Linear Matrix Inequalities for Robust Strictly Positive Real Design

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Abstract

A necessary and sufficient condition is proposed for the existence of a polynomial $p(s)$ such that the rational function $p(s)/q(s)$ is robustly strictly positive real when $q(s)$ is a given Hurwitz polynomial with polytopic uncertainty. It turns out that the whole set of candidates $p(s)$ is a convex subset of the cone of positive semidefinite matrices, resulting in a straightforward strictly positive real design algorithm based on linear matrix inequalities.

Keywords

Strictly positive real, Polynomial, Linear matrix inequalities.

1 Introduction

Strictly Positive Real (SPR) rational functions[3] play a central role in systems theory. To ensure asymptotic stability of an adaptive output error algorithm for identification, it is required that a certain transfer function be SPR [Anderson et al., 1990]. SPR functions are also closely related to the concept of passive system in nonlinear control, see e.g. [Marquez and Damaren, 1995] and the references therein. Considering uncertain systems unavoidably leads to the notion of robust SPR functions, i.e. functions whose SPR property is kept under the presence of uncertainty in the system parameters.

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[3]In a continuous-time setting, SPR rational functions are functions with strictly positive real part along the imaginary axis.

There exist several methods to analyze robust SPRness of systems with parameter uncertainty. Just to mention a few, an analysis based on Kharitonov's theorem was first proposed in [Dasgupta and Bhagwat, 1987]. Polytopic uncertainty affecting either the numerator or the denominator of the transfer function was then considered in [Šiljak, 1989]. Latest achievements on robust SPR analysis with polytopic uncertainty affecting both numerator and denominator are reported in [Xie and Fu, 1996], see also the references given there.

Robust SPR design methods have also been developed, generally as an extension of the analysis results. A necessary and sufficient phase restriction condition for the existence of a numerator polynomial $p(s)$ such that the rational function $p(s)/q(s)$ is SPR when denominator $q(s)$ belongs to a set of stable polynomials is given in [Anderson et al., 1990] but only in the discrete-time case. In the continuous-time case, necessary and sufficient conditions are known for $p(s)$ rational or $p(s)$ polynomial of degree less than or equal to 3. Sufficient conditions for the existence of a polynomial $p(s)$ were later on proposed in [Betser and Zeheb, 1993], based on Lyapunov equations and the Lefschetz-Kalman-Yakubovich lemma. Relationships between the SPR design problem (in the absence of uncertainty) and the polynomial, or algebraic approach to systems control [Kučera, 1979] were unveiled in [Marquez and Damaren, 1995]. Using Bézout identity and the Kalman-Yakubovich lemma, a parametrization of all candidate numerator polynomials $p(s)$ is described when denominator $q(s)$ is a fixed, given Hurwitz[4] polynomial without uncertainty. Recently, the Hermite-Biehler theorem and interlacing properties of roots of even and odd parts of polynomials $p(s)$ and $q(s)$ were used to derive a frequency domain approach to the robust SPR design problem [Marquez and Agathoklis, 1998]. The problem of existence of $p(s)$ is not raised however, and the method is essentially graphical and based on a trial-and-error scheme.

In this paper, we address the problem of robust SPR design in the case of polytopic uncertainty (probably the most general way of modeling uncertainty, since it encompasses the so-called interval uncertainty generally considered in the robust SPR literature [Betser and Zeheb, 1993, Marquez and Agathoklis, 1998]). We derive a necessary and sufficient condition for existence of a continuous-time numerator polynomial ensuring robust SPRness, thus solving the problem raised in [Anderson et al., 1990]. Using the notion of positivity of polynomials (see [Šiljak, 1971] for an algebraic treatment and [Šiljak and Šiljak, 1998] for extensions to uncertain polynomials) and recent results on cone duality [Nesterov, 2000], we show that the whole set of candidate numerator polynomials belongs to a convex set in the cone of positive semidefinite matrices. This set is described by a linear matrix inequality (LMI, see [Boyd et al., 1994]). Numerically efficient semidefinite programming software such as SeDuMi [Sturm, 1999], now widely used in systems control, can therefore be used to perform robust SPR design. In contrast with the rather involved approaches described in [Betser and Zeheb, 1993] or [Marquez and Agathoklis, 1998], our design algorithm can readily be implemented, resulting in a systematic, computationally efficient robust SPR design method.

[4] A polynomial is Hurwitz when all its roots lie in the open left half-plane.

