On figure 4 we represent the three-dimensional convex feasible set. Numerically, we observe that the minimum area is achieved at the boundary points $a_3 = \pm\frac{1}{32}$ on the plane $a_5 = a_7 = 0$.

3.2 Planar rotors

In this section, we revisit the problem of minimizing the area in the class of rotors by using the same scheme as in section 3.1.

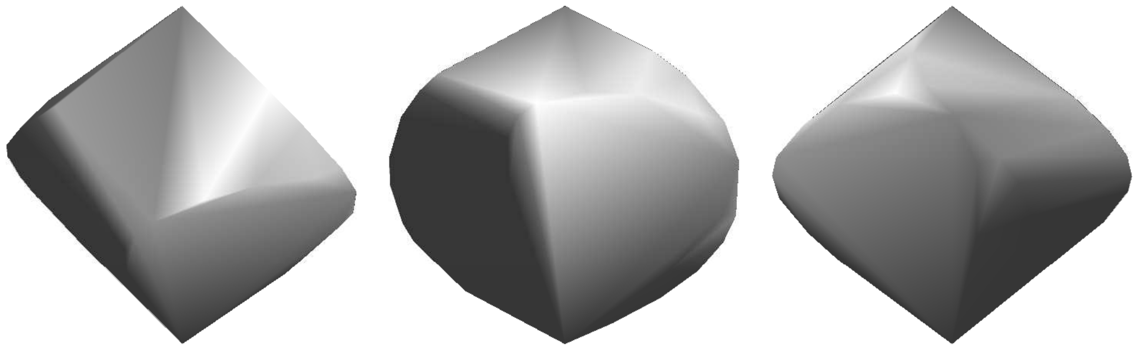

Figure 4: Three different views of the set of admissible Fourier coefficients for constant width convex bodies with trigonometric polynomial support functions of seventh degree.

A rotor is a generalization of a constant width body of $\mathbb{R}^2$ in an n-gon (that is a regular polygon with n sides). Let P a polygon with n sides. We say that a regular polygon P with n sides is tangential to a convex body K if $K \subset P$ and every side of P has a non-empty intersection with K. Geometrically speaking, a rotor K in an n-gon is a convex body such that each tangential regular polygon P has the same perimeter. It can be proved [4, 10] that a convex body K is a rotor of an n-gon if and only if its support function p is a 2π-periodic function of class $C^{1,1}$ such that:

$$\begin{cases} p(\theta) - 2\cos\delta p(\theta+\delta) + p(\theta+2\delta) = 1, \ \forall\, \theta \in \mathbb{R}, \\ p + p'' \geq 0 \text{ a.e.}, \end{cases} \tag{20}$$

where $\delta = \frac{2\pi}{n}$. The radius r of the inscribed circle to a tangential polygon P of a rotor K satisfies $4r\sin^2(\frac{\pi}{n}) = 1$. In the case $n = 4$, relations (20) yield relations (13) and $r = \frac{1}{2}$, corresponding to a body of constant unit width, studied in the previous section. The Fourier series of the support function of a rotor is:

$$p(\theta) = r + \sum_{k \in J} p_k e^{ik\theta},$$

where $J = (n\mathbb{Z}+1) \cup (n\mathbb{Z}-1)$, see [4]. Equivalently, we have:

$$\begin{aligned} p(\theta) = r + 2\mathrm{Re}(p_1 e^{i\theta}) \quad &+ \quad 2\sum_{k\geq 1} \left(a_{kn+1}\cos(kn+1)\theta + b_{kn+1}\sin(kn+1)\theta\right) \\ &+ \quad 2\sum_{k\geq 1} \left(a_{kn-1}\cos(kn-1)\theta + b_{kn-1}\sin(kn-1)\theta\right). \end{aligned}$$

It has been proved that the rotor of minimal area is the regular-trammel rotor described by Goldberg [4, 9]. The minimizers for $n = 3$ and $n = 5$ and the corresponding radii of curvature are represented on figure 5. Hereafter, we give the value of the radius of curvature for the theoretical minimizer on $[0, \frac{2\pi}{n}]$, see [4, 9] for more details:

$$\rho = \sum_{0 \leq j \leq n-1} r_j \mathbb{I}_{[\tau_j, \tau_{j+1}[}, \ r_j = \frac{r}{\cos(\frac{\pi}{n})}(\cos(\pi/n) - \cos((2j+1)\pi/n)), \ \tau_j = \frac{2j\pi}{n(n-1)}. \tag{21}$$

When $n=3$ the minimum area is $\frac{\pi}{3}-\frac{\sqrt{3}}{2}\simeq 0.18117$. When $n=5$ the minimum area is $\frac{\pi}{16\sin(\pi/5)^4}-\frac{10(\cot(3\pi/20)-1)-\pi(1+\cot(\pi/5)^2)}{32\sin(\pi/5)^2\cos(\pi/5)^2}\simeq 1.5713$. The exact value of the minimum area for any regular rotor can be found in [4] via Fourier series.

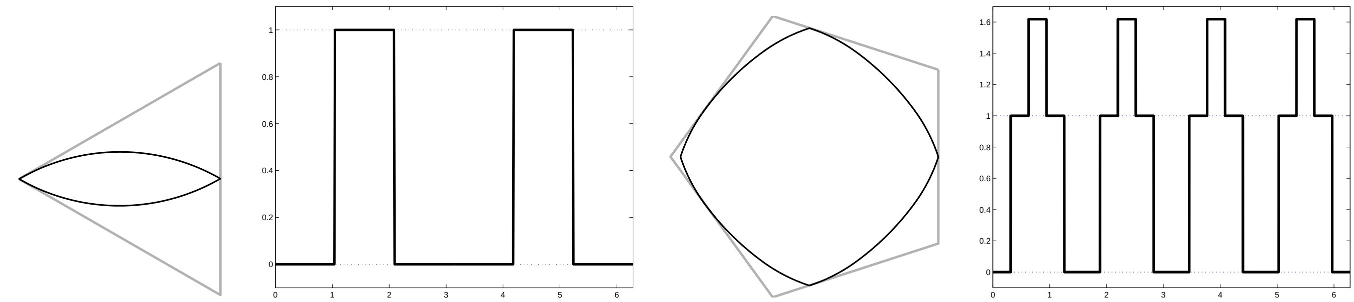

Figure 5: Optimal rotors (black) inscribed in polygons (gray) and radii of curvature for $n=3$ (left) and $n=5$ (right).

