On semidefinite representations of plane quartics

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Abstract

This note focuses on the problem of representing convex sets as projections of the cone of positive semidefinite matrices, in the particular case of sets generated by bivariate polynomials of degree four. Conditions are given for the convex hull of a plane quartic to be exactly semidefinite representable with at most 12 lifting variables. If the quartic is rationally parametrizable, an exact semidefinite representation with 2 lifting variables can be obtained. Various numerical examples illustrate the techniques and suggest further research directions.

Keywords

semidefinite programming; polynomials; algebraic plane curves

1 Introduction

Following the pioneering work of Nesterov and Nemirovskii [10], it is known that a significant collection of convex sets are representable as linear sections or projections of the cone of positive semidefinite matrices (or semidefinite cone for short). A classification has been proposed in [1], and advanced optimization modeling software such as YALMIP [9] exploits this knowledge to generate semidefinite programming problems from general convex programming problems, thus allowing the use of powerful interior-point algorithms based on self-concordant barrier functions [10]. The dictionnary proposed in [1] is far from being comprehensive, however, and further efforts are required to identify convex sets which can be efficiently represented via the semidefinite cone. This note focuses on the particular case of the convex hull of algebraic plane curves of degree four.

Let $x \in \mathbb{R}^2 \mapsto p(x)$ be a bivariate polynomial of degree 4, a quartic, and let

$$\mathcal{C} = \{x \in \mathbb{R}^2 : p(x) = 0\} \tag{1}$$

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be the corresponding algebraic curve. Alternatively, we can use the homogenization $\mathbf{x} \in \mathbb{P}^2 \mapsto \mathbf{p}(\mathbf{x})$, defined in the projective plane $\mathbb{P}^2$, and such that $\mathbf{p}(\mathbf{x}) = \mathbf{x}_0^{\deg p} p(\mathbf{x}_1/\mathbf{x}_0, \mathbf{x}_2/\mathbf{x}_0)$ and $p(x) = \mathbf{p}(1, x_1, x_2)$. Algebraic curve (2) can be defined equivalently in the projective plane as

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{P}^2 : \mathbf{p}(\mathbf{x}) = 0\}.$$

See the appendix for more information on algebraic sets in the projective plane.

Let

$$\mathcal{P} = \operatorname{conv} \mathcal{C} \tag{2}$$

denote the convex hull of curve $\mathcal{C}$, a semi-algebraic set. We assume that $\mathcal{P}$ has a non-empty interior, and we denote by $\partial\mathcal{P}$ the boundary of $\mathcal{P}$. For notational convenience, we also assume that the sign of $p(x)$ is such that the semialgebraic set $\{\mathbf{x} \in \mathbb{P}^2 : \mathbf{p}(\mathbf{x}) \geq 0\}$ is included in $\mathcal{P}$.

As stated above, we are interested in modeling $\mathcal{P}$ as a projection of a linear section of the semidefinite cone:

$$\{\mathbf{x} \in \mathbb{P}^2 : \exists \mathbf{y} \in \mathbb{P}^m : \sum_{i=0}^{2} F_i \mathbf{x}_i + \sum_{j=0}^{m} G_j \mathbf{y}_j \succeq 0\}$$

where F_i, G_j are real symmetric matrices of size n to be found, and $\succeq 0$ means positive semidefinite. Following the terminology of [1], we are seeking a semidefinite representation of $\mathcal{P}$. The $\mathbf{y}_j$ are called lifting variables, or liftings for short, and the set is obtained by projecting the feasible set of a linear matrix inequality (LMI).

Note that lifting variables are necessary in the above description, already in the particular case when $\{\mathbf{x} \in \mathcal{P}^2 : \mathbf{p}(\mathbf{x}) \geq 0\}$ is a convex set. Indeed, a generic line passing through set $\mathcal{P}$ cuts its boundary $\mathcal{C}$ at real points only twice, which is less than the degree of $\mathbf{p}$, and this implies that $\mathcal{P}$ is not rigidly convex in the sense of [5]. It follows that $\mathcal{P}$ cannot be represented without lifting variables ($m = 0$), or, equivalently, as a linear section of the semidefinite cone.

2 Primal moment approximations

In [7], Jean-Bernard Lasserre proposed a hierarchy of semidefinite, or LMI relaxations for polynomial optimization, based on the theory of moments and the dual representation of non-negative polynomials as sum-of-squares (SOS). In particular [8], Lasserre's relaxation of order k for set $\mathcal{P}$ is given by

$$\mathcal{P}_k = \{\mathbf{x} \in \mathbb{P}^2 : \exists \mathbf{y} \in \mathbb{P}^m : \mathbf{y}_{00} = \mathbf{x}_0, \mathbf{y}_{10} = \mathbf{x}_1, \mathbf{y}_{01} = \mathbf{x}_2, M_k(\mathbf{y}) \succeq 0, M_{k-2}(\mathbf{p}\mathbf{y}) = 0\}$$

with

$$\mathbf{y} = [\mathbf{y}_{00}\ \mathbf{y}_{10}\ \mathbf{y}_{01}\ \mathbf{y}_{20}\ \mathbf{y}_{11}\ \mathbf{y}_{02}\ \mathbf{y}_{30} \cdots \mathbf{y}_{0,2k}] \in \mathbb{P}^m$$

a (truncated) vector of bivariate moments with $m+1 = (k+1)(2k+1)$ entries,

$$M_k(\mathbf{y}) = \begin{bmatrix} \mathbf{y}_{00} & \mathbf{y}_{10} & \mathbf{y}_{01} & \mathbf{y}_{20} & & \\ \mathbf{y}_{10} & \mathbf{y}_{20} & \mathbf{y}_{11} & \mathbf{y}_{30} & & \\ \mathbf{y}_{01} & \mathbf{y}_{11} & \mathbf{y}_{02} & \mathbf{y}_{21} & & \\ \mathbf{y}_{20} & \mathbf{y}_{30} & \mathbf{y}_{21} & \mathbf{y}_{40} & & \\ & & & & \ddots & \\ & & & & & \mathbf{y}_{0,2k} \end{bmatrix}$$

a (truncated) moment matrix, and $M_k(\mathbf{py})$ a (truncated) localising matrix. For example, if $p(x) = 1 - 2x_1^3 x_2$ we have $\mathbf{p}(\mathbf{x}) = \mathbf{x}_0^4 - 2\mathbf{x}_1^3\mathbf{x}_2$ and

$$M_k(\mathbf{py}) = \begin{bmatrix} \mathbf{y}_{00} - 2\mathbf{y}_{31} & \mathbf{y}_{10} - 2\mathbf{y}_{41} & \mathbf{y}_{01} - 2\mathbf{y}_{32} & & \\ \mathbf{y}_{10} - 2\mathbf{y}_{41} & \mathbf{y}_{20} - 2\mathbf{y}_{51} & \mathbf{y}_{11} - 2\mathbf{y}_{42} & & \\ \mathbf{y}_{01} - 2\mathbf{y}_{32} & \mathbf{y}_{11} - 2\mathbf{y}_{42} & \mathbf{y}_{02} - 2\mathbf{y}_{33} & & \\ & & & \ddots & \\ & & & & \mathbf{y}_{0,2k} - 2\mathbf{y}_{3,1+2k} \end{bmatrix}.$$

In particular if $k = 2$ (the first relaxation in the hierarchy for quartic sets), the localising matrix is the scalar linear form $M_0(\mathbf{py}) = \mathbf{y}_{00} - 2\mathbf{y}_{31}$ obtained by lifting monomials of p.

