A review of "Optimization of Polynomials in Non-Commutating Variables" by Sabine Burgdorf, Igor Klep and Janez Povh. Springer Brief in Matematics, Springer, 2016.

This is a very timely contribution to the already vast literature on polynomial optimization, see [J. B. Lasserre. Moments, positive polynomials and their applications. Imperial College Press Optimization Series, Imperial College Press, London, 2010] for a comprehensive treatment, and [J. B. Lasserre. An introduction to polynomial and semi-algebraic optimization. Cambridge Texts in Applied Mathematics. Cambridge University Press, Cambridge, 2015] for an introduction. The focus of the book under review is on non-commutative (NC) polynomial optimization, where polynomials have matrix unknowns. These matrix polynomials arise in many areas (e.g. control theory, quantum information theory, statistical physics). They depend only on the system structure, they do not change with the size of the matrices involved, and for this reason they are sometimes referred to as dimension-free. More generally, we speak of free analysis and free real algebraic geometry.

The book covers the basics of NC polynomial optimization, building on elementary material on algebra and analysis as well as on more advanced concepts such as the Gelfand-Naimark-Segal construction. On the one hand, readers already familiar with commutative polynomial optimization will find it very useful to draw parallels between commutative results and their NC counterparts. They will recognize standard material such as moment and localizing matrices, or the flat extension results by Curto and Fialkow. On the other hand, readers who are not (yet) familiar with commutative polynomial optimization will benefit from the condensed yet elementary treatment of the book which focuses on the essential results. Actually, it turns out that most of the NC results have clean and simple statements, while their commutative counterparts are somehow harder, or at least more subtle. For example, Helton's sums of squares theorem states that a NC polynomial is positive semidefinite if and only if it is a sum of Hermitian squares, whereas it has been known since Hilbert that the commutative version fails, namely they are polynomials which are non-negative yet not sums of squares of polynomials.

This short yet very accessible book written by three leading experts in the field contains many examples, including explicit reproducible computations with their open source Matlab toolbox NCSOStools. It can be recommended to young researchers with an interest in polynomial optimization (commutative or not), but also to more experienced readers who want to get acquainted quickly with the essential ideas and may be willing to extend the application domains of this currently fast growing branch of mathematical optimization.

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