Following the same scheme as in the case of constant width bodies, we now consider the set of trigonometric polynomials p of degree $m=Nn+1$ which satisfy (20) and are given by:

$$\begin{aligned} p(\theta) = r + 2\mathrm{Re}(p_1 e^{i\theta}) &+ 2\sum_{k=1}^{N}\left(a_{kn+1}\cos(kn+1)\theta + b_{kn+1}\sin(kn+1)\theta\right) \\ &+ 2\sum_{k=1}^{N}\left(a_{kn-1}\cos(kn-1)\theta + b_{kn-1}\sin(kn-1)\theta\right). \end{aligned} \tag{22}$$

The area of a rotor is

$$A(p) = \pi r^2 + 2\pi\sum_{k=1}^{N}\left(1-(kn+1)^2\right)\left(a_{kn+1}^2+b_{kn+1}^2\right)+\left(1-(kn-1)^2\right)\left(a_{kn-1}^2+b_{kn-1}^2\right)$$

and finding the rotor of minimal area can be then stated as follows:

$$\begin{array}{ll} \min_p & A(p) \\ \text{s.t.} & (20) \text{ and } (22). \end{array}$$

In view of the Fourier decomposition of p on J, we consider the set of hermitian matrices Q of size $Nn+1$ satisfying the affine equations:

$$T_0\cdot Q=\frac{1}{2},\ T_1\cdot Q=0,\ T_{kn\pm l}\cdot Q=0,\ l\neq\pm 1,\ 1\leq kn\pm l\leq Nn+1. \tag{23}$$

The previous problem is then equivalent to:

$$\begin{array}{ll} \min_Q & \mathcal{A}(Q) \\ \text{s.t.} & Q\succeq 0 \text{ and } (23) \end{array} \tag{24}$$

The convex bodies obtained by this procedure for $m=30$ are represented on figure 6. Tables 3 and 4 report the areas of the rotors obtained by this procedure for $n=3$ and $n=5$, respectively. The coordinates of the n vertices of the tangential polygon to a rotor K can be computed by the formula ($0\leq k\leq n-1$):

$$\begin{cases} x_k = \frac{1}{\sin\delta}\left(p(k\delta)\sin(k+1)\delta - p((k+1)\delta)\sin k\delta\right), \\ y_k = \frac{1}{\sin\delta}\left(p((k+1)\delta)\cos k\delta - p(k\delta)\cos(k+1)\delta\right). \end{cases}$$

It is obtained by intersecting the n support lines of equation $x\cos k\delta + y\sin k\delta = p(k\delta)$ (see [3],[4]).

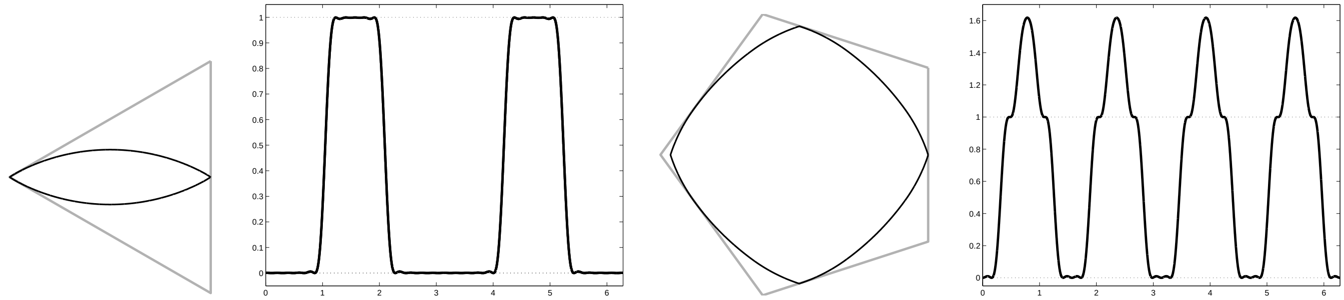

Figure 6: Optimal rotors (black) inscribed in polygons (gray) and radii of curvature of degree 30 for $n = 3$ (left) and $n = 5$ (right).

degree	1	2-3	4-7	8-9	10-13	14-15
area	0.33333	0.29089	0.22973	0.21416	0.20129	0.19699
degree	16-19	20-21	22-25	26-28	29-32	
area	0.19205	0.19007	0.18797	0.18696	0.18582	

Table 3: Optimal rotor areas for $n = 3$.

degree	1-3	4-13	14-15	16-23	24-35
area	1.6450	1.5901	1.5865	1.5807	1.5763

Table 4: Optimal rotor areas for $n = 5$.

3.3 Constant width bodies of revolution

In this section, we consider the problem of minimizing the surface within the class of constant width bodies of revolution. We briefly recall how the problem is parametrized in terms of support functions. We then compute the functional for support functions which are trigonometric polynomials, and we apply the same SDP parametrization to obtain a numerical solution of the problem.

It is standard that any convex body K obtained by rotation of a constant width planar body K_0 around one of its axis a symmetry Δ can be represented by a 2π-periodic support function p of class $C^{1,1}$ that satisfies

$$\begin{cases} p(\theta) + p(\theta + \pi) = 1, \ \forall \theta \in \mathbb{R} \\ p(\theta) = p(\pi - \theta), \ \forall \theta \in \mathbb{R} \\ p + p'' \geq 0 \text{ a.e.} \end{cases} \tag{25}$$

More details on this parametrization and on the following computations can be found in [3],[14]. Without loss of generality, if Δ is the z-axis, then the boundary of K is parametrized by

$$\begin{cases} x(\theta, \varphi) = (p(\theta)\cos\theta - p'(\theta)\sin\theta)\cos\varphi, \\ y(\theta, \varphi) = (p(\theta)\cos\theta - p'(\theta)\sin\theta)\sin\varphi, \\ z(\theta, \varphi) = p(\theta)\sin\theta + p'(\theta)\cos\theta, \end{cases} \tag{26}$$

with $\theta \in [0, 2\pi]$, $\varphi \in [0, \pi]$. Notice that any function p satisfying (25) must be such that

$p(0) = \frac{1}{2}$, $p'(\frac{\pi}{2}) = 0$ and

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(\theta)\cos\theta d\theta = 1. \tag{27}$$