2 Problem Statement

We are given a polynomial $q(s)$ subject to polytopic uncertainty, i.e. $q(s)$ belongs to the convex hull of a polytope with given vertices $q^1(s), \ldots, q^N(s)$ of degree n and can be written as

$$q(s) = \lambda^1 q^1(s) + \cdots + \lambda^N q^N(s)$$

where $\lambda^i > 0$ and $\lambda^1 + \cdots + \lambda^N = 1$.

Given polytopic polynomial $q(s)$, we are seeking a polynomial $p(s)$ of the same degree n such that $p(s)$ and $q(s)$ be coprime and the rational function

$$G(s) = \frac{p(s)}{q(s)}$$

be robustly SPR, i.e.

- $q(s)$ is robustly Hurwitz, and
- $\mathrm{Re}\, G(\mathrm{j}\omega) > 0$ for all real ω.

Let us first focus on the first requirement that denominator polynomial $q(s)$ be robustly Hurwitz, i.e. all its roots stay in the open left half-plane for all admissible values of λ_i. It turns out that checking robust stability of polynomials is a well-studied problem in systems control. The edge theorem states that checking robust stability of a whole polytope is equivalent to checking stability of its edges. This can be done efficiently with a polynomial-time algorithm [Sideris, 1991]. In the case of interval uncertainty (a special case of polytopic uncertainty) robust stability analysis can be carried out with Kharitonov's theorem and its variations [Anderson et al., 1987].

Assuming that the first condition for robust SPRness is satisfied, in the sequel we will focus on the second requirement that the real part of rational function $G(s)$ be positive along the imaginary axis.

3 Preliminaries

Robust Non-negativity

First we pursue the approach of [Stipanović and Šiljak, 2000] to convert the problem of robust SPRness of $G(s)$ into the problem of robust non-negativity of a polynomial.

Since we assume that denominator $q(s)$ is robustly Hurwitz, $q(\mathrm{j}\omega)$ never vanishes for any real ω and we can write

$$\mathrm{Re}\, G(\mathrm{j}\omega) = \frac{1}{2}\left(\frac{p(\mathrm{j}\omega)}{q(\mathrm{j}\omega)} + \frac{p(-\mathrm{j}\omega)}{q(-\mathrm{j}\omega)}\right) = \frac{1}{2}\frac{p(\mathrm{j}\omega)q(-\mathrm{j}\omega) + q(\mathrm{j}\omega)p(-\mathrm{j}\omega)}{q(\mathrm{j}\omega)q(-\mathrm{j}\omega)}.$$

The rational function $G(s)$ is robustly SPR as soon as

$$\mathrm{Re}\, G(\mathrm{j}\omega) \geq \varepsilon \tag{1}$$

for some (arbitrary small) strictly positive scalar ε. So we define the real polynomial

$$r(\omega) = \frac{1}{2}\left(p(\mathrm{j}\omega)q(-\mathrm{j}\omega) + q(\mathrm{j}\omega)p(-\mathrm{j}\omega)\right) - \varepsilon q(\mathrm{j}\omega)q(-\mathrm{j}\omega) \tag{2}$$

and we can state the following

Lemma 1 *Assuming polynomial $q(s)$ is robustly stable, rational function $G(s)$ is robustly SPR if and only if polynomial $r(\omega)$ is robustly non-negative for some (arbitrarily small) $\varepsilon > 0$, i.e. $r(\omega) \geq 0$ for all real w.*

Non-negative Polynomials

Second we recall a parametrization of non-negative polynomials obtained in [Nesterov, 2000] via cone duality, see also [Lasserre, 2001] for a proof based on the theory of moments. The idea behind this parametrization can be traced back to [Shor, 1987] and it is also related to Hilbert's 17th problem about the sum-of-squares decomposition of polynomials, see Chapter 4 in [Parrilo, 2000]. This parametrization was used recently in [Genin et al., 2000] for filter design.

The set of even degree polynomials that are non-negative along the real axis is a convex subset of the cone of positive semidefinite matrices, as captured by the following

Lemma 2 *The polynomial*

$$r(\omega) = r_0 + r_1\omega + \cdots + r_{2n}\omega^{2n}$$

satisfies $r(\omega) \geq 0$ for all real ω if and only if

$$r_k = \sum_{i+j=k} Q_{ij}, \quad k = 0, 1, \ldots, 2n \tag{3}$$

for some positive semidefinite symmetric matrix Q of size $n+1$ with entries Q_{ij}, i.e.

$$Q = Q^T = [Q_{ij}]_{i,j=0,1,\ldots,n} \succeq 0. \tag{4}$$

In equality (3) it is assumed that $Q_{ij} = 0$ when i or j are outside their definition range.

