Positive trigonometric polynomials for strong stability of difference equations[1]

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Abstract

We follow a polynomial approach to analyse strong stability of continuous-time linear difference equations with several delays Upon application of the Hermite stability criterion on the discrete-time homogeneous characteristic polynomial, assessing strong stability amounts to deciding positive definiteness of a multivariate trigonometric polynomial matrix. This latter problem is addressed with a converging hierarchy of linear matrix inequalities (LMIs). Numerical experiments indicate that certificates of strong stability can be obtained at a reasonable computational cost for state dimension and number of delays not exceeding 4 or 5.

Keywords: strong stability, spectral radius, trigonometric polynomials, LMI.

1 Introduction

In general, spectrum-based analysis of time-delay systems can be handled in the same way it is done for delay-free systems. Although the spectrum is infinite, stability is determined by the rightmost eigenvalues, more precisely by the sign of the spectral abscissa, the maximum real part of the eigenvalues. For retarded systems, the spectral abscissa is nonsmooth but continuous in all parameters of the system, including time delays, see [27]. However, it results from [14, 2, 9, 10], that, in general, it is not the case for neutral systems and kernel operators – the so-called associated difference equations, see also [20, 21, 22]. It is well-known that the spectral abscissa of the difference equation is not

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continuous in delays. Thus, arbitrarily small changes in the delay values can destroy stability. Moreover, it can even happen that the number of unstable roots increases stepwise from zero to infinity. In order to handle this hypersensitivity of the stability of the difference equation with respect to delay values, the concept of *strong stability* was introduced by [10]. Let us remark that the *strong stability* concept has recently been generalized by [23] toward difference equations with dependencies in the delays.

As stability of its kernel operator is a necessary condition for stability of a neutral system, all the hypersensitivity stability issues are carried over to the stability of neutral systems. Thus the *strong stability* test should always be performed to guarantee practical stability of neutral systems. However, as will be shown later in the text, the *strong stability* test is rather complex. So far, a coarse numerical implementation of the test without guarantee or certificate has been used as a rule, see e.g. [21, 28]. Even though this *brute force* based approach works in most cases, it might fail due to approximation errors in the numerical scheme. As the main result of this paper we propose a more rigorous *strong stability test* that is based on a polynomial approach, relying on the numerical solution of a hierarchy of linear matrix inequalities (LMIs).

In the field of time-delay systems, LMIs are usually used as stability determining criteria resulting from the Lyapunov time-domain approach, see e.g. [24] or [19], among many others.

1.1 Problem statement

We consider a continuous-time difference equation in the following form

$$x(t) + \sum_{k=1}^{m} H_k x(t - \tau_k) = 0, \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $H_k \in \mathbb{R}^{n \times n}$ are real coefficient matrices and $\tau_k > 0, k = 1, \ldots m$ are the time delays. Stability of the equation (1) is determined by the position of the roots of the following characteristic equation

$$\det \left(sI + \sum_{k=1}^{m} H_k e^{-s\tau_k} \right) = 0. \tag{2}$$

Equation (1) is asymptotically stable if and only if all the infinitely many roots of (2) are located in the open left half plane. As mentioned in the introductory section, it has been observed already by [14, 2], see also [9, 10, 21], that the stability can be destroyed by even infinitesimally small changes in the delays. This was the main motivation for defining the concept of *s*trong stability with respect to time delays. The difference equation (1) is *s*trongly stable, if it is asymptotically stable for given delay values and the stability is robust against delay changes. In [10] (Theorem 2.2 and Corollary 2.2), a criterion for determining strong stability is stated as follows:

Proposition 1 *Delay difference equation (1) is strongly stable if and only if*

$$\gamma_0 := \max_{\theta \in [0,\ 2\pi]^m} \ \rho\left(\sum_{k=1}^{m} H_k e^{-i\theta_k}\right) < 1, \tag{3}$$

where ρ denotes the spectral radius, i.e. the maximum modulus of the eigenvalues. The notation $\theta \in [0,\ 2\pi]^m$ means that θ is a vector of $\mathbb{R}^m$ whose entries θ_k, $k = 1, \ldots, m$ belong to the interval $[0,\ 2\pi]$.

Furthermore, if $\gamma_0 > 1$ then equation (1) is exponentially unstable for rationally independent[1] *delays.*

Notice that the quantity γ_0 does not depend on the value of the delays, i.e. exponential stability locally in the delays is equivalent with exponential stability globally in the delays [10]. In this sense, condition (3) can be interpreted as a robust stability condition where the uncertainties are complex parameters of unit magnitude, see [4].

