A review of the book "Real Solutions to Equations from Geometry" by Frank Sottile, University Lecture Series, Vol. 57, American Mathematical Society, Providence, Rhode Island, 2011.

The book reports on latest achievements in real algebraic geometry, and namely on upper and lower bounds on the number of real solutions of systems of polynomial equations. The focus in on equations coming from toric varieties and Grassmannians, and on some geometric problems with all solutions real.

The book is structured as follows. The main results described in the book are summarized in the overview Chapter 1, which contains also some motivational and historical material, as well as a synthetic description of a few problems from geometry and the Boris and Michael Shapiro conjecture on the Wronski map from the real Grassmannian to real projective space. Chapter 2 reviews classical results on real roots of univariate polynomials, namely Descartes's bounds on the number of positive solutions, Descartes's rule of signs, Sturm's theorem on real root counting and the Budan-Fourier theorem giving a bound on the number of roots in an interval. The remainder of the book deals with extensions of the results of Chapter 2 to the multivariate case, which is significantly harder. Chapters 3 to 6 deal with upper bounds, whereas Chapters 7 and 8 deal with lower bounds. Chapter 9 describes problems from enumerative real geometry. The final Chapters 10 to 14 focus on more advanced and recent material on the Shapiro conjecture and its ramifications.

Chapters 3 and 4 describe the polyhedral bounds of Kushnirenko and Bernstein, which depend on the geometry of the support of the polynomial, i.e. the set of exponents of monomials with nonzero coefficients. Chapter 5 describes Khovanskii's fewnomial bound which is a multivariate extension of Descartes's bound. It provides an upper bound on the number of positive solutions of a system of N polynomial equations in N variables with a fixed (presumably small) number of nonzero monomials. The proof is topological in nature, and the bound is not sharp. Attempts to derive sharper bounds, and to generalize Descartes's rule of signs to the multivariate case, are described in the delightful section 5.2, and especially example 5.6 (2 equations of degree 6 in 2 variables with 5 positive real solutions) showing how difficult it is to find systems with many real solutions. Chapter 6 collects improvements of Khovanskii's bound exploiting Gale duality.

Chapter 7 describe lower bounds for sparse polynomial systems, or said differently, results that guarantee the existence of real solutions of systems of polynomial equations. Applications to special polynomial systems constructed from partially ordered sets or geometric problems of Schubert calculus are then described in Chapter 8.

Chapter 9 describes a collection of problems from enumerative geometry, the science (or art) of counting geometric figures subject to given geometric constraints. These problems are sources of examples of polynomial systems of equations with all solutions real: e.g. there are 3264 real plane conics tangent to 5 given conics in general position. Schubert calculus, involving linear spaces meeting other linear spaces, is briefly touched upon in Section 9.3.

Readers interested in more advanced material, and results revolving around the Michael and Boris Shapiro conjecture, will find an updated account in the final Chapters 10 to 14. Research efforts along these lines cover a broad range of mathematical techniques, including for example a study of maximally inflected curves (curves with all of its flexes real), see Section 13.2.

Generally speaking, the book is written in a technical, but very accessible way. The author is an internationally recognized expert in the field, and I am convinced that his book can be a source of inspiration for newcomers to real algebraic geometry, as well as a timely update of latest results for more experienced scholars. I am particularly impressed by the author's ability to convey visually some technical ideas with the help of splendid computer-generated figures. His book offers a fresh, visual and colorful approach to real algebraic geometry.

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