Note also that by symmetry there are redundant constraints in the equation $M_{k-2}(\mathbf{py}) = 0$ arising in the definition of $\mathcal{P}_k$. For notational convenience, we however stick with this matrix notation.

By definition, $\mathcal{P} \subset \cdots \subset \mathcal{P}_3 \subset \mathcal{P}_2$, and hence we have a hierarchy of embedded outer semidefinite representable approximations for $\mathcal{P}$.

3 Dual SOS approximations

Let

$$\mathcal{F} = \{\mathbf{f} \in \mathbb{P}^2 \,:\, \mathbf{f}(\mathbf{x}) = \mathbf{f}_0\mathbf{x}_0 + \mathbf{f}_1\mathbf{x}_1 + \mathbf{f}_2\mathbf{x}_2 \geq 0 \;\forall \mathbf{x} \in \mathcal{P}\}$$

denote the dual cone of $\mathcal{P}$ in the sense that if $\mathbf{f} \in \mathcal{F}$ then the half-plane $\{\mathbf{x} : \mathbf{f}(\mathbf{x}) \geq 0\}$ contains $\mathcal{P}$. It follows that either $\mathbf{x} \in \mathcal{P}$ or there exists $\mathbf{f} \in \mathcal{F}$ such that $\mathbf{f}(\mathbf{x}) < 0$. Since $\mathcal{P}$ has a non-empty interior, these are mutually exclusive statements. A geometric interpretation is that $\mathcal{P}$ is the intersection of the (generally infinite number of) half-planes generated by all separating elements, i.e. $\mathcal{P} = \{\mathbf{x} \in \mathbb{P}^2 \,:\, \mathbf{f}(\mathbf{x}) \geq 0 \;\forall \mathbf{f} \in \mathcal{F}\}$. Set $\mathcal{F}$ admits the following equivalent representation, dehomogenized with respect to x:

$$\mathcal{F} = \{\mathbf{f} \in \mathbb{P}^2 \,:\, f(x) = \mathbf{f}_0 + \mathbf{f}_1 x_1 + \mathbf{f}_2 x_2 \geq 0 \;\forall x \in \mathcal{P}\}.$$

Let $\mathbb{R}[x]_k$ denote the set of polynomials of degree at most k and $\mathbb{S}[x]_{2k}$ the set of polynomials that can be written as sums of squares (SOS) of polynomials of $\mathbb{R}[x]_k$. Let

$$\mathcal{F}_k = \{\mathbf{f} \in \mathbb{P}^2 \,:\, f(x) = s_0(x) + s_1(x)p(x),\, s_0(x) \in \mathbb{S}[x]_{2k},\, s_1(x) \in \mathbb{R}[x]_{2(k-2)}\}. \tag{3}$$

Since SOS polynomials are only a subset of non-negative polynomials [12], it holds $\mathcal{F} \supset \cdots \supset \mathcal{F}_3 \supset \mathcal{F}_2$.

Lemma 1 $\mathcal{P}_k = \{x \in \mathbb{R}^2 \,:\, f(x) \geq 0 \;\forall f \in \mathcal{F}_k\}$.

Proof: The condition that an element $x^* \in \mathbb{R}^2$, or, equivalently, an element $\mathbf{x}^* \in \mathbb{P}^2$, belongs to $\mathcal{P}_k$ is the existence of a vector $\mathbf{y}$ satisfying the primal semidefinite constraints

$$\begin{array}{l} M_k(\mathbf{y}) = \sum_\alpha A_\alpha \mathbf{y}\alpha \succeq 0 \\ M_{k-2}(\mathbf{py}) = \sum_\alpha B_\alpha \mathbf{y}_\alpha = 0 \\ \mathbf{y}_{00} = \mathbf{x}_0^*,\, \mathbf{y}_{10} = \mathbf{x}_1^*,\, \mathbf{y}_{01} = \mathbf{x}_2^*. \end{array} \tag{4}$$

Given two symmetric matrices A and X of the same size, define the inner product $\langle A, X\rangle = \text{trace}(AX)$. Build the Lagrangian $L(\mathbf{y}, X, \mathbf{f}) = -\langle M_k(\mathbf{y}), X_0\rangle - \langle M_{k-2}(\mathbf{py}), X_1\rangle + \mathbf{f}_0(\mathbf{y}_{00} - \mathbf{x}_0^*) + \mathbf{f}_1(\mathbf{y}_{10} - \mathbf{x}_1^*) + \mathbf{f}_2(\mathbf{y}_{01} - \mathbf{x}_2^*)$ with $X_0 \succeq 0$ and the corresponding dual function $\inf_{\mathbf{y}} L(\mathbf{y}, X, \mathbf{f})$ to be maximized. Gathering the terms depending on $\mathbf{y}$, the semidefinite problem dual to (4) consists in finding X and $\mathbf{f}$ maximizing the linear function $-\mathbf{f}(\mathbf{x}^*)$ subject to

$$\begin{array}{l} \langle A_{00}, X_0\rangle + \langle B_{00}, X_1\rangle = \mathbf{f}_0 \\ \langle A_{10}, X_0\rangle + \langle B_{10}, X_1\rangle = \mathbf{f}_1 \\ \langle A_{01}, X_0\rangle + \langle B_{01}, X_1\rangle = \mathbf{f}_2 \\ \langle A_\alpha, X_0\rangle + \langle B_\alpha, X_1\rangle = 0,\ |\alpha| > 1 \\ X_0 \succeq 0. \end{array} \tag{5}$$

Notice that feasibility of dual problem (5) amounts to the existence of an SOS representation for the affine expression $f(x) = s_0(x) + s_1(x)p(x)$ with $s_0(x) \in \mathbb{S}[x]_{2k}$ and $s_1(x) \in \mathbb{R}[x]_{2(k-2)}$, with respective Gram matrices X_0 and X_1, see e.g. [7].