Let us remark that by homogeneity, the expression of γ_0 can be simplified to

$$\gamma_0 = \max_{\theta \in [0,\ 2\pi]^{m-1}} \ \rho\left(\sum_{k=1}^{m-1} H_k e^{-i\theta_k} + H_m\right). \tag{4}$$

Here are some properties of the quantity γ_0, see [22, 21], for more details:

1. Stability of the difference equation (1) with rationally independent delays implies strong stability, and vice versa
2. In the case of one delay ($m = 1$),
   $$\gamma_0 = \rho(H_1).$$
3. In the case of a scalar equation ($n = 1$),
   $$\gamma_0 = \sum_{k=1}^{m} |H_k|.$$
4. A sufficient, but as a rule conservative, condition for strong stability is given by
   $$\sum_{k=1}^{m} \|H_k\| < 1$$
   where $\|.\|$ denotes the matrix Euclidean norm, i.e. the maximum singular value.

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[1] The m numbers $\tau = (\tau_1, \ldots, \tau_m)$ are rationally independent if and only if $\sum_{k=1}^{m} n_k \tau_k = 0,\ n_k \in \mathbb{Z}$ implies $n_k = 0,\ \forall k = 1, \ldots, m$. For instance, two delays τ_1 and τ_2 are rationally independent if their ratio is an irrational number.

Let us remark that the main motivation for assessing strong stability of the difference equation (1) comes from the stability analysis of neutral systems, e.g., in the following form

$$\frac{d}{dt}\left(x(t)+\sum_{k=1}^{m} H_k x(t-\tau_k)\right) = A_0\; x(t) + \sum_{j=1}^{p} A_j x(t-\vartheta_j), \tag{5}$$

where the left hand side of (1) is involved in the derivative term in (5). Analysing the behavior of the high frequency spectra of (1) and (5), it can be shown that the strong stability of equation (1) is a necessary condition for strong stability of neutral system (5), see e.g. [9, 21].

1.2 Computational issues

The problem of solving (3) can be formulated as an optimization task with the objective to find the global maximum of spectral radius over $\theta \in [0,\ 2\pi]^m$. However, in general the objective function $\rho(\theta)$ is nonconvex, i.e. it can have multiple local maxima. Besides, the function can be nonsmooth (e.g. at the points where the spectral radius is determined by more than either one single eigenvalue or a couple of complex conjugate eigenvalues). The fact that the function is nonsmooth precludes the use of standard optimization procedures. Instead, nonsmooth optimization methods can be used, such as gradient sampling, see [5, 25]. However, even though these methods can handle the problem of nonsmoothness, they converge to local extrema as a rule. As suboptimal solutions are not sufficient (the global maximum of the spectral radius is needed) a brute force method has been used to solve the task so far, see [21, 23, 28]. In the first step, each dimension of $[0,\ 2\pi]^m$ is discretized to N points. Then evaluation of (3) consists in solving N^m times $n \times n$ eigenvalue problems. Hence, the overall cost of one evaluation of γ_0 is $O\left(N^m n^3\right)$, see [28]. With the simplified expression (4), the computational costs reduces to $O\left(N^{m-1} n^3\right)$. Obviously, the complexity of the computation grows considerably with the number of delays in the difference equation. Moreover, the risk of missing global extrema due to sparse or inappropriate gridding cannot be avoided.

2 Strong stability and Hermite's condition

Consider the characteristic polynomial

$$p(z) = \det(z_0 I_n + \sum_{k=1}^{m} z_k H_k), \tag{6}$$

which is homogeneous of degree n in $m+1$ variables z_k, $k = 0, 1, \ldots, m$.

Based on (3), considering $z_k = e^{j\theta_k}, \theta_k \in [0, 2\pi], k = 1, .., m$, the difference equation (1) is strongly stable if and only if the univariate polynomial

$$z_0 \rightarrow p(z)$$

is discrete-time stable,i.e. it has all its roots in the open unit disk.

In order to deal with stability of this polynomial, we use a stability criterion based on the Hermite matrix. It is a Hermitian matrix of dimension n whose entries are quadratic in the coefficients of the polynomial. The Hermite matrix $z_1, \ldots, z_m \to H(z)$ is therefore a trigonometric polynomial matrix in m variables $z_1, \ldots, z_m$.