Indeed, one has $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(\theta)\cos\theta d\theta = \int_0^{\frac{\pi}{2}} p(\theta)\cos\theta d\theta + \int_{-\frac{\pi}{2}}^{0} p(\theta)\cos\theta d\theta$, and $\int_{-\frac{\pi}{2}}^{0} p(\theta)\cos\theta d\theta = -\int_{\frac{\pi}{2}}^{\pi}(1-p(\theta))\cos\theta d\theta = 1 - \int_0^{\frac{\pi}{2}} p(\theta)\cos\theta d\theta$, which gives (27). The surface of revolution $S(p)$ of K is then given by:

$$S(p) = 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(p(\theta)+p''(\theta))(p(\theta)\cos\theta - p'(\theta)\sin\theta)d\theta = 2\pi\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(p^2(\theta)-\frac{1}{2}p'^2(\theta))\cos\theta d\theta \tag{28}$$

see [3]. Using (25), the previous expression can be written on $[0,\frac{\pi}{2}]$ as follows:

$$S(p) = 4\pi\int_0^{\frac{\pi}{2}}(p^2(\theta)-p'^2(\theta))\cos\theta d\theta + 2\pi - 4\pi\int_0^{\frac{\pi}{2}} p(\theta)\cos\theta d\theta.$$

Therefore, for the ball of radius $\frac{1}{2}$, $p(\theta) = \frac{1}{2}$ and $S(p) = \pi$, whereas for the rotated Reuleaux triangle, the support function on $[0,\frac{\pi}{2}]$ is:

$$\begin{cases} p(\theta) = \frac{1}{2}\cos\theta, \ \theta\in[0,\frac{\pi}{3}], \\ p(\theta) = 1+\frac{1}{2}\cos\theta - \cos(\theta-\frac{\pi}{3}), \ \theta\in[\frac{\pi}{3},\frac{\pi}{2}] \end{cases}$$

and $S(p) = 2\pi - \frac{\pi^2}{3} \approx 2.9933$.

Using (25), the decomposition of p into Fourier series writes:

$$p(\theta) = \frac{1}{2} + 2\sum_{k\geq 0} b_{2k+1}\sin(2k+1)\theta. \tag{29}$$

The surface of revolution can be then rewritten in terms of Fourier coefficients of p. Consider for $j,k\geq 0$ the real:

$$\nu_{jk} = \frac{32(-1)^{j+k}jk(1+j+k+jk)}{(2j+2k+3)(2j+2k+1)(2j-2k+1)(2j-2k-1)}.$$

Proposition 3.2. *If K is a constant width body of revolution obtained by rotation of K_0, and p is the support function of K_0, then the surface of K is:*

$$S(p) = \pi + 8\pi\sum_{j,k\geq 0}\nu_{jk}b_{2j+1}b_{2k+1}. \tag{30}$$

Proof. Let $\alpha_j = 2b_{2j+1}$, $j\geq 0$. From (28), we have:

$$S(p) = 2\pi\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}((p(\theta)-\frac{1}{2})^2 - \frac{1}{2}p'^2(\theta))\cos\theta d\theta + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(\theta)\cos\theta d\theta - \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos\theta d\theta\right).$$

A straightforward computation shows that:

$$\begin{cases} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}(p(\theta)-\frac{1}{2})^2\cos\theta d\theta = \sum_{j,k\geq 0}\lambda_{j,k}\alpha_{2j+1}\alpha_{2k+1}, \\ \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p'^2(\theta)\cos\theta d\theta = \sum_{j,k\geq 0}(2j+1)(2k+1)\mu_{j,k}\alpha_{2j+1}\alpha_{2k+1}, \end{cases}$$

with

$$\lambda_{j,k} = -\frac{2(-1)^{j+k}((2j+1)^2 + (2k+1)^2 - 1)}{(2j+2k+3)(2j+2k+1)(2j-2k+1)(2j-2k-1)},$$

and

$$\mu_{j,k} = -\frac{4(-1)^{j+k}(4jk+2j+2k+1)}{(2j+2k+3)(2j+2k+1)(2j-2k+1)(2j-2k-1)}.$$

Next we can verify that $\nu_{jk} = \lambda_{jk} - \frac{(2j+1)(2k+1)}{2}\mu_{jk}$ and that $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} p(\theta)\cos\theta d\theta - \frac{1}{4}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\theta d\theta = 1/2$ by using (27). Relation (30) follows readily. □

Let us now state the optimization problem that we consider in the case of constant width bodies of revolution. Consider the set of trigonometric polynomials p of degree $m = 2N+1$ which satisfy (25) and whose Fourier series is given by:

$$p(\theta) = \frac{1}{2} + 2\sum_{k=0}^{N} b_{2k+1}\sin(2k+1)\theta. \tag{31}$$

The surface of the body is

$$S(p) = \pi + 8\pi \sum_{j,k=0}^{N} \nu_{jk} b_{2j+1} b_{2k+1},$$

and hence the optimization problem we consider is

$$\begin{array}{ll} \min_p & S(p) \\ \text{s.t.} & (25) \text{ and } (31). \end{array} \tag{32}$$

The convex body of revolution obtained by this procedure for $m = 49$ is represented on figure 7. The optimal surfaces for various degrees are reported in table 5.

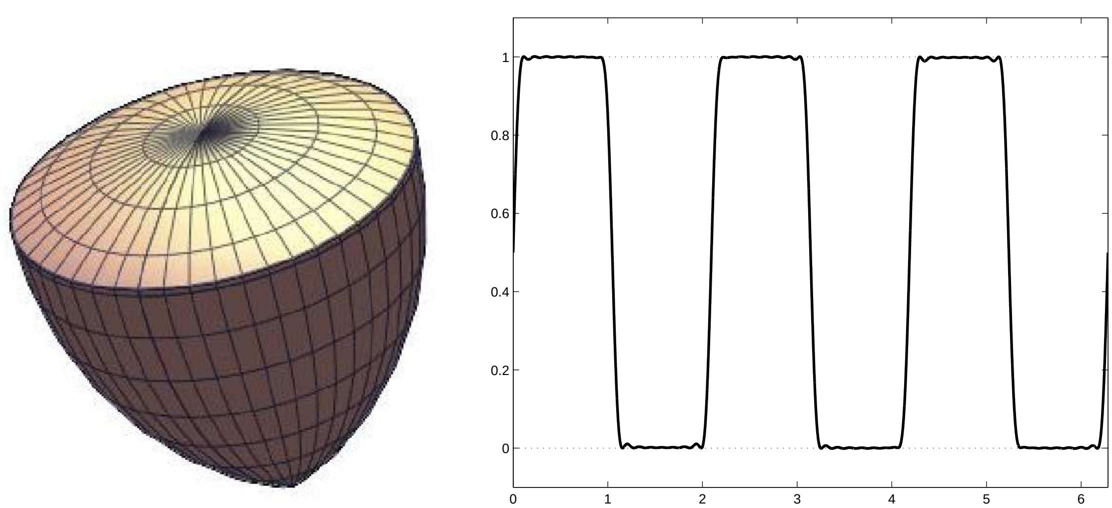

Figure 7: Convex body of revolution and radius of curvature of degree 39.

degree	3-5	7	9-11	13	15-17	19	21-23
surface	3.0518	3.0512	3.0218	3.0215	3.0105	3.0102	3.0050
degree	25	27-29	31	33-35	37	39	
surface	3.0049	3.0020	3.0019	3.0001	3.0000	2.9989	

Table 5: Optimal surfaces of constant width convex bodies of revolution.