Weak duality [2, Section 5.8.1] informs us that the optimal value of the dual objective function $-\mathbf{f}(\mathbf{x}^*)$ s.t. constraints (5) is always greater than or equal to zero. If this value is strictly positive, i.e. if there exists an $\mathbf{f}$ such that $\mathbf{f}(\mathbf{x}^*) < 0$, then primal problem (4) is infeasible. In turn, this implies that $\mathbf{x}^* \notin \mathcal{P}_2$. Conversely, if $\mathbf{x}^* \in \mathcal{P}_2$ then for all $\mathbf{f}$ feasible for problem (5), i.e. for all $\mathbf{f} \in \mathcal{F}_k$, it holds $\mathbf{f}(\mathbf{x}^*) \geq 0$. $\square$.

4 Exactness

Given any element $\mathbf{f} \in \mathcal{F}$, define the quartic

$$p_f(x) = f(x) - p(x).$$

Lemma 2 *The first relaxation is exact, i.e. $\mathcal{P}$ is semidefinite representable as $\mathcal{P}_2$ (with at most 12 lifting variables) if and only if, for all $\mathbf{f} \in \mathcal{F}$, $p_f(x) \geq 0$ for all x.*

Proof: Given any $\mathbf{f} \in \mathcal{F}$, the inequality $p_f(x) \geq 0$ implies that $p_f(x) = s_0(x) \in \mathbb{S}[x]_4$, since bivariate quartics are non-negative if and only if they are polynomial SOS [12]. Since $\mathbf{f} \in \mathbb{P}^2$ by homogeneity we can choose $s_1(x) = 1$ (without loss of generality) such that $f(x) = s_0(x) + s_1(x)p(x)$, and thus $\mathbf{f} \in \mathcal{F}_2$. Therefore $\mathcal{F} = \mathcal{F}_2$ and hence $\mathcal{P} = \mathcal{P}_2$. The total number of liftings is equal to 15 (the number of monomials of a trivariate quartic), subject to 3 equality constraints, leaving 12 degrees of freedom. $\square$.

Define

$$\mathcal{F}^* = \{\mathbf{f} \in \mathbb{P}^2 : f(x) \geq 0\ \forall x \in \partial\mathcal{P}\}.$$

as the subset of $\mathcal{F}$ consisting only of lines which are tangents to $\mathcal{P}$.

Lemma 3 *$\mathcal{P} = \mathcal{P}_2$ if and only if, for all $\mathbf{f} \in \mathcal{F}^*$, $p_f(x) \geq 0$ for all x.*

Proof: Let $x^f \in \partial\mathcal{P}$ be the solution of the convex (but possibly non-smooth) problem of maximizing the linear function $f^T x$ subject to the constraint that x belongs to $\mathcal{P}$. The line $f^T(x -$

$x^f) = 0$ is the tangent to $\mathcal{P}$ at $x = x^f$. Let $\mathbf{f}_0 = -f^T x^f$, so that $f(x) = f^T(x - x^f) = \mathbf{f}_0 + \mathbf{f}_1 x_1 + \mathbf{f}_2 x_2 \geq 0$ for all $x \in \mathcal{P}$, and the corresponding element $\mathbf{f}$ belongs to $\mathcal{F}$. Given such an element, assume that the corresponding polynomial $p_f(x)$ is non-negative. Any other element $\mathbf{f}^* \in \mathcal{F}$ such that $(\mathbf{f}_1^*, \mathbf{f}_2^*) = f$ has a constant value $\mathbf{f}_0^*$ which is larger than $\mathbf{f}_0$, and hence the corresponding polynomial $p_{f^*}(x)$ is also non-negative. $\square$

Checking the condition of Lemma 3 implies sweeping out over a real parameter (an angle), and for each value, finding the tangent to $\mathcal{P}$ and checking non-negativity of the bivariate quartic $p_f(x)$ (e.g. by solving a semidefinite programming problem). This can be computationally demanding, and it makes sense to derive more tractable sufficient conditions ensuring $\mathcal{P} = \mathcal{P}_2$ or $\mathcal{P} \neq \mathcal{P}_2$.

Let $\nabla p(x) \in \mathbb{R}[x]_3^2$ denote the gradient of $p(x)$.

Lemma 4 *If $\mathcal{P}$ is bounded, then $\mathcal{P} = \mathcal{P}_2$ if and only if, for all $\mathbf{f} \in \mathcal{F}^*$, $p_f(x) \geq 0$ for all x such that $\nabla p(x) = f$.*

Proof: If $\mathcal{P}$ is bounded, then $p(x) \to -\infty$ and hence $p_f(x) \to +\infty$ when $\|x\| \to +\infty$. The polynomial $p_f(x)$ then achieves its minimum when its gradient vanishes, i.e. when $\nabla p(x) = f$. $\square$

Generically, there is a finite number (at most 4) of real points x satisfying $\nabla p(x) = f$, and the sign of $p_f(x)$ should be tested only at these points, which is a significant saving over assessing global non-negativity of $p_f(x)$ as in Lemmas 2 or 3. If $\mathcal{P}$ is not bounded, we also have to check the sign of $p_f(x)$ at infinity.

Lemma 5 *If $p(x)$ is concave, then $\mathcal{P} = \mathcal{P}_2$.*

Proof: If $p(x)$ is concave, then $p_f(x) = f(x) - p(x)$ is convex. This polynomial has a unique minimum x^f when its gradient $\nabla p_f(x) = f - \nabla p(x)$ vanishes, with $f = (\mathbf{f}_1, \mathbf{f}_2)$. Given f, the point x^f solution to $f = \nabla p(x)$ is the point along the boundary $\partial\mathcal{P}$ at which the line $f^T(x - x^f) = 0$ is tangent to $\mathcal{P}$. Since $p(x)$ is concave and $\mathcal{P}$ has non-empty interior, it cannot happen that $\nabla p(x)$ vanishes along $\partial\mathcal{P}$. Therefore $\partial\mathcal{P}$ is necessarily smooth, and point x^f always exists for any f. At this point, polynomial $p_f(x)$ vanishes. Since this is the unique minimum and $p_f(x)$ is convex, it follows that $p_f(x)$ is globally non-negative. $\square$

Testing concavity of $p(x)$ is equivalent to testing negative semidefiniteness of its Hessian. The Hessian is a 2-by-2 bivariate quadratic matrix, and it is negative semidefinite if and only if its trace is non-positive and its determinant is non-negative. The first condition is trivial to test, whether the second condition can be tested by semidefinite programming.

Lemma 6 *If $\partial\mathcal{P}$ is non-smooth, then $\mathcal{P} \neq \mathcal{P}_2$.*

Proof: As in the proof of Lemma 3, given a direction f, let $x^f \in \partial\mathcal{P}$ be the solution of the convex problem of maximizing the linear function $f^T x$ subject to the constraint that x belongs to $\mathcal{P}$. Suppose that $\partial\mathcal{P}$ is non-smooth at x^f, which implies that $p(x^f) = 0$ and $\nabla p(x^f) = 0$. Consider the Taylor expansion of the quartic $p_f(x) = f(x) - p(x)$ around $x = x^f$, which reads

$p_f(x) = p_f(x^f) + \nabla p_f(x^f)^T(x - x^f) + \cdots = f^T(x - x^f) + \cdots$ where the dots indicate terms of degree two or higher. This polynomial has a non-zero first-order term, and hence it cannot be globally non-negative. From Lemma 2, it follows that $\mathcal{P} \neq \mathcal{P}_2$. □

Lemma 6 states that smoothness of $\partial\mathcal{P}$ is necessary for the first relaxation to be exact. However, it says nothing about semidefinite representability of $\mathcal{P}$ in general.