Derived by Charles Hermite in 1854, the Hermite matrix criterion is a symmetric version of the Routh-Hurwitz criterion for assessing stability of a polynomial. It says that a polynomial $p(z) = p_0 + p_1 z + \cdots + p_n z^n$ has all its roots in the open upper half of the complex plane if and only if its Hermite matrix $H(p)$ is positive definite. Note that $H(p)$ is n-by-n, Hermitian and quadratic in coefficients p_k, so that the above necessary and sufficient stability condition is a quadratic matrix inequality (QMI) in coefficient vector $p = [p_0 \, p_1 \, \cdots \, p_n]$.

The standard construction of the Hermite matrix goes through the notion of Bézoutian, a particular form of the resultant. A bivariate polynomial is constructed, from which a quadratic term is factored out, yielding a quadratic form shaped by the Hermite matrix. The construction is explained e.g. in [11] and references therein. See especially [18] which explains that a discrete-time Hermite matrix, sometimes called Schur-Cohn or Schur-Fujiwara matrix, can be obtained similarly. The discrete-time Hermite matrix is also quadratic in the p_k, and it is positive definite if and only if polynomial $p(z)$ has all its roots in the open unit disk.

Zdeněk Hurák pointed out that there is a much simpler construction of the Hermite matrix in the discrete-time case. The construction can be traced back to Issai Schur [26], and it is explained in [1]. Entrywise formulas are also described in [3, Theorem 3.13]. Let

$$S_1(p) = \begin{bmatrix} p_n & p_{n-1} & p_{n-2} & \\ 0 & p_n & p_{n-1} & \\ 0 & 0 & p_n & \\ & & & \ddots \end{bmatrix} \qquad S_2(p) = \begin{bmatrix} p_0 & p_1 & p_2 & \\ 0 & p_0 & p_2 & \\ 0 & 0 & p_0 & \\ & & & \ddots \end{bmatrix}$$

be n-by-n upper-right triangular Toeplitz matrices. Then

$$H(p) = S_1^T(p) S_1(p) - S_2^T(p) S_2(p).$$

Strong stability of the difference equation is hence equivalent to positive definiteness of the Hermite matrix of the univariate characteristic polynomial, which is a multivariate trigonometric polynomial matrix in $z_1, \ldots, z_m$. We express this constraint as

$$H(z_1, \ldots, z_m) \succ 0. \tag{7}$$

3 Positivity of trigonometric polynomials

As shown in the previous section, the key ingredient in our approach to strong stability of difference equation is assessing positivity of multivariate trigonometric polynomials. This topic has been subject to recent studies, and the recent monograph [7] is a good introduction focusing on signal processing applications.

In this section we start with a scalar multivariate trigonometric polynomial, formulate its positivity test as a minimization problem, describe an LMI hierarchy yielding an asymptotically converging monotonically increasing sequence of lower bounds. We also describe a hierarchy of eigenvalue problems (linear algebra, much simpler computationally that LMI methods) to generate a hierarchy of upper bounds.

Then we extend these results to matrix polynomials, and describe the hierarchy of LMI problems that must be solved to guarantee positivity of a trigonometric matrix polynomial at the price of solving a hierarchy of convex problems, the decision variables being entries of a Gram matrix yielding a sum-of-squares decomposition for the matrix polynomial.

3.1 Minimising trigonometric polynomials

A trigonometric polynomial has the form $h(z) = \sum_\alpha h_\alpha z^\alpha$ where integer vector $\alpha \in \mathbb{N}^n$ is a multi-index such that $z^\alpha = \prod_{i=1}^n z_i^{\alpha_i}$, complex vector $z \in \mathbb{C}^n$ contains indeterminates such that $z_i = e^{j\theta_i}$ for some $\theta \in [0, 2\pi]^n$, and complex numbers $h_\alpha \in \mathbb{C}$ are coefficients. We use the notation $z \in \mathbb{T}^n$ to capture the constraint that each variable $z_k \in \mathbb{C}$ belongs to the unit disk $\mathbb{T}$.

We consider real trigonometric polynomials such that $h(z) = h^*(z)$ where the star denotes complex conjugation. These are such that $\sum_\alpha h_\alpha z^\alpha = \sum_\alpha h^*_\alpha z^{-\alpha}$ and hence $h_\alpha = h^*_{-\alpha}$.

Since $h(z)$ maps $\mathbb{T}^n$ onto $\mathbb{R}$, we are interested in solving the problem

$$h_{\min} = \min_{z \in \mathbb{T}^n} h(z).$$

3.2 Hierarchy of lower bounds via SDP

In this section we construct a monotonically increasing sequence of lower bounds on $h_{\min}$ that converges asymptotically. Each bound can be computed by solving an LMI, a convex semidefinite programming (SDP) problem.