3.4 Optimization problem with a relaxation of the width constraint

In this section, we consider the problem of minimizing the area in a class of convex bodies for which the width lies within two values w_1 and w_2 with $w_1 < w_2$ and we use the SDP parametrization to obtain numerical solutions. Such a contraint is a natural extension of constant width bodies and has been studied in [1],[2]. A set satisfying such an inequality constraint is not necessarily strictly convex, and its support function is not necessarily of class C^1, but Lipschitz continuous (see [19]). Let us note $C^{0,1}$ the set of all Lipschitz continuous functions which are 2π-periodic. In this section, we consider the subset F of $C^{0,1}$ of all functions p satisfying:

$$\begin{cases} w_1 \leq p(\theta) + p(\theta+\pi) \leq w_2, \ \forall \theta \in \mathbb{R}, \\ \exists \ \theta_1 \in [0, 2\pi], \ p(\theta_1) + p(\theta_1 + \pi) = w_1, \\ \exists \ \theta_2 \in [0, 2\pi], \ p(\theta_2) + p(\theta_2 + \pi) = w_2, \\ p + p'' \geq 0, \ \text{a.e.} \end{cases} \tag{33}$$

The function p is the support function of a convex body K for which the width function $w(\theta) = p(\theta) + p(\theta + \pi)$ takes values within $I = [w_1, w_2]$, and the two extremal values of I are achieved. Notice that the set F does not define a convex subset of $C^{0,1}$. Consider the set of trigonometric polynomials p of degree $m \leq N$:

$$p(\theta) = p_0 + 2\sum_{k=1}^{N}(a_k \cos k\theta + b_k \sin k\theta) \tag{34}$$

which satisfy (33). If θ_1 and θ_2 are fixed, the two pointwise constraints of (33) are linear with respect to the Fourier coefficients of p. The convexity constraint can be then reformulated as above as a semidefinite constraint. The two inequality constraints on the width function are nonnegativity constraints on trigonometric polynomials, and they can also be transformed into semidefinite constraints.

Our goal is to study the optimization problem:

$$\begin{array}{ll} \min_p & J(p) \\ \text{s.t.} & (33) \text{ and } (34) \end{array} \tag{35}$$

where $J(p)$ represents plus or minus the area or perimeter of a convex body K whose support function is p.

We successively consider the next four problems:

- (P1): maximization of the perimeter s.t. (33) (linear objective)

- (P2): minimization of the perimeter s.t. (33) (linear objective)
- (P3): maximization of the area s.t. (33) (convex quadratic objective)
- (P4): minimization of the area s.t. (33) (concave quadratic objective)

As far as we know, these problems have been studied only in [1, 2]. The authors studied (P1), (P2) and (P3) and described convex bodies solving these problems. Optimality was proven by computing the support function associated to each convex body and by proving that the Hamilton-Jacobi equation is satisfied (sufficient conditions for optimality).

This section aims at finding numerically the optimal shape described in [1, 2]. From the numerical experiments made for (P4), we are able to make a conjecture concerning a solution of this problem.

It is always possible to let $\theta_1 = 0$ (for which the minimal width w_1 is achieved) by performing a rotation. The value of θ_2 (for which the maximal width w_2 is achieved) cannot be fixed and we currently do not know how to handle this in the optimization procedure. Here we consider the simpler instance consisting in letting $\theta_2 = \frac{\pi}{2}$, and we take $w_1 = 1$, $w_2 = 2$.

3.4.1 Maximization of the perimeter

In the following, let $\alpha = \arccos(w_1/w_2)$. For problem (P1), the result proved in [1, 2] is the following. A maximizer of the perimeter under the additional constraint (33) is a convex body K represented by its support function p such that

$$\begin{cases} p(0) + p(\pi) = w_1, \\ p(\theta) + p(\theta + \pi) = w_2, \ \theta \in [\alpha, \pi - \alpha], \\ p(\theta) + p''(\theta) = p(\theta + \pi) + p''(\theta + \pi) = 0, \ \theta \in [0, 2\pi] \backslash [\alpha, \pi - \alpha], \end{cases}$$

and it is unique up to a translation. In the direction $\theta = 0$, the width of K is w_1, whereas for $\theta \in [\alpha, \pi - \alpha]$, the width of K is w_2. Geometrically speaking, the boundary of K, ∂K, has four corners (when $p(\theta) + p''(\theta) = 0$). Moreover, $\{\theta \in [0, 2\pi) \mid w(\theta) = w_1\}$ is a singleton, hence ∂K is reduced to a line-segment in the direction $\theta = 0$ and $\theta = \pi$. The boundary of K therefore contains strictly convex parts (when $w(\theta) = w_2$) and line-segment ($\theta = 0, \pi$). The value of the perimeter for a maximizer is $L_M = 2\sqrt{w_2^2 - w_1^2} + 2w_2 \arcsin(w_1/w_2) \simeq 5.55840$.

From a numerical point of view, these properties are confirmed (see figure 8). The radius of curvature obtained numerically is close to a Dirac measure for $\theta = 0, \pi$ (corresponding to a line-segment) and is close to zero in four disjoint intervals (corresponding to four corners).

degree	3	4-5	6-7	8-9	10-11	12-13	14-15	
perimeter	4.71239	5.00361	5.04898	5.16139	5.23920	5.25526	5.32387	
degree	16-17	18-19	20-21	22-23	24-25	26-27	28-29	30
perimeter	5.33480	5.36358	5.37960	5.38821	5.40448	5.41384	5.42019	5.43213

Table 6: Numerical solution of the perimeter maximization problem.