Testing smoothness of $\partial\mathcal{P}$ is equivalent to finding all singular points of $\mathcal{C}$ (this can be done by solving the system of polynomial equations $p(x) = \nabla p(x) = 0$) and testing whether they lie in the interior of $\mathcal{P}$ or not.

5 Examples

5.1 Egg

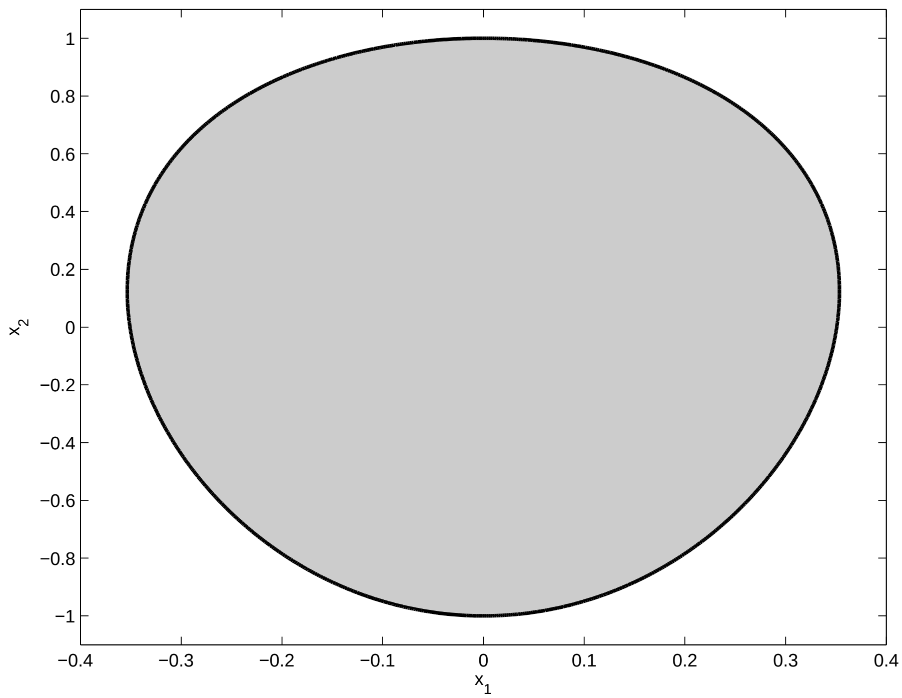

Figure 1: Egg modeled by a 6-by-6 LMI with 12 liftings.

Let $p(x) = 1 - 8x_1^2 - (x_1^2 - x_2)^2$ describe an egg curve. With the `algcurves` package for Maple we can check that curve $\mathcal{C}$ has genus zero with a triple singular point $\mathbf{x} = (0, 0, 1)$ at infinity:

```
> with(algcurves):
> p:=1-8*x1^2-(x1^2-x2)^2:
> genus(p,x1,x2);
                              0
> singularities(p,x1,x2);
                    {[[0, 1, 0], 2, 3, 2]}
```

The Hessian of $p(x)$ is given by

$$\begin{bmatrix} -16-4x_2-12x_1^2 & 4x_1 \\ 4x_1 & -2 \end{bmatrix}$$

and its evaluation at $x = (0, -5)$ shows that it is indefinite and hence that quartic $p(x)$ is not concave, so that Lemma 5 cannot be applied.

Let us test the sign condition of Lemma 2 for a given direction $\mathbf{f} \in \mathcal{F}^*$. Choose e.g. the point $x^f = (0, 1) \in \mathcal{C}$ at which $\nabla p(x^f) = (0, -2)$ and hence $\mathbf{f} = (2, 0, -2)$. We have $p_f(x) = 2 - 2x_2 - p(x) = 1 - 2x_2 + 8x_1^2 + (x_1^2 - x_2)^2$. With YALMIP [9] we could find the SOS decomposition $p_f(x) = (1 - x_2 + x_1^2)^2 + 6x_1^2$ certifying non-negativity of $p_f(x)$:

```
>> sdpvar x1 x2
>> pf=1-2*x2+8*x1ˆ2+(x1ˆ2-x2)ˆ2;
>> [sol,v,Q]=solvesos(sos(pf));
>> Q{1}
ans =
    1.0000   -1.0000    1.0000         0
   -1.0000    1.0000   -1.0000         0
    1.0000   -1.0000    1.0000         0
         0         0         0    6.0000
>> eig(Q{1})'
ans =
    0.0000    0.0000    3.0000    6.0000
>> sdisplay(v{1}')
ans =
    '1'    'x2'    'x1ˆ2'    'x1'
>> sdisplay(clean(pf-v{1}'*Q{1}*v{1},sqrt(eps)))
0
```

Sweeping over all points along $\partial\mathcal{P} = \mathcal{C}$, we can check that the condition of Lemma 2 is satisfied, and the convex set $\mathcal{P} = \mathcal{P}_2$ is represented on Figure 1.

5.2 Bean

Let $p(x) = x_1(x_1^2 + x_2^2) - x_1^4 - x_1^2x_2^2 - x_2^4$ define the so-called bean quartic. The curve has genus zero, and a triple singular point at the origin $\mathbf{x} = (1, 0, 0)$.

At this point the gradient $\nabla p(x)$ vanishes, whereas the curve $\mathcal{P}$ has a tangent $f(x) = x_1$. Polynomial $p_f(x) = x_1 - p(x)$ has a non-zero linear term, and hence it cannot be non-negative. By Lemma 6 we have $\mathcal{P}$ strictly included in $\mathcal{P}_2$.

Inspection of the constant and linear terms in the expression (3) arising in the definition of $\mathcal{P}_k$ in Lemma 1 shows that actually $\mathcal{P}$ is strictly included in $\mathcal{P}_k$ for all k.