First note that

$$h_{\min} = \min_\mu \int_{\mathbb{T}^n} h(z) d\mu(z) \tag{8}$$

where the minimisation is over all probability measures defined on the sigma-algebra of the multidisk $\mathbb{T}^n$, see Chapter 5 in [16]. For an intuitive understanding of relation (8), consider a collection of distinct points $z_i \in \mathbb{T}^n$, $i = 1, \ldots, N$ and the linear programming problem

$$\begin{array}{ll} \min_w & \sum_{i=1}^N w_i h(z_i) \\ \text{s.t.} & \sum_{i=1}^N w_i = 1, \;\; w_i \geq 0, \;\; i = 1, \ldots, N \end{array}$$

whose objective function is a normalized weighted sum of evaluations of polynomial $h(z)$, and let N tend to infinity.

Let us express the polynomial $h(z)$ as a Hermitian quadratic form

$$h(z) = b_k^*(z) X_k b_k(z) \tag{9}$$

where $b_k(z)$ is a vector basis of trigonometric polynomials of degree up to k, e.g. containing monomials z^α, $\alpha \geq 0$, $\max_{i=1,\ldots,m} \alpha_i \leq k$. Matrix X_k is called the Gram matrix of polynomial $h(z)$ in basis $b_k(z)$.

As soon as k is fixed, finding a matrix $X_k \succeq 0$ satisfying (9) can be cast into an SDP feasibility problem which amounts to expressing polynomial $h(z)$ as a sum-of-squares (SOS) of trigonometric polynomials of degree k.

Now defining

$$\begin{array}{rll} \underline{h}_k = & \sup & \underline{h} \\ & \text{s.t.} & h(z) - \underline{h} = b_k^*(z) X_k b_k(z) \text{ for some } X_k \succeq 0 \end{array} \tag{10}$$

it follows by construction that $\underline{h}_k \leq \underline{h}_{k+1}$, i.e. the sequence of lower bounds is monotonically increasing.

Lemma 1 *The sequence of lower bounds converges asymptotically to the minimum of the polynomial, i.e.* $\lim_{k\to\infty} \underline{h}_k = h_{\min}$.

Proof: A result of functional analysis by M. Putinar, transposed to trigonometric polynomials [7, Theorems 3.5 and 4.11], states that $h(z) > 0$ if and only if there exists a finite integer d and a positive semidefinite Hermitian matrix $X_d \succeq 0$ such that (9) holds for $k = d$. Given an arbitrarily small $\epsilon > 0$, polynomial $h(z) - h_{\min} + \epsilon$ is strictly positive on $\mathbb{T}^n$, and hence for a sufficient high value of d it has a sum-of-squares representation $h(z) - h_{\min} + \epsilon = b_d^*(z) X_d b_d(z)$ for some Gram matrix $X_d \succeq 0$. Letting ϵ tend to zero and d tend to infinty, convergence of the sequence of lower bounds follows. Note that an alternative proof of Putinar's theorem for trigonometric polynomials, building on ideas of operator theory, can be found in [6], where it is shown that factorable trigonometric polynomials (i.e. those which can be expressed as sum-of-squares) are dense, and in particular, strictly positive polynomials are factorable. □

3.3 Hierarchy of upper bounds via EVP

In this section we show that we can construct a monotonically decreasing sequence of upper bounds on $h_{\min}$ that converges asymptotically. Each bound can be computed by solving an eigenvalue problem (EVP)

In problem (8) let us consider that measure μ is absolutely continuous w.r.t. measure ν, the probability measure supported uniformly on the multidisk. Let us further restrict the class of measures by considering that there exists a trigonometric polynomial $q_k(z) = \sum_{0\leq\alpha\leq k} q_{k\alpha} z^\alpha = q_k^* b_k(z)$ of total degree k such that $\mu_k(dz) = q_k^*(dz) q_k(dz) \nu(dz)$, with $\lim_{k\to\infty} \mu_k = \mu$ since $\mathbb{T}^n$ is compact. Let $y_\alpha = \int_{\mathbb{T}^n} z^\alpha d\nu(z)$ denote the moment of order α of ν. Finally, let us define

$$\overline{h}_k = \min_{\mu_k} \int_{\mathbb{T}^n} h(z) d\mu_k(z)$$

as an optimisation problem over this restricted class of measures.