Proof: To prove the result, notice that relation (3) implies that

$$r(\omega) = [1 \quad \omega \quad \cdots \quad \omega^n] Q [1 \quad \omega \quad \cdots \quad \omega^n]^T$$

hence positive semidefiniteness of Q implies non-negativity of $r(\omega)$ for all ω. Conversely, the existence of Q for any non-negative polynomial $r(\omega)$ follows from the existence of a decomposition as a sum of squares $r(\omega) = \sum_k t^k(\omega)t^k(\omega)$ where $t^k(w) = t_0^k + t_1^k\omega + \cdots$ are polynomials of degree at most n. Then, the matrix Q with entries $Q_{ij} = \sum_k t_i^k t_j^k$ satisfies both linear equalities (3) and positive semidefinite inequality (4). □

Equalities (3) together with the positive semidefiniteness constraint (4) form a set of linear matrix inequalities (LMIs, see [Boyd et al., 1994]). Consequently, if the entries of $r(\omega)$ are linear in some design parameters, we can handle these parameters in a convex way over the cone of positive semidefinite matrices. This is precisely the idea pursued in the next section.

4 Main Result

One can check easily that odd power coefficients of polynomial $r(\omega)$ in equation (2) are zero. Even power coefficients are linked with the coefficients of $p(s)$ and $q(s)$ by the set of equalities

$$r_{2k} = \sum_{i+j=2k} (-1)^{k+i}(p_i q_j - \varepsilon q_i q_j), \quad k = 0, 1, \ldots, n$$

which can also be written as

$$\begin{bmatrix} r_0 \\ r_2 \\ r_4 \\ \vdots \\ r_{2n-2} \\ r_{2n} \end{bmatrix} = \begin{bmatrix} q_0 & 0 & 0 & \cdots & & & 0 \\ -q_2 & q_1 & -q_0 & & & & \\ q_4 & -q_3 & q_2 & & & & \\ \vdots & & & \ddots & & & \vdots \\ & & & & -q_n & q_{n-1} & -q_{n-2} \\ 0 & & & \cdots & 0 & 0 & q_n \end{bmatrix} \left(\begin{bmatrix} p_0 \\ p_1 \\ p_2 \\ \vdots \\ p_{n-1} \\ p_n \end{bmatrix} - \varepsilon \begin{bmatrix} q_0 \\ q_1 \\ q_2 \\ \vdots \\ q_{n-1} \\ q_n \end{bmatrix} \right).$$

The above matrix is square of size $n+1$. Its lower-right corner of size n is generally referred to as the Hurwitz matrix[5] of $q(s)$, see e.g. [Barnett, 1983]. It is a well-known result in linear system theory that this matrix is non-singular when $q(s)$ is Hurwitz. In this case, coefficient q_0 cannot be zero and the above linear system of equations has a (unique) solution vector $[p_0 \quad p_1 \ \cdots \ p_n]^T$ associated with any given left-hand side vector $[r_0 \quad r_2 \ \cdots \ r_{2n}]^T$.

We have shown in the preceding section that robust SPRness of $G(s)$ is equivalent to robust non-negativity of $r(\omega)$, provided $q(s)$ is robustly Hurwitz. We also proved that non-negativity of a polynomial is equivalent to feasibility of an LMI in the polynomial coefficients. Consequently, we can now derive easily the following main result.

Theorem 1 *Given a robustly Hurwitz uncertain polynomial $q(s)$ with polytopic uncertainty described by vertices $q^1(s), \ldots, q^N(s)$, there exists a polynomial $p(s)$ making rational function $G(s) = p(s)/q(s)$ robustly SPR if and only if for some (arbitrary small) $\varepsilon > 0$ the coefficients p_i of $p(s)$ satisfy the convex LMI*

$$\begin{array}{l} \sum_{i+j=2k}(-1)^{k+i}(p_i q_j^m - \varepsilon q_i^m q_j^m) = \sum_{i+j=2k} Q_{ij}^m, \quad k = 0, 1, \ldots, n \\ Q^m = (Q^m)^T = [Q_{ij}^m]_{i,j=0,1,\ldots,n} \succeq 0, \quad m = 1, \ldots, N. \end{array} \tag{5}$$

[5]Up to some row or column sign changes.