On Figure 2 we see embedded semidefinite representable sets $\mathcal{P}_k$ for $k = 2, 3, 4, 5$ (thin lines and shaded regions) and the convex set $\mathcal{P}$ (thick line). It seems that $\mathcal{P}$ is smooth but actually there

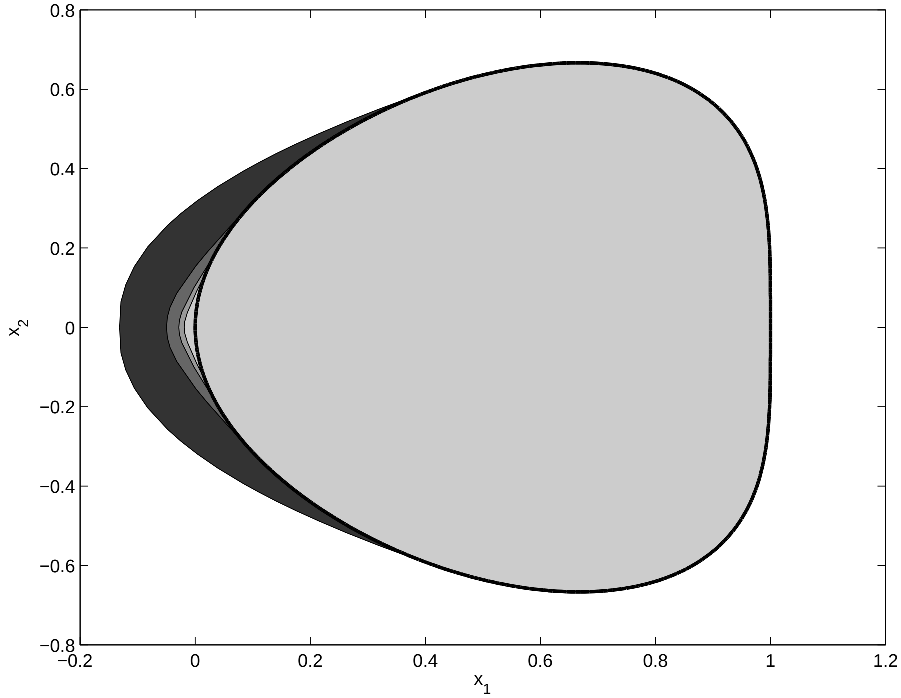

Figure 2: Bean curve (thick line) and its first four embedded outer semidefinite approximations (thin lines and shaded regions).

is a singularity at the origin. We see the global consequences of the pointwise singularity on the shape of the sets $\mathcal{P}_k$.

For interested readers, semidefinite sets $\mathcal{P}_k$ can be visualized with the following Matlab script, mixing features from GloptiPoly 3 [6] and YALMIP [9]:

```
mpol x 2
p = x(1)*(x(1)^2+x(2)^2)-x(1)^4-x(1)^2*x(2)^2-x(2)^4;
k = 3; % relaxation order
P = msdp(p==0, k);
[F,h,y] = myalmip(P);
plot(F,y(1:2)); % projection on first degree moments
```

The impact in polynomial optimization of the non-exactness of semidefinite relaxations can be observed with the help of the following Matlab script using GloptiPoly 3:

```
bounds = [];
for k = 2:10
 P = msdp(min(x(1)), p==0, k);
 [status,obj] = msol(P);
 bounds = [bounds obj];
end
```

We obtain the following sequence of lower bounds on the minimum value of x_1 such that $p(x)$

vanishes: -0.1315, -0.02915, -0.009705, -0.01022, -0.009319, -0.009320, -0.009646, -0.009346, $-0.009371, \ldots$ We expect the sequence to converge from below to zero, the genuine minimum, but numerically, it stagnates around $-9 \cdot 10^{-3}$.

5.3 Water drop

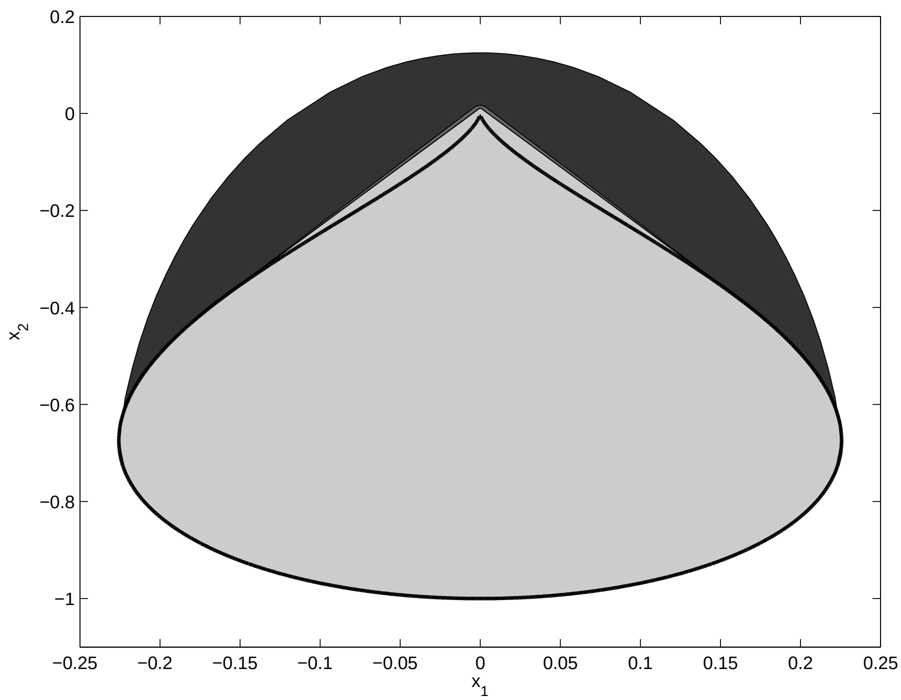

Figure 3: Water drop quartic (thick line) and its first four embedded semidefinite approximations (thin lines and shaded regions).

Let $p(x) = -x_1^2 - x_2^3 - (x_1^2 + x_2^2)^2$ define a water drop quartic. The curve has genus two, with a singular point (a cusp) at the origin. On Figure 3 we see semidefinite representable embedded sets $\mathcal{P}_k$ for $k = 2, 3, 4, 5$ (thin lines and shaded regions) and the non convex curve $\mathcal{C}$ (thick line). The set $\mathcal{P}_5$ and the convex hull $\mathcal{P} = \text{conv}\,\mathcal{C}$ are almost undistinguishable. However, as for Example 5.2, it can be shown that $\mathcal{P}_k$ cannot be equal to $\mathcal{P}$ for k finite.

5.4 Lemniscate

Let $p(x) = x_1^2 - x_2^2 - (x_1^2 + x_2^2)^2$ define a lemniscate, a curve of genus zero, with singular points at the origin and at the infinite complex points $\mathbf{x} = (0, \pm i, 1)$. Even though $\mathcal{C}$ is singular, the boundary $\partial\mathcal{P}$ is smooth.

Sweeping over all directions f indicates that $p_f(x)$ is always non-negative, and hence that $\mathcal{P} = \mathcal{P}_2$, see Figure 5.4. Here, the singularity of $\mathcal{C}$ is in the interior of $\mathcal{P}$, and it does not prevent the first relaxation to be exact.