With these notations

$$\int h(z)d\mu_k(z) = \int h(z)q_k^*(z)q_k(z)d\nu(z) =$$

$$= \int h(z)q_k^* b_k(z)b_k^*(z)q_k d\nu(z)$$

is the same as

$$q_k^* \left(\int h(z)b_k(z)b_k^*(z)d\nu(z) \right) q_k = q_k^* M_k(h\, y)q_k$$

where $M_k(h\, y)$ is called the localising matrix of order k of measure ν w.r.t. polynomial h, see [16]. Its rows and columns are indexed by multi-indices β and γ respectively, and its entry (β, γ) is equal to $\sum_\alpha h_\alpha y_{\alpha-\beta+\gamma}$. Therefore matrix $M_k(h\, y)$ can be obtained from the moments of ν, and hence it is given. It is positive definite.

If $h(z) = 1$, matrix $M_k(y)$ is called the moment matrix of order k of measure ν. Its entry (β, γ) is equal to $y_{-\beta+\gamma}$, and hence matrix $M_k(y)$ is given as well. Since μ_k is a probability measure

$$\int d\mu_k = \int q_k^* q_k d\nu_k = q_k^* M_k(y) q_k = 1$$

and hence

$$\begin{array}{rll} \overline{h}_k = & \min_{q_k} & q_k^* M_k(h\, y) q_k \\ & \text{s.t.} & q_k^* M_k(y) q_k = 1. \end{array}$$

It follows by construction that $\overline{h}_k \geq \overline{h}_{k+1}$, i.e. the sequence of upper bounds is monotonically decreasing.

Lemma 2 *The sequence of upper bounds converges asymptotically to the minimum of the polynomial, i.e.* $\lim_{k\to\infty} \overline{h}_k = h_{\min}$.

Proof: The polynomial $h(z) - h_{\min}$ is nonnegative on the compact set $\mathbb{T}^n$, and by [17, Theorem 3.2] – reformulated in the trigonometric case – this is equivalent to $M_k((h - h_{\min})\, y) \succeq 0$ for all k. Convergence then follows from [17, Theorem 4.1]. $\Box$

Finally, let us point out that given positive definite Hermitian matrices A and B, optimisation problem $\min_v v^* A v$ s.t. $v^* B v = 1$ can be solved via linear algebra. Indeed, let $\bar{z}$ denote an eigenvalue of the pencil $zB - A$, and let v denote the corresponding unit eigenvector. Then vector $\bar{v} = (v^* B v)^{-\frac{1}{2}} v$ is such that $\bar{v}^* B \bar{v} = 1$ and $\bar{v}^* A \bar{v} = \bar{z}$. Minimising this quantity then amounts to finding the minimum eigenvalue of pencil $zB - A$.

3.4 Polynomial matrices

The above results on scalar polynomials can be extended directly to polynomial matrices by considering a matrix basis instead of a vector basis to build the Hermitian matrix representation (9).

In the context of our strong stability analysis problem, the core idea is then to replace the (typically difficult) Hermite matrix positivity condition (7) with a hierarchy of tractable SDP problems. We write

$$\begin{array}{rll} \underline{h}_k = & \sup & \underline{h} \\ & \text{s.t.} & H(z) - \underline{h} = (b_k(z) \otimes I_n)^* X_k (b_k(z) \otimes I_n) \\ & & X_k \succeq 0 \end{array} \tag{11}$$

as an LMI relaxation of order k of positivity condition (7).

If $\underline{h}_k > 0$ for some k, then it implies that (7) is satisfied.

If $\underline{h}_k \leq 0$ for some k, then we cannot conclude directly, but we can try to extract from the dual (moment) SDP problem a certificate that indeed matrix $H(z)$ cannot be positive definite, see [13] even though the trigonometric polynomial matrix case is not developed in this reference. If we cannot extract useful information from the dual problem, we have to increase the value of k and solve the next LMI in the hierarchy.

4 Complexity

Let M denote the size of the Gram matrix X_k in SDP problem (11). If we use an interior-point method, the worst-case complexity of one Newton iteration for an SDP problem in a cone of that size is $O(M^6)$. Experiments reveals that the practical complexity is approximately $O(M^4)$.

The number of monomials of m variables of degree k in basis $b_k(z)$ is equal to $(k+1)^m$. Polynomial $p(z)$ has m variables and degree n so degree k in (9) should be such that $2k \geq n$. Note that we can have $2k > n$ since higher-degree terms may cancel in the right handside of equation (9).