Proof: Invoking lemma 2, for each vertex index m, the above inequalities (5) ensure that each polynomial $r^m(\omega) = \frac{1}{2}(p(\mathrm{j}\omega)q^m(-\mathrm{j}\omega) + q^m(\mathrm{j}\omega)p(-\mathrm{j}\omega)) - \varepsilon q^m(\mathrm{j}\omega)q^m(-\mathrm{j}\omega)$ is non-negative for all real ω. Using matrix notation and Schur complement [Boyd et al., 1994] non-negativity of $r^m(\omega)$ is equivalent to positive semidefiniteness of the 2×2 Hermitian matrix

$$R^m(\omega) = \left[\begin{array}{cc} \frac{1}{2}(p(\mathrm{j}\omega)q^m(-\mathrm{j}\omega) + q^m(\mathrm{j}\omega)p(-\mathrm{j}\omega)) & \varepsilon q^m(\mathrm{j}\omega) \\ \varepsilon q^m(-\mathrm{j}\omega) & \varepsilon \end{array} \right].$$

Since matrix $R^m(\omega)$ is linear in coefficients of $q^m(s)$, this implies that matrix $R(\omega) = \lambda^1 R^1(\omega) + \cdots + \lambda^N R^N(\omega)$ is positive semidefinite as well, and hence that the uncertain polynomial $r(\omega) = \lambda^1 r^1(\omega) + \cdots + \lambda^N r^N(\omega)$ is non-negative for all admissible values of λ_i. Together with robust Hurwitzness of $q(s)$, this implies robust SPRness of $G(s)$, in virtue of lemma 1. Conversely, if $G(s)$ is robustly SPR, then there necessarily exist positive semidefinite matrices Q^m that ensure non-negativity of each vertex polynomial $r^m(\omega)$. $\Box$

Theorem 1 provides a necessary and sufficient condition of existence of a numerator polynomial $p(s)$ such that $G(s) = p(s)/q(s)$. The whole set of candidate polynomials $p(s)$ is described in a convex way by LMI (5). Efficient interior-point methods are available for solving LMIs, so the conditions are constructive, as illustrated by the numerical examples in the next section.

Note that the designer can either (a) set ε to some positive value from the outset, and then solve LMI (5) to check whether inequality (1) holds; or (b) let $\varepsilon > 0$ be a design parameter – since it appears linearly in LMI (5) – that can be maximized for example.

Moreover, one can freely incorporate additional constraints on coefficients p_i of $p(s)$, as soon as they can be formulated as LMIs. For example, one can seek a monic polynomial $p(s)$ when enforcing $p_n = 1$.

5 Numerical Examples

No uncertainty

We consider the Hurwitz polynomial

$$\begin{aligned} q(s) \;=\; & 463 + 5165s + 26917s^2 + 87007s^3 + 195260s^4 + \\ & 322607s^5 + 405929s^6 + 396806s^7 + 304551s^8 + \\ & 184140s^9 + 87420s^{10} + 32245s^{11} + 9059s^{12} + \\ & 1874s^{13} + 269s^{14} + 24s^{15} + s^{16} \end{aligned}$$

of degree 16 and we want to find a monic polynomial $p(s)$ of the same degree such that the rational function $G(s) = p(s)/q(s)$ be SPR.

We set $\epsilon = 0.5$. Using the software SeDuMi [Sturm, 1999] we obtain after about 1 second of CPU time on a Sun Sparc 5 workstation the monic polynomial

$$\begin{array}{rcl} p(s) & = & 538.277 + 5223.17s + 24896.3s^2 + 76594.8s^3 + 168888s^4 + \\ & & 281275s^5 + 364292s^6 + 372861s^7 + 303816s^8 + \\ & & 197128s^9 + 101181s^{10} + 40500.0s^{11} + 12339.8s^{12} + \\ & & 2751.37s^{13} + 418.352s^{14} + 36.8136s^{15} + s^{16} \end{array}$$

whose coefficients solve LMI (5). One can check on figure 1 that

$$\begin{array}{l} \mathrm{Re}\, G(0) \approx 1.16 \\ \max_\omega \mathrm{Re}\, G(\mathrm{j}\omega) \approx 2.25 \quad \text{for} \quad |\omega| \approx 6.75 \\ \min_\omega \mathrm{Re}\, G(\mathrm{j}\omega) \approx 0.51 \geq \varepsilon \quad \text{for} \quad |\omega| \approx 0.98 \\ \lim_{|\omega| \to \infty} \mathrm{Re}\, G(\mathrm{j}\omega) = 1 \end{array}$$

so that $G(s)$ is indeed strictly positive real.