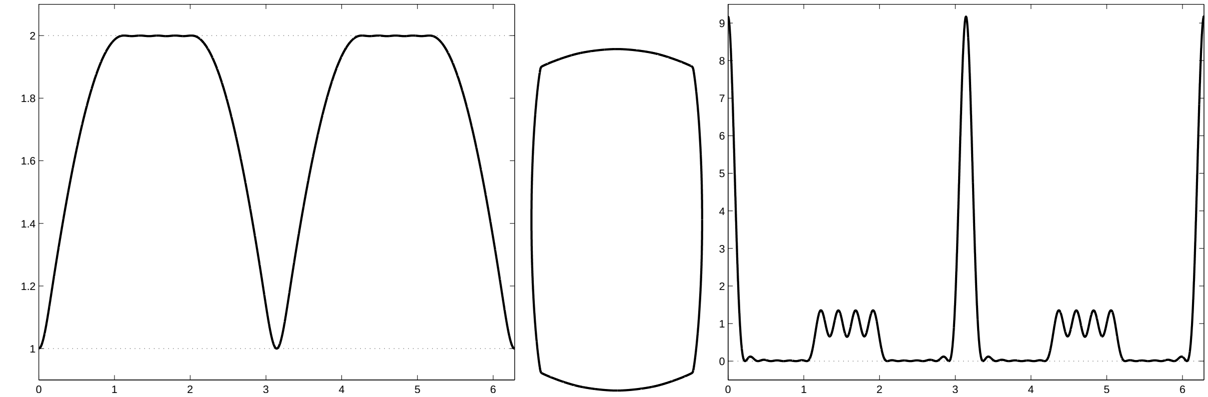

Figure 8: Numerical solution of the perimeter maximization problem for degree 30: width, body and curvature.

3.4.2 Minimization of the perimeter

For problem (P2), the result proved in [1, 2] is the following. A minimizer of the perimeter under the additional constraint (33) is a convex body represented by a support function p such that :

$$\begin{cases} p(0) + p(\pi) = w_2, \\ p(\theta) + p(\theta + \pi) = w_1, \ \theta \in [\alpha, \pi - \alpha], \\ p(\theta) + p''(\theta) = p(\theta + \pi) + p''(\theta + \pi) = 0, \ \theta \in [0, 2\pi] \backslash [\alpha, \pi - \alpha], \end{cases}$$

and it is unique up to a translation. Notice that (P1) and (P2) are analogous as the perimeter is linear with respect Fourier coefficients. It is therefore natural to obtain a solution of (P2) by permutation of w_1 and w_2. The value of the perimeter for the minimizer is $L_m = 2\sqrt{w_2^2 - w_1^2} + 2w_1 \arcsin(w_1/w_2) \simeq 4.51129$.

As in section 3.4.1, the numerical results confirm the result obtained in [1, 2].

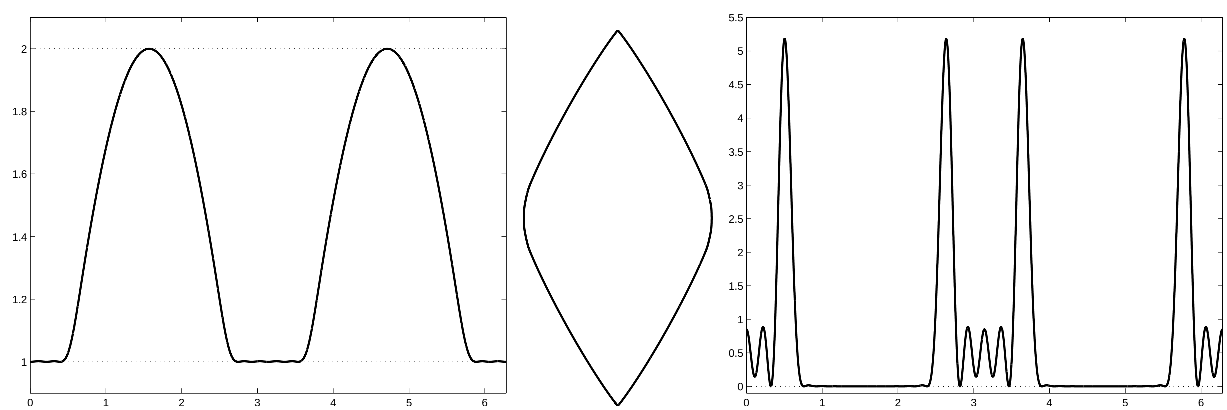

Figure 9: Numerical solution of the perimeter minimization problem for degree 30: width, body and curvature.

degree	3-5	6-7	8-9	10-11	12-13	14-15	16-17
perimeter	4.71239	4.64968	4.58365	4.58164	4.568315	4.54714	4.53457
degree	18-19	20-21	22-23	24-25	26-27	28-29	30
perimeter	4.53376	4.53137	4.526218	4.52528	4.52446	4.52157	4.51987

Table 7: Numerical solution of the perimeter minimization problem.

3.4.3 Maximization of the area

For problem (P3), the result proved in [1, 2] is the following. A solution of (P3) is the intersection of a disc of diameter w_2 and the domain inside two parallel lines symmetric with respect to the center of the disc and at a distance w_1. The value of the area evaluated for the maximizer is $A_M = 1/2(w_1\sqrt{w_2^2 - w_1^2} + w_2^2 \arcsin(w1/w2)) \simeq 1.9132$. Again, the numerical results (see Figure 10) confirm the theoretical result. Notice the analogy between the numerical solution obtained in sections 3.4.1 and 3.4.2 although theoretical solutions of P1 and P3 are different. The numerical problem which is solved in this section is however slightly different as the theoretical one as both θ_1 and θ_2 have been fixed.

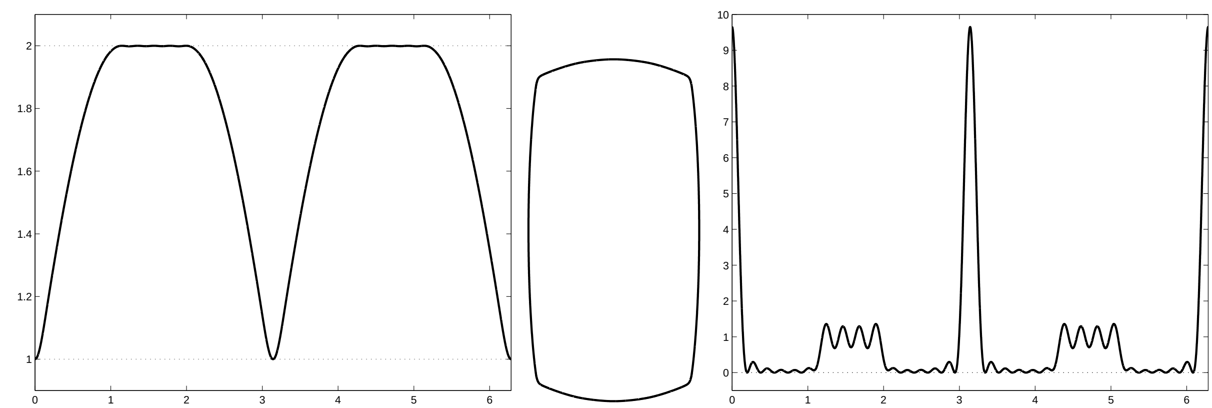

Figure 10: Numerical solution of the area maximization problem for degree 30: width, body and curvature.

degree	3	4-5	6-7	8-9	10-11	12-13	14-15	16-17
area	1.4726	1.6472	1.6815	1.7388	1.7718	1.7844	1.8098	1.8173
degree	18-19	20-21	22-23	24-25	26-27	28-29	30	
area	1.8273	1.8364	1.8411	1.8467	1.8524	1.8552	1.8595	

Table 8: Numerical solution of the area maximization problem.