If we choose $k = n/2$ or $k = (n+1)/2$ depending on whether n is even or not, in terms of complexity $M = O(n^{m+1})$. The overall complexity of our SDP approach to strong stability analysis therefore grows exponentially in the number of delays m, and polynomially in the number of states n. However the exponent of this polynomial growth is quite large. In comparison, the gridding approach mentioned at the beginning of the paper has a complexity which also grows exponentially in the number of delays, but the dependence on the number of states is only cubic. However, contrary to the SDP approach, the gridding approach does not provide guarantees.

Whereas it is clear that our approach scales badly with the number of delays, there is still the possibility of decreasing the overall computational burden by exploiting the problem structure. In particular, if the (complex) matrices H_k, $k = 1, \ldots, m$ can be simultaneously block-triangularized by a non-singular (complex) similarity transformation

$$U^{-1} H_k U = \begin{bmatrix} H_k^{11} & H_k^{12} \\ 0 & H_k^{22} \end{bmatrix}, \quad k = 1, \ldots, m$$

then the computation of the spectral radius in problem (3) boils down to the computation of two spectral radii of smaller matrices, i.e.

$$\rho(\sum_k H_k e^{-i\theta_k}) = \rho(\sum_k U^{-1}H_kUe^{-i\theta_k}) = \max\left(\rho(\sum_k H_k^{11}e^{-i\theta_k}),\ \rho(\sum_k H_k^{22}e^{-i\theta_k})\right)$$

and a similar reduction can then be attempted on each individual diagonal blocks. We could not find a readily available implementation of such a block-triangularization algorithm, and we leave this option for further research.

5 Examples

Preliminary numerical examples indicate that the EVP approach of paragraph 3.3 yields a sequence of bounds which converges slowly (sublinearly). This is why in this section we focus exclusively on the SDP approach of paragraph 3.2.

We implemented a collection of Matlab functions to manipulate trigonometric polynomials, Hermite matrices, and formulate SDP problems corresponding to positivity checks. The functions are available for download[2] and they provide the following functionalities:

- `sampledet.m` - given a collection of matrices H_k, $k = 0, \ldots, m$, this function computes the coefficients of the multivariate polynomial $p(z) = \det(H_0 + H_1z_1 + \cdots + H_mz_m)$; it proceeds by sampling and interpolation, as described in [12]
- `trigoherm.m` - computes the Hermite matrix of a homogenized multivariate polynomial; it uses the formula of [3, Theorem 3.13] adapted to complex coefficients
- `trigohermgram.m` - computes the SDP problem corresponding to the positivity test for a given Hermitian multivariate polynomial matrix; the SDP problem is given in SeDuMi's input format

$$\begin{array}{llcll} \min & c^Tx & \quad & \max & b^Ty \\ \text{s.t.} & Ax = b & & \text{s.t.} & z = c - A^Ty \\ & x \in K & & & z \in K \end{array}$$

  where $x \in \mathbb{R}^N$, $y \in \mathbb{R}^M$, and K is the cone of positive semidefinite matrices of size $S = \sqrt{N}$.

Some instrumental functions are also provided, namely `genmon.m` which generates powers of monomials and `locmon.m` which locates a monomial in a Gram matrix. Besides, the function `bfssde.m` is available to evaluate (3) by brute force, as explained in subsection 1.2

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[2] `homepages.laas.fr/henrion/software/trigopoly.tar.gz`