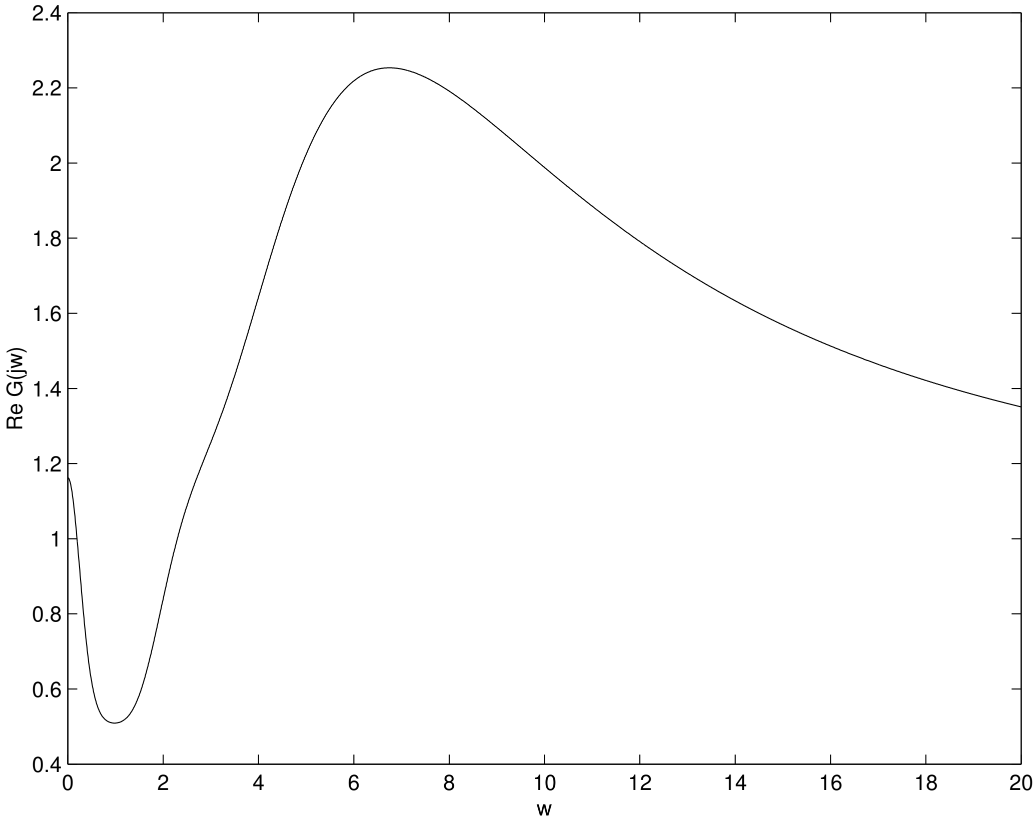

Figure 1: Real part of $G(j\omega)$ as a function of ω.

Interval Uncertainty

Now consider the interval polynomial studied in [Betser and Zeheb, 1993]

$$q(s) = [48, 50] + [80, 88]s + [42, 56]s^2 + [11, 13]s^3 + s^4.$$

Using a simplified version of Kharitonov's theorem [Anderson et al., 1987], we can check that $q(s)$ is robustly Hurwitz. Then we seek a fixed polynomial $p(s)$ of degree 4 such that $G(s) = p(s)/q(s)$ is robustly SPR, i.e. SPR for all possible values of $q(s)$ in the uncertainty range.

We build a polytope with $2^4 = 16$ vertices $q^1(s), \ldots, q^{16}(s)$ and we solve LMI problem (5) where we set $\varepsilon = 1$. With the SeDuMi software we obtain, after slightly more than 1 second of CPU time on a Sun Sparc 5 workstation, the polynomial

$$p(s) = 56.6138 + 135.368s + 121.633s^2 + 42.3733s^3 + s^4$$

that ensures robust SPRness of all the vertex rational functions $G^m(s) = p(s)/q^m(s)$ for all $m = 1, \ldots, 16$. On figure 2 we can check that the family of plants $G^m(s)$ is