3.4.4 Minimization of the area

For problem (P4), we expect to find a similar shape as for (P3), although (P3) is convex whereas (P4) is not convex due to the concave quadratic objective function. The numerical computations are depicted on Figure 3.4.4.

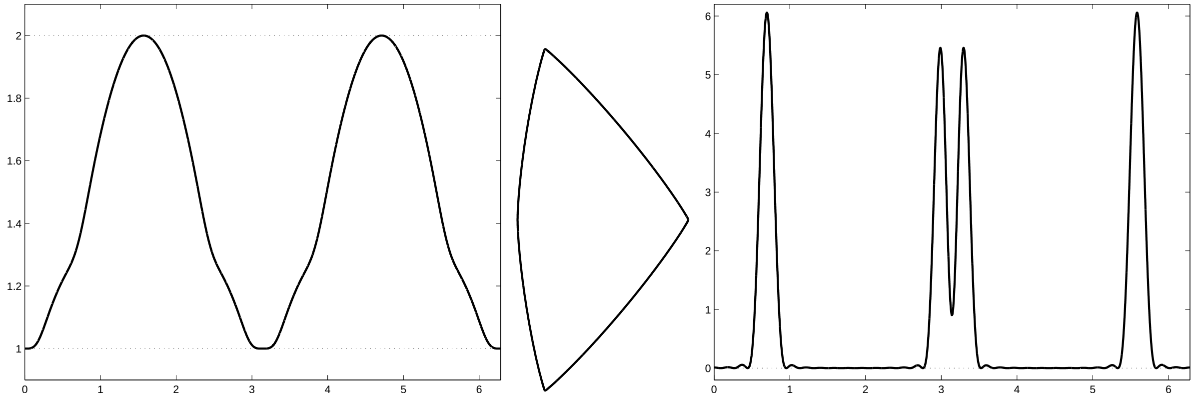

Figure 11: Numerical solution of the area minimization problem for degree 30: width, body and curvature.

degree	3-4	5	6	7	8-9	10	11
area	1.4726	1.4081	1.3937	1.3349	1.3091	1.3060*	1.2563
degree	12	13	14	15	16	17	18
area	1.2562	1.2319	1.2243	1.2299*	1.2246*	1.1978*	1.1966
degree	19	20	21	22	23	24	25
area	1.1936	1.1783*	1.1659*	1.1636	1.1484*	1.1640*	1.1465*
degree	26	27	28	29	30		
area	1.1325	1.1319	1.1353*	1.1281*	1.1271*		

Table 9: Numerical solution of the area minimization problem. A star means that PENBMI cannot guarantee convergence (the solution is feasible but not necessary locally optimal).

3.4.5 Comments on (P1), (P2), (P3), (P4)

From a theoretical point of view, both problems (P1), (P2) are linear with respect to p. The constraints given by (33) can be decomposed into a convex set $\mathcal{C}_1$ (defined by $p + p'' \geq 0$, $w_1 \leq w \leq w_2$) and a non-convex set $\mathcal{C}_2$ (defined by $\min_\theta w(\theta) = w_1$, $\max_\theta w(\theta) = w_2$). It is standard that a minimizer p (or a maximizer) of the perimeter cannot take values in the interior of $\mathcal{C}_1$ on a set of strictly positive measure. It can be then expected that a minimizer p will be such that for a.e. $\theta \in [0, 2\pi]$, $p(\theta) + p''(\theta) = 0$ or $w(\theta) \in \{w_1, w_2\}$. Moreover, for the minimization problem, the set $\{\theta \in [0, 2\pi] \mid w(\theta) = w_2\}$ must be of zero measure. We can therefore expect that the boundary of a minimizer consists of line-segments (separated by w_2) and strictly convex parts such that $w(\theta) = w_1$. As the width function w is continuous, there exists a certain interval on which $p(\theta) + p''(\theta) = 0$ (corresponding to a corner) connecting a line-segment and a strictly convex part. The numerical experiments performed in the previous section confirm these remarks (both for (P1) and (P2)).

Similar remarks can be made for (P3) and (P4). First, notice that for (P3), the functional is convex. If the constraint $\min_\theta w(\theta) = w_1$ is removed, the disc of diameter w_2 would solve (P3) by the isoperimetric inequality. It can be then expected that the boundary of a maximizer contains arcs of circle of radius $w_2/2$ and also line-segments deparated by w_1 to ensure the constraint $\min_\theta w(\theta) = w_1$. Problem (P4) is concave and is more tedious to study. From the numerical computations, we conjecture that a minimizer is a convex polygon with at most four sides.

4 Convergence of the method

This section is devoted to the study of the convergence of the method which has been previously discussed. We investigate the case of planar constant width bodies (section 3.1). The proof can be transposed to the case of rotors (see remark 4.2).

In the following, let H^1 denote the set of 2π-periodic functions whose restrictions to compact subdomains Ω of $\mathbb{R}$ lie in $L^2(\Omega)$ with weak partial derivatives in $L^2(\Omega)$. The Fourier coefficients $c_k = a_k - ib_k$ of a function $p \in H^1$ are defined as in section 2.1. Let F the set of 2π-periodic functions $p \in C^{1,1}(\mathbb{R},\mathbb{R})$ which satisfy:

$$\begin{cases} p(\theta) = \frac{1}{2} + \sum_{k\geq 1} a_{2k+1}\cos(2k+1)\theta + b_{2k+1}\sin(2k+1)\theta, \\ p + p'' \geq 0, \end{cases} \tag{36}$$

and F_N the subset of F containing the trigonometric polynomials such that:

$$\begin{cases} p(\theta) = \frac{1}{2} + \sum_{1\leq k\leq N} a_{2k+1}\cos(2k+1)\theta + b_{2k+1}\sin(2k+1)\theta, \\ p + p'' \geq 0. \end{cases} \tag{37}$$

By compactness, there exists a minimum p_N of a functional J (e.g. the area) in F_N whose Fourier series is given for each N by (37). Recall that p_R denotes the support function of the Reuleaux triangle and p_R is the unique minimizer of J in F (up to a translation) according to the Blaschke-Lebesgue theorem. Moreover, $\rho_R = p_R + p_R''$ is given by (14) and we have:

$$p_R(\theta) = p_R(0)\cos\theta + p_R'(0)\sin\theta + \int_0^\theta \sin(\theta - t)\rho_R(t)dt. \tag{38}$$

As $F_N \subset F$, we have $A(p_N) \geq A(p_R)$ for all N. The aim of this section is to prove the following theorem.