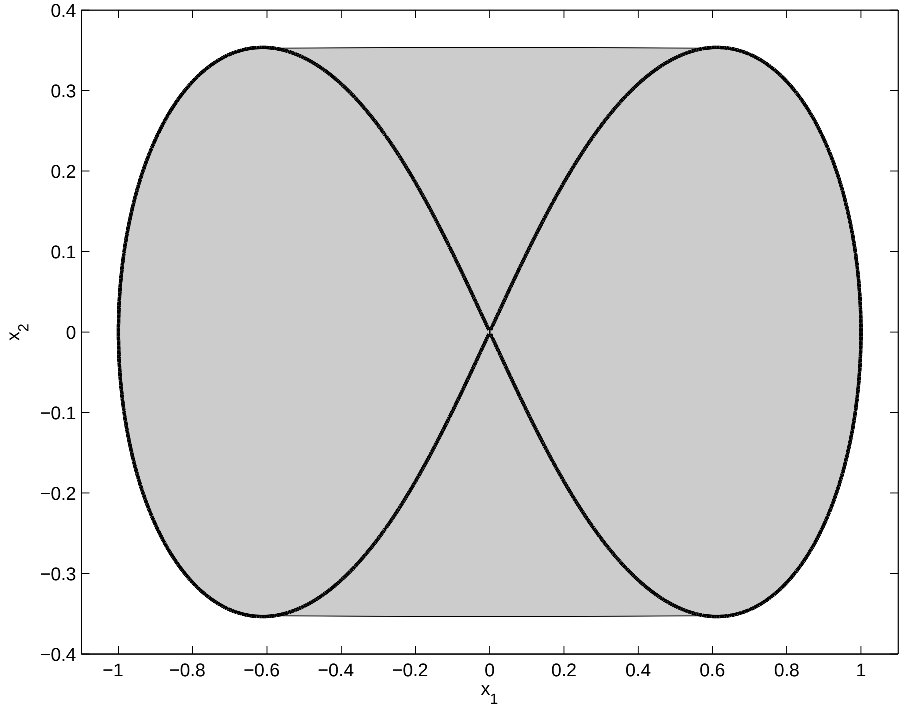

Figure 4: Lemniscate (thick line) and its convex hull modeled by a 6-by-6 LMI with 12 liftings.

5.5 Folium

Let $p(x) = -x_1(x_1^2 - 2x_2^2) - (x_1^2 + x_2^2)^2$ define a folium, a curve of genus zero, with a triple singular point at the origin. As in Example 5.4, the singularity of $\mathcal{C}$ is the interior of $\mathcal{P}$, hence the boundary $\partial\mathcal{P}$ is smooth, and we may expect that the first relaxation is exact.

However, along the direction $\mathbf{f} = (1, 3/4, 3\sqrt{2}/2)$ corresponding to a bitangent of $\mathcal{C}$ (obtained computationally by finding singular points of the dual curve), polynomial $p_f(x) = f(x) - p(x)$ has a strictly negative minimum achieved at $x \approx (0.2040, -0.8762)$.

On Figure 5 we see curve $\mathcal{C}$ (thick line) and semidefinite sets $\mathcal{P}_2$ (exterior thin line, dark shaded region) and $\mathcal{P}_3$ (interior thin line, shaded region). We observe that $\mathcal{P} = \operatorname{conv}\mathcal{C}$ is strictly included in $\mathcal{P}_2$, whereas, apparently, $\mathcal{P}_3 = \mathcal{P}$ (but we are not able to prove this identity).

5.6 Smooth and convex

Let $p(x) = x_1 + x_1^2 - 2x_1^4 - x_2^4$ define a convex quartic curve of genus three. Note that $p(x)$ is not concave. There is no singularity, so we may expect that the first relaxation is exact. However, for the tangent $f(x) = x_1$, polynomial $p_f(x) = f(x) - p(x)$ is not non-negative (consider e.g. its sign along the line $x_2 = 0$) and hence $\mathcal{P} \neq \mathcal{P}_2$. As in Example 5.5, apparently $\mathcal{P} = \mathcal{P}_3$.

This example was prepared with the help of Bill Helton and Jiawang Nie.

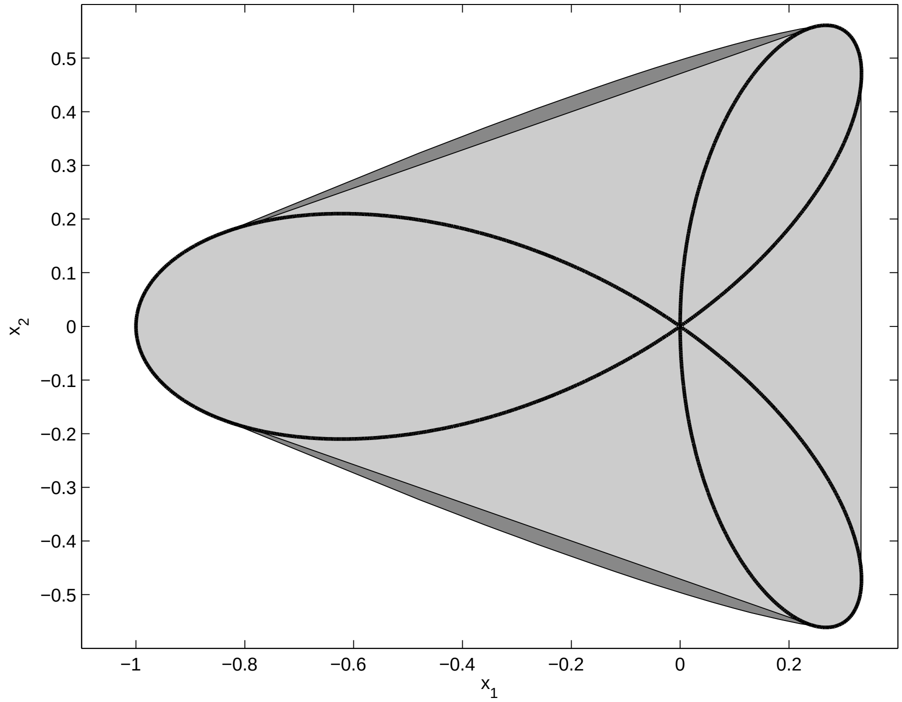

Figure 5: Folium curve and its first two embedded outer semidefinite approximations (thin lines and shaded regions).

5.7 Fermat

The Fermat quartic $p(x) = 1 - x_1^4 - x_2^4$ (also called the TV-screen quartic) is smooth, so its convex hull is semidefinite representable as a 6-by-6 LMI with 12 liftings. However, inspection reveals that there is a semidefinite representation with only 2 liftings:

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{P}^2 \,:\, \exists \mathbf{y} \in \mathbb{P} \,:\, \begin{bmatrix} \mathbf{x}_0 + \mathbf{y}_0 & \mathbf{y}_1 & & & & \\ \mathbf{y}_1 & \mathbf{x}_0 - \mathbf{y}_0 & & & & \\ & & \mathbf{x}_0 & \mathbf{x}_1 & & \\ & & \mathbf{x}_1 & \mathbf{y}_0 & & \\ & & & & \mathbf{x}_0 & \mathbf{x}_2 \\ & & & & \mathbf{x}_2 & \mathbf{y}_1 \end{bmatrix} \succeq 0\}.$$

However, we do not know how to derive this reduced representation with 2 liftings from the generic representation with 12 liftings.