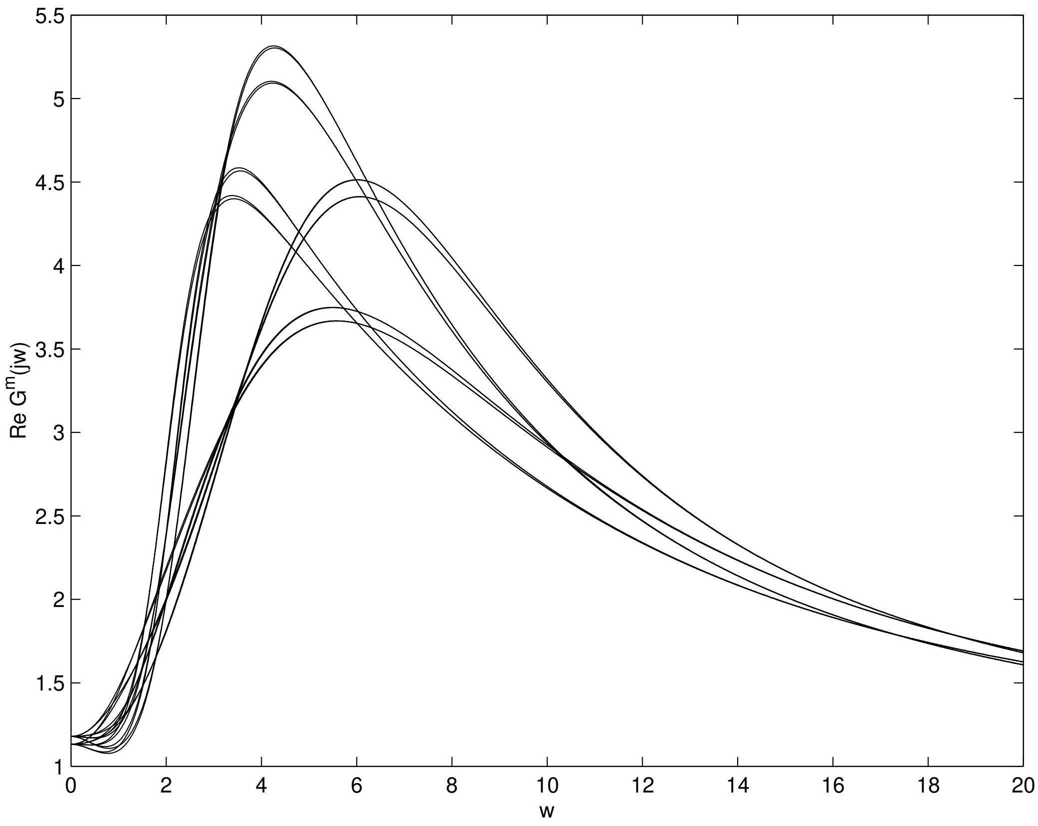

Figure 2: Real parts of $G^m(\mathrm{j}\omega)$ for $m = 1, \ldots, 16$ as functions of ω

indeed robustly SPR. The minimum value of Re $G(\mathrm{j}\omega)$ for all admissible uncertainty is approximately equal to $1.08 \geq \varepsilon$ for $\omega \approx 0.78$ and it is attained by the vertex polynomial $50 + 88s + 42s^2 + 11s^3 + s^4$.

6 Conclusion

We have proposed a straightforward numerical method to design a polynomial $p(s)$ such that the rational function $p(s)/q(s)$ is robustly strictly positive real for all admissible values of uncertain Hurwitz polynomial $q(s)$. The approach is based on semidefinite programming, or optimization over linear matrix inequalities, for which several powerful software are available now.

Only real coefficient polynomials are studied in this paper, but complex coefficients can be handled similarly. Since the set of candidates $p(s)$ is described by a matrix inequality which is linear in the design coefficients, one can readily incorporate additional convex design constraints using this formulation. One can also think of optimizing some criterion subject to the design constraints. Moreover, the approach is not limited to continuous-time systems and SPRness along the imaginary axis. Any one-dimensional stability boundary (circle, line, segment etc.) can be handled in a similar fashion. Extensions of this work to polynomial matrices can also be considered.

Finally, recall that numerical methods involving polynomials expressed in the monomial base $\{1, s, s^2, \ldots, s^n\}$ generally feature poor numerical stability for sufficiently high values of n, see the concluding remarks in [Genin et al., 2000]. Indeed, Hankel matrices such as those appearing implicitely in LMI problem (5) are always badly conditioned. Therefore it should be insightful to study the numerical improvements brought by alternative polynomial basis, such as Chebyshev polynomials for example.

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