Theorem 4.1. *The sequence p_N converges to p_R in H^1 and $A(p_N)$ tends to $A(p_R)$ when N tends to infinity.*

The theorem is proved in two steps (Lemma 4.1 and Lemma 4.2).

Remark 4.1. *Let ρ_R^N the partial sum of the Fourier series of ρ_R. By Fourier series theory, it is standard that ρ_R^N converges to ρ_R in the L^2-norm when N goes to infinity and this convergence is not uniform as ρ_R is discontinuous (Gibbs phenomenon). Moreover, the infimum of ρ_R^N is strictly negative and converges to a certain value $\lambda < 0$ (the exact value of λ can be computed exactly in terms of the jump of ρ_R at the discontinuity). As a consequence, no function p_R^N solution of $p + p'' = \rho_R^N$ belongs to F_N.*

Lemma 4.1. *Let p_N a minimizer of A in F_N. There exists $p \in F$ such that up to a subsequence, p_N converges to p in H^1. Moreover, $A(p_N)$ tends to $A(p)$ when N tends to infinity.*

Proof. Let $\alpha := \inf_N A(p_N) \geq 0$. We have for all $N > 0$:

$$\alpha \leq A(p_N) = \frac{\pi}{2} + \sum_{1\leq k\leq N} (1-(2k+1)^2)(a_{2k+1}^2 + b_{2k+1}^2).$$

From this inequality, we deduce that p_N is bounded in H^1, and consequently, there exist $p \in H^1$ such that, up to a subsequence, p_N weakly converges to p in H^1. Moreover, by the Cauchy-Schwarz inequality, there exists $C > 0$ such that for all $N \geq 0$

$$\sup_{\|\phi\|_{L^2}\leq 1} \left| \int_0^{2\pi} (p_N + p_N'')\phi \right| = \sup_{\|\phi\|_{L^2}\leq 1} \left| \int_0^{2\pi} p_N\phi - p_N'\phi' \right| \leq C.$$

It follows that $\rho_N := p_N + p_N''$ is bounded in L^2. Up to a subsequence, we may assume that ρ_N weakly converges to a certain function $\rho \in L^2$. Let (α, β) be taken arbitrarily in $\mathbb{R}^2$ and set $p_N(0) = \alpha$, $p_N'(0) = \beta$ for all $N > 0$ (the area A is invariant with respect to translations). We then have:

$$p_N(\theta) = \alpha\cos\theta + \beta\sin\theta + \int_0^\theta \rho_N(t)\sin(\theta - t)dt, \tag{39}$$

and

$$p_N'(\theta) = -\alpha\sin\theta + \beta\cos\theta + \int_0^\theta \rho_N(t)\cos(\theta - t)dt. \tag{40}$$

By weak convergence of ρ_N, we obtain that for all $\theta \in [0, 2\pi]$:

$$p_N(\theta) \to \tilde{p}(\theta) := \alpha\cos\theta + \beta\sin\theta + \int_0^\theta \rho(t)\sin(\theta - t)dt. \tag{41}$$

Clearly, $\tilde{p}$ is of class C^1, and similarly, we get for all $\theta \in [0, 2\pi]$:

$$p_N'(\theta) \to \tilde{p}'(\theta) := -\alpha\sin\theta + \beta\cos\theta + \int_0^\theta \rho(t)\cos(\theta - t)dt. \tag{42}$$

As the sequence $\|\rho_N\|_{L^2}$ is bounded, we obtain by (39) that p_N is uniformly bounded on $[0, 2\pi]$. By Lebesgue's theorem, the sequence p_N strongly converges to p in L^2 and thus $\tilde{p} = p$. Using (40), we obtain similarly that p_N' strongly converges to p' in L^2. It remains to prove that p is the support function of a planar constant width body. As for all $k \neq 0$ we have $c_{2k}(p_N) = 0$, we obtain at the limit $c_{2k}(p) = 0$. Moreover, for all non-negative function φ of class C^2 with compact support on $[0, 2\pi]$, it holds

$$\int_0^{2\pi} (p + p'')\varphi = \lim_{N\to\infty} \int_0^{2\pi} (p_N + p_N'')\varphi,$$

and the latter integral is non-negative as $p_N \in F_N$. Hence, $p + p'' \geq 0$ and p is in F. The convergence of $A(p_N)$ to $A(p)$ is a consequence of the convergence of p_N to p in H^1. □

The next lemma is the main result and establishes the convergence in H^1 of a sequence $h_N \in F_N$ to p_R.

Lemma 4.2. *There exists a sequence of trigonometric polynomials $h_N \in F_N$ converging to p_R in H^1 and such that $A(h_N)$ tends to $A(p_R)$ when N tends to infinity.*

Proof. Let $\varepsilon > 0$. We first prove the existence of a continuous periodic function ρ_ε, C^1-piecewise such that:

$$\begin{cases} \rho_\varepsilon(\theta) + \rho_\varepsilon(\theta + \pi) = 1, \ \forall \theta \in \mathbb{R}, \\ \varepsilon \leq \rho_\varepsilon(\theta) \leq 1 - \varepsilon, \\ \|\rho_\varepsilon - \rho_R\|_{L^2} \leq \sqrt{2}\varepsilon, \\ \int_0^{2\pi} \rho_\varepsilon(\theta) e^{i\theta} d\theta = 0. \end{cases} \tag{43}$$

There exists a continuous mapping $\overline{\rho}_\varepsilon : [0, \pi] \to \mathbb{R}$, C^1-piecewise such that:

$$\begin{cases} \varepsilon \leq \overline{\rho}_\varepsilon(\theta) \leq 1 - \varepsilon, \ \forall \theta \in [0, \pi], \\ \int_0^\pi (\overline{\rho}_\varepsilon - \rho_R)^2 \leq \varepsilon^2, \\ \overline{\rho}_\varepsilon(0) = \overline{\rho}_\varepsilon(\pi) = \frac{1}{2}. \end{cases} \tag{44}$$