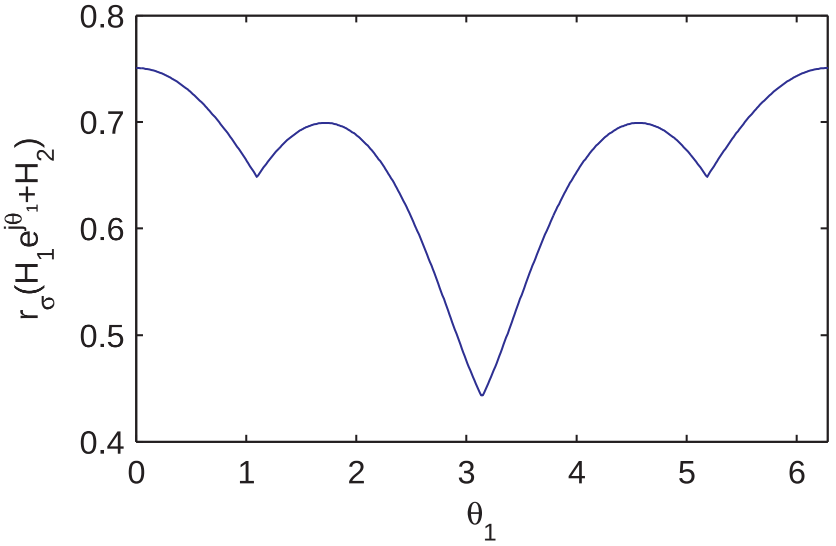

Figure 1: Spectral radius $\rho(\theta_1)$ for the example in Subsection 5.1

5.1 Three states, two delays

We adopt the illustrative example from [21] with $n = 3$, $m = 2$, where

$$H_1 = \begin{bmatrix} 0 & 0.2 & -0.4 \\ -0.5 & 0.3 & 0 \\ 0.2 & 0.7 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} -0.3 & -0.1 & 0 \\ 0 & 0.2 & 0 \\ 0.1 & 0 & 0.4 \end{bmatrix}$$

for which `bfssde.m` (with $N = 360$) provides $\gamma_0 = 0.7507$ in less then 0.1 seconds under Matlab 7.7 on our Linux PC equipped with Intel Xeon 2.67GHz CPU with 8GB RAM. On Fig. 1 shows the spectral radius as a function of θ_1.

The following Matlab script assesses stability of the corresponding difference equation by first building the determinantal polynomial, then the corresponding Hermite matrix, then the SDP problem, and eventually by solving the SDP problem with SeDuMi, a primal-dual interior-point solver:

```
H1=[0 0.2 -0.4;-0.5 0.3 0;0.2 0.7 0];
H2=[-0.3 -0.1 0;0 0.2 0;0.1 0 0.4];
p=sampledet({eye(3),H1,H2}); % evaluate determinant
p=p(:,abs(p(1,:))>1e-8); % remove small coefficients
H=trigoherm(p); % compute Hermite matrix
[A,b,c,K]=trigohermgram(H); % build SDP problem
[x,y,info]=sedumi(A,b,c,K); % solve SDP problem
```

The resulting SDP problem has size $N = 2304$, $M = 225$ and a positive semidefinite Gram matrix of size $S = 48$ is found after less than 0.1 seconds with SeDuMi 1.3.

We can also specify the strong stability radius γ_0, defined in (3), as a second input argument to function `trigoherm`. Internally, the polynomial is scaled appropriately and positivity of the Hermite matrix is assessed:

```
H=trigoherm(p,0.750);
[A,b,c,K]=trigohermgram(H);
[x,y,info]=sedumi(A,b,c,K);
```

With the above sequence the SDP problem is found feasible. Changing the first instruction to

```
H=trigoherm(p,0.751);
```

makes the resulting SDP problem infeasible, and this is certified by SeDuMi which returns a dual Farkas vector. As discussed at the end of paragraph 3.4, further analysis is required to conclude that indeed the Hermite matrix cannot be positive definite. We leave a comprehensive treatment of this case for further work.

5.2 Four states, three delays

We consider a system with $n = 4$, $m = 3$, where

```
H1=[-0.15 0 0.32 0;0 -0.07 0 0.05;
    0.08 0 0.04 0;0.2 0.03 0 -0.13];
H2=[-0.02 0.12 0 0.25;0 -0.05 0.04 0;
    0 0.23 0 -0.3;0.19 0 0.28 -0.09];
H3=[0 0 -0.03 0.14;0.01 -0.04 0 0;
    0 0 0.09 0.26; 0.05 -0.27 -0.06 0];
```

for which `bfssde.m` (with $N = 360$) provides $\gamma_0 = 0.6028$ in 4.5 seconds, see Fig. 2 with the distribution of the spectral radius with respect to values of θ_1 and θ_2.

The resulting SDP problem has size $N = 250000$, $M = 5840$ and a positive semidefinite Gram matrix of size $S = 500$ is found after approximately 6 minutes of CPU time, certifying that the spectral radius is less than one.