6 Rational curves

In this section we restrict the class of $\mathcal{C}$ to algebraic curves of genus zero [3], i.e. curves which admit a polynomial parametrization

$$\mathcal{C} = \{\mathbf{x} \in \mathbb{P}^2 \,:\, \exists \mathbf{t} \in \mathbb{P} \,:\, \mathbf{x}_i = \mathbf{p}_i(\mathbf{t}),\, i = 0,1,2\} \tag{6}$$

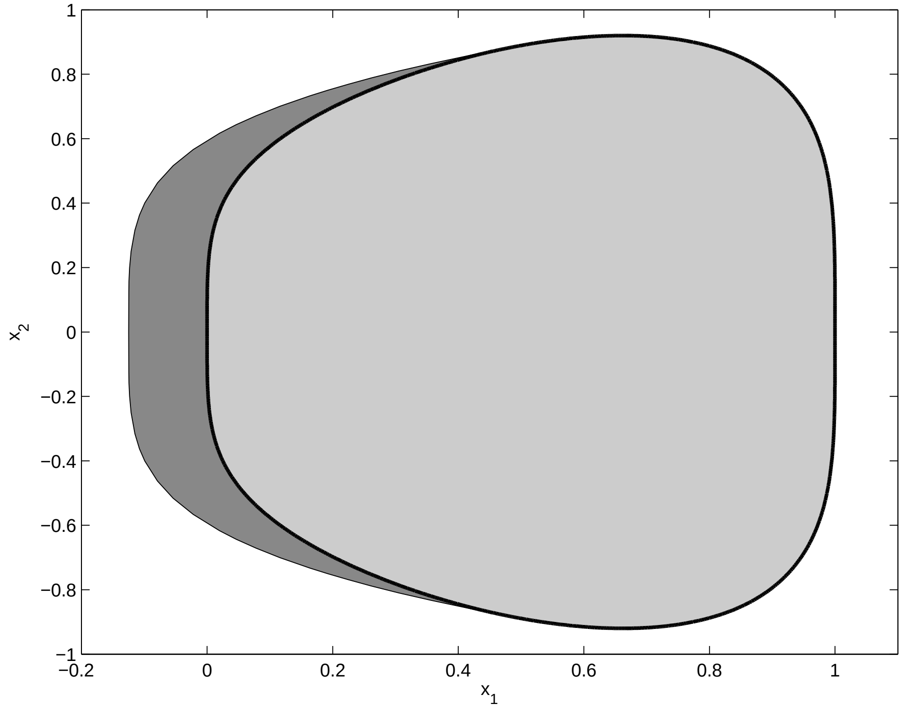

Figure 6: Smooth convex quartic and its first two embedded outer semidefinite approximations (thin lines and shaded regions).

with $\mathbf{p}_i(\mathbf{t})$ bivariate quartic forms in $\mathbf{t} \in \mathbb{P}$. Given $p(x)$ in implicit representation (1), there are algorithms to compute the $\mathbf{p}_i(\mathbf{t})$ in the above explicit representation, see e.g. the Maple package `algcurves` for an implementation.

Let us define the Hankel moment matrix

$$M_2(\mathbf{y}) = \sum_{\alpha}^{4} \mathbf{y}_\alpha H_\alpha = \begin{bmatrix} \mathbf{y}_0 & \mathbf{y}_1 & \mathbf{y}_2 \\ \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 \\ \mathbf{y}_2 & \mathbf{y}_3 & \mathbf{y}_4 \end{bmatrix}$$

and, given the quartic $\mathbf{p}(\mathbf{t}) = \sum_\alpha^4 \mathbf{p}_\alpha \mathbf{t}^\alpha$, the localising linear form

$$M_0(\mathbf{p}\mathbf{y}) = \sum_{\alpha}^{4} \mathbf{p}_\alpha \mathbf{y}_\alpha.$$

Lemma 7 $\mathcal{P} = \{\mathbf{x} \in \mathbb{P}^2 \,:\, \exists \mathbf{y} \in \mathbb{P}^4 \,:\, \mathbf{x}_i = M_0(\mathbf{p}_i\mathbf{y}),\, i = 0,1,2,\, M_2(\mathbf{y}) \succeq 0\}$.

Proof: We closely follow the proof of Lemma 2. Given $\mathbf{f} \in \mathcal{F}$ and $\mathbf{x} \in \mathcal{C}$, consider the bivariate form $\mathbf{g}(\mathbf{t}) = \mathbf{f}^T\mathbf{x} = \sum_{i=0}^2 \mathbf{f}_i\mathbf{x}_i = \sum_{i=0}^2 \mathbf{f}_i\mathbf{p}_i(\mathbf{t}) = \sum_\alpha^4 (\mathbf{g}_\alpha^T\mathbf{f})\mathbf{t}^\alpha$ with $\mathbf{t} \in \mathbb{P}$ and coefficients $\mathbf{g}_\alpha \in \mathbb{R}^3$. This non-negative bivariate form is always SOS [12], hence there exists a 3-by-3 matrix X such that

$$\begin{array}{l} \mathrm{trace}(H_\alpha X) = \mathbf{g}_\alpha^T\mathbf{f},\ \alpha = 0,1,\ldots,4 \\ X \succeq 0, \end{array}$$

hence the dual formulation of Lemma 7.□

An alternative, more direct proof suggested by Roland Hildebrand, consists in viewing $\mathcal{P}$ as the image through a linear mapping of the convex hull of a Veronese variety, namely the image of the nonlinear map sending $\mathbf{t} \in \mathbb{P}$ into $[\mathbf{t}_0^4\ \mathbf{t}_0^3\mathbf{t}_1\ \mathbf{t}_0^2\mathbf{t}_1^2\ \mathbf{t}_0\mathbf{t}_1^3\ \mathbf{t}_1^4] \in \mathbb{P}^4$, see [3]. This Veronese variety is also called sometimes the moment curve, and its convex hull is indeed the cone of positive semidefinite Hankel matrices.

Jean-Bernard Lasserre informed me that Pablo Parrilo presented a related semidefinite representation of the convex hull of rational plane curves at a workshop at Banff, Canada, in October 2006. At the time of writing of these notes (March 2008), the result is not available in electronic or printed form, however.

Note that the relations $\mathbf{x}_i = M_0(\mathbf{p}_i\mathbf{y})$ in Lemma 7 form a consistent linear system of 3 equations with 5 indeterminates, so the number of lifting variables can always be reduced to 5-3=2.