Notice that the restriction of ρ_R to $[0, \pi]$ is symmetric with respect to $\theta = \frac{\pi}{2}$. It is therefore possible to assume that $\overline{\rho}_\varepsilon$ is also symmetric with respect to $\theta = \frac{\pi}{2}$ and that:

$$\int_0^{\frac{\pi}{2}} \overline{\rho}_\varepsilon(\theta) \sin\theta d\theta = \frac{1}{2}. \tag{45}$$

By symmetry, $\overline{\rho}_\varepsilon$ satisfies:

$$\int_0^\pi \overline{\rho}_\varepsilon(\theta) \cos\theta d\theta = 0, \tag{46}$$

and by (45) one has:

$$\int_0^\pi \overline{\rho}_\varepsilon(\theta) \sin\theta d\theta = 1. \tag{47}$$

Let us now define ρ_ε on $[0, 2\pi]$ by:

$$\begin{cases} \rho_\varepsilon(\theta) = \overline{\rho}_\varepsilon(\theta), \ \theta \in [0, \pi], \\ \rho_\varepsilon(\theta) = 1 - \overline{\rho}_\varepsilon(\theta - \pi), \ \theta \in [\pi, 2\pi]. \end{cases}$$

By periodicity, the function ρ_ε is extended to $\mathbb{R}$. By (44), we have $\overline{\rho}_\varepsilon(0) = 1 - \overline{\rho}_\varepsilon(\pi)$, and consequently ρ_ε is continuous for $\theta = \pi$ and $\theta = 2\pi$ and defines a continuous and C^1-piecewise mapping. By construction ρ_ε takes values within $[\varepsilon, 1 - \varepsilon]$ and satisfies $\rho_\varepsilon(\theta + \pi) + \rho_\varepsilon(\theta) = 1$ for all $\theta \in \mathbb{R}$. Now we have by (46) and (47):

$$\int_0^{2\pi} \rho_\varepsilon(\theta) e^{i\theta} d\theta = 2 \int_0^\pi \overline{\rho}_\varepsilon(\theta) e^{i\theta} d\theta - 2i = 0.$$

Finally:

$$\|\rho_\varepsilon - \rho_R\|_{L^2}^2 = 2 \int_0^\pi (\overline{\rho} - \rho_R)^2 \leq 2\varepsilon^2,$$

and we get (43).
Now consider q_N the partial sum of the Fourier series of ρ_ε. As ρ_ε is continuous and C^1-piecewise, q_N uniformly converges to ρ_ε on $\mathbb{R}$, and there exists N_0 such that for all $N \geq N_0$:

$$\|q_N - \rho_\varepsilon\|_{L^\infty} \leq \frac{\varepsilon}{2}.$$

Thus, q_N takes values within $[\frac{\varepsilon}{2}, 1 - \frac{\varepsilon}{2}]$ for $N \geq N_0$, and by the triangular inequality, one has:

$$\|q_N - \rho_R\|_{L^2} \leq \beta \frac{\varepsilon}{2},$$

where $\beta = \sqrt{2} + \pi$. To finish the proof, let us define h_N as the solution of $h_N + h_N'' = q_N$ such that:

$$h_N(\theta) = p_R(0)\cos\theta + p_R'(0)\sin\theta + \int_0^\theta \sin(\theta - t) q_N(t) dt.$$

A standard computation shows that:

$$h_N'(2\pi) - h_N'(0) - i(h_N(2\pi) - h_N(0)) = \int_0^{2\pi} q_N(\theta) e^{i\theta} d\theta.$$

By (43) the first Fourier coefficient of ρ_ε is equal to zero, therefore the last quantity is equal to zero which ensures that h_N is 2π-periodic. By construction $0 \le q_N = h_N + h_N'' \le 1$ and we have for all $\theta \in \mathbb{R}$: $h_N(\theta) + h_N(\theta + \pi) = 1$ (by the decomposition of q_N on the harmonics of order $2k+1$). Thus, h_N belongs to F_N. By using (38) and the Cauchy-Schwarz inequality, we get:

$$\|h_N - p_R\|_{L^2} \le \pi\beta\varepsilon,$$

and consequently h_N converges to p_R in L^2. Similarly, h_N' converges to p_R' in L^2. Finally, $A(h_N)$ tends to $A(p_R)$ when N tends to infinity which ends the proof of the Lemma. □

Proof of theorem 4.1. By Lemma 4.1, the sequence of minimizers p_N converges to $p \in F$ in H^1. As $h_N \in F_N$, we have:

$$A(p_R) \le A(p_N) \le A(h_N).$$

From Lemma 4.2, $A(h_N)$ tends to $A(p_R)$. Hence, $A(p_N)$ tends to $A(p) = A(p_R)$ and by uniqueness of the minimizer of A in F, we obtain $p(\theta) = p_R(\theta) + (\alpha - p_R(0))\cos\theta + (\beta - p_R'(0))\sin\theta$, $\forall\theta \in \mathbb{R}$.□

Remark 4.2. *The result of Theorem 4.1 can be extended in the case of planar rotors of a regular polygon P_n. The width constraint is replaced by (20), and the Fourier decomposition of a rotor involves harmonics of orders $kn+1$ and $kn-1$. The proof mimics that of Lemma 4.2. We believe that the result of Theorem 4.1 can be transposed to the problem of section 3.3. Although the admissible set in section 3.3 is a subset of F_N, the functional is non-autonomous and the proof must be modified.*

5 Conclusion and perspectives

This paper aims at giving a numerical approach to investigate optimization problems in the class of convex bodies. As far as we know, the transformation of the convexity constraint into a semidefinite programming problem has not been previously studied in this framework. This method seems to be efficient in the context of planar constant width bodies and rotors, even though minimization of a concave functional on a convex set is a difficult problem in principle. The approximation can be made precise quickly (a few seconds on a standard PC) by increasing the number of Fourier coefficients to more than 30.

We believe this work opens a promising way to investigate other optimization problems in the class of planar and space convex bodies. To extend this method in $\mathbb{R}^3$ the Fourier series decomposition must be replaced by bivariate spherical harmonics. Nevertheless, the decomposition of non-negative spherical harmonics into sum-of-squares is more subtle in this case.

The question of finding a numerical relevant method to investigate Meissner's conjecture [6] on the constant breadth body of minimal volume remains a challenging problem, as recalled in [12].

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