5.3 Four states, four delays

We conclude with an example with $n = 4$, $m = 4$ and the matrices

```
H1=[0.1 0 0 -0.2;pi/5 -0.1 0 -0.3;
    0 0 0.03 2;0 -exp(-1) 0 0.23]
H2=[0 0 0 0.0456;0 -0.33 0.11 0;
    0 1 0.2 0;0 -exp(-3) 0.176 0.73]
H3=[0.1 0.65 0 0.42;0.087 -pi/8 -0.1 0;
    0 -0.063 0 0.72;0.076 0.1 0 -0.23]
H4=[-0.678 0 0 -0.4;-0.0983 0 0 0;
    0 0.0763 0 0.2;-exp(-5) 0 0.36 0]
```

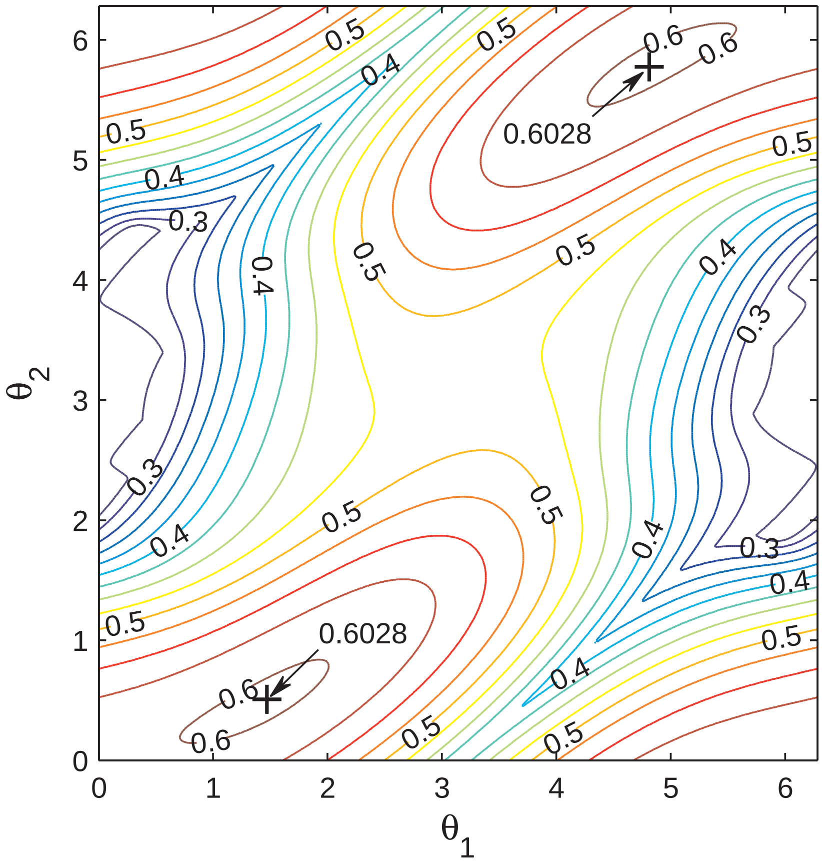

Figure 2: Spectral radius $\rho(\theta_1, \theta_2)$ for the example in Subsection 5.2

for which `bfssde.m` (with $N = 360$) provides $\gamma_0 = 1.7649$ in more than 30 minutes.

The resulting SDP problem has size $N = 6250000$, $M = 52496$ and a positive semidefinite Gram matrix of size $S = 2500$. This problem cannot not be solved on our computer, SeDuMi issues an out of memory error message. In this case, we may to try to exploit the problem structure (sparsity) to generate a smaller SDP problem, but this is out of the scope of this paper.

6 Conclusions

In the context of neutral time-delay systems, strong stability of difference equations is generally assessed numerically with a brute force gridding approach. A parallel can be draw with the μ-analysis approach to robustness of linear systems, see e.g. [29] where brute force gridding can yield misleading results and should be replaced, if possible, with more rigorous certificates of robustness.

In this paper, using the Hermite stability criterion for discrete-time polynomials the problem of assessing strong stability is reformulated as the problem of deciding positive definiteness of a trigonometric matrix polynomial of size equal to the state dimension and number of variables equal to the number of delays. This decision problem is hard, but it can be approached through a converging hierarchy of tractable semidefinite programming (SDP) or linear matrix inequality (LMI) relaxations.

As far as we know, the idea of using the Hermite matrix in this context of strong stability is new. In [4] the problem of assessing robust stability of a matrix is also approached with a converging hierarchy of LMI relaxations, but the LMI problems correspond to the search of a Lyapunov certificate of stability, with polynomial dependence on the uncertain parameters. In our approach, the role of a Lyapunov certificate is played by the Hermite matrix itself, and its positive definiteness is equivalent to strong stability.

Numerical experiments reveal that the approach is limited to small state dimension and a small number of delays, as expected. However, it seems to be possible to lower the overall computational burden and exploit the system structure (when present) by using algorithms of numerical linear algebra for simultaneous block triangularization of a collection of matrices.

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