6.1 Folium revisited

Consider again the folium quartic of Example 5.5. With the `algcurves` package of Maple, we obtain the following rational parametrization: $p_0(t) = 1+2t_1^2+t_1^4$, $p_1(t) = -1+2t_1^2$, $p_2(t) = -t_1+2t_1^3$.

The lifting variables in the representation of Lemma 7 satisfy the linear system of equations $\mathbf{x}_0 = \mathbf{y}_0+2\mathbf{y}_2+\mathbf{y}_4$, $\mathbf{x}_1 = -\mathbf{y}_0+2\mathbf{y}_2$, $\mathbf{x}_2 = -\mathbf{y}_1+2\mathbf{y}_3$. From this we derive $\mathbf{y}_2 = \frac{1}{2}(\mathbf{x}_1+\mathbf{y}_0)$, $\mathbf{y}_3 = \frac{1}{2}(\mathbf{x}_2+\mathbf{y}_1)$, $\mathbf{y}_4 = \mathbf{x}_0-\mathbf{x}_1-2\mathbf{y}_0$ that we can report in the Hankel matrix constraint to produce a semidefinite representation of the convex hull of $\mathcal{C}$ with 2 liftings:

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{P}^2 \,:\, \exists \mathbf{y} \in \mathbb{P} \,:\, \begin{bmatrix} 2\mathbf{y}_0 & 2\mathbf{y}_1 & \mathbf{x}_1+\mathbf{y}_0 \\ 2\mathbf{y}_1 & \mathbf{x}_1+\mathbf{y}_0 & \mathbf{x}_2+\mathbf{y}_1 \\ \mathbf{x}_1+\mathbf{y}_0 & \mathbf{x}_2+\mathbf{y}_1 & 2\mathbf{x}_0-2\mathbf{x}_1-4\mathbf{y}_0 \end{bmatrix} \succeq 0\}.$$

6.2 Bean revisited

Consider again the bean quartic of Example 5.2. A rational parametrization (6) is given by $p_0(t) = 1+t_1^2+t_1^4$, $p_1(t) = 1+t_1^2$ and $p_2(t) = t_1+t_1^3$, from which follows a semidefinite representation with 2 liftings:

$$\mathcal{P} = \{\mathbf{x} \in \mathbb{P}^2 \,:\, \exists \mathbf{y} \in \mathbb{P} \,:\, \begin{bmatrix} \mathbf{y}_0 & \mathbf{y}_1 & \mathbf{x}_1-\mathbf{y}_0 \\ \mathbf{y}_1 & \mathbf{x}_1-\mathbf{y}_0 & \mathbf{x}_2-\mathbf{y}_1 \\ \mathbf{x}_1-\mathbf{y}_0 & \mathbf{x}_2-\mathbf{y}_1 & \mathbf{x}_0-\mathbf{x}_1 \end{bmatrix} \succeq 0\}.$$

7 Conclusion

This note investigated semidefinite representations of convex plane quartics, and more specifically the exactness of the first semidefinite relaxation in Lasserre's hierarchy. Also described was an elementary exact semidefinite representation of the convex hull of rationally parametrized quartics. Exactness conditions followed from the well-known fact that non-negative polynomials can be represented as sum-of-squares in the bivariate quartic case and in the univariate case. It follows that the exactness result of Section 6 is valid for rationally parametrizable curves of arbitrary degree and dimension, but the exactness result of Section 4 is limited to plane quartics. Also unclear is what kind of conditions should be enforced to ensure exactness of the second, third, and in general higher-order relaxations.

In [8], Lasserre proposed sufficient conditions for convex semialgebraic sets to be semidefinite representable. The conditions are algebraic in nature, strongly connected with the polynomials used to model the set, and related with the particular SOS representation of Lemma 1. We also notice that Lemma 5 can be found in Example 3 in [8] where it is proved using Karush-Kuhn-Tucker optimality conditions. In [4], Helton and Nie derived sufficient conditions in terms of negative definiteness of the Hessian along the tangent space along the boundary of the set, provided the gradient does not vanish along this boundary. In contrast with these general statements, our focus in this note was more on computational aspects and explicit examples, the driving force being that if we do not understand well the simplest non-trivial case (plane quartics) it is likely that we will not understand more complicated configurations. It is expected that our examples can provide further motivation for studying alternative semidefinite representations, see e.g. [11]. For instance, we are not aware of any exact semidefinite representation of the convex hull of the water drop quartic of Example 5.3.

As illustrated in Example 5.7, a given quartic may have different semidefinite representations with a different number of lifting variables. Given a representation, it could be interesting to design a systematic algorithm to remove redundant lifting variables. Similarly, the problem of finding a representation with a minimum number of lifting variables seems to be open.

Since projections of LMIs are convex semialgebraic sets, and the essential difficulty when building semidefinite representations seems to be singularities (points at which the gradient vanishes), one may be tempted to conjecture that convex regions delimited by higher degree smooth curves admit an exact semidefinite representation. Helton and Nie go even farther in the conclusion of [4] by conjecturing that every convex semialgebraic set is semidefinite representable.

Appendix

In this paper we use projective spaces $\mathbb{P}^k$ over the field $\mathbb{R}$, together with affine spaces $\mathbb{R}^k$. By projective space $\mathbb{P}^k$, we mean the set of all one-dimensional subspaces of $\mathbb{R}^{k+1}$. Equivalently, $\mathbb{P}^k$ is the quotient space of equivalence classes of $\mathbb{R}^{k+1}-0$ under the equivalence relation $(\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_k)\sim(a\mathbf{x}_0,a\mathbf{x}_1,\ldots,a\mathbf{x}_k)$ for all nonzero $a\in\mathbb{R}$. Projective space $\mathbb{P}^k$ is a compact space under the Zariski topology where a closed set is defined as the zero set of homogeneous polynomials. When $\mathbf{x}_0\neq 0$, it holds $(\mathbf{x}_0,\mathbf{x}_1,\ldots,\mathbf{x}_k)\sim(1,\mathbf{x}_1/\mathbf{x}_0,\ldots,\mathbf{x}_k/\mathbf{x}_0)$, and we can identify a point $(x_1,\ldots,x_k)\in\mathbb{R}^k$ with a point $(1,x_1,\ldots,x_k)\in\mathbb{P}^k$. Then the affine space $\mathbb{R}^k$ is the open subset of the projective space $\mathbb{P}^k$ defined by $\mathbf{x}_0\neq 0$. Points $(0,\mathbf{x}_1,\ldots,\mathbf{x}_k)$ with $\mathbf{x}_0=0$ corresponds to points at infinity, and $\mathbb{P}^k$ can be also viewed as the affine space $\mathbb{R}^k$ extended with points at infinity. See e.g. [3, Chapter 1] for an elementary introduction.

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