On parameter-dependent Lyapunov functions for robust stability of linear systems

Didier Henrion, Denis Arzelier, Dimitri Peaucelle, Jean-Bernard Lasserre

***Abstract*— For a linear system affected by real parametric uncertainty, this paper focuses on robust stability analysis via quadratic-in-the-state Lyapunov functions polynomially dependent on the parameters. The contribution is twofold. First, if n denotes the system order and m the number of parameters, it is shown that it is enough to seek a parameter-dependent Lyapunov function of given degree $2nm$ in the parameters. Second, it is shown that robust stability can be assessed by globally minimizing a multivariate scalar polynomial related with this Lyapunov matrix. A hierarchy of LMI relaxations is proposed to solve this problem numerically, yielding simultaneously upper and lower bounds on the global minimum with guarantee of asymptotic convergence.**

I. Introduction

Robustness analysis for linear systems affected by structured real parametric uncertainty is an active field of research. Even though it is known that most of these problems are NP-hard [5], it does not imply that no efficient numerical methods can be designed to solve them in practice.

In the technical literature, a distinction is made between quadratic and robust stability. Quadratic stability means stability for any (possibly infinite) time-variation of the uncertain parameters, whereas robust stability means stability for every possible (but frozen) values of the uncertain parameters. Quadratic stability results are generally based on convex optimization over linear matrix inequalities (LMI), see [6]. When assessing quadratic stability, a quadratic-in-the-state Lyapunov function is sought which is independent of the uncertain parameters. Quadratic stability obviously implies robust stability, but the converse is not true. As a consequence, quadratic stability can prove overly conservative, or pessimistic [3]. On the other hand, robust stability cannot in general be assessed using convex optimization. Only in very special cases standard tools of numerical linear algebra can be used to conclude about robust stability. Such examples include Kharitonov's theorem and the edge theorem, see [3] for a good overview.

In order to reduce the gap between robust stability and quadratic stability, researchers attempts for more than one decade have been aimed at reducing the conservatism of LMI methods. In order to go beyond parameter-independent Lyapunov functions, LMI techniques were proposed to derive quadratic-in-the-state Lyapunov functions which are affine in the parameters [8], [16], [20], quadratic in the parameters [25], and recently polynomial in the parameters [4]. Checking robust stability with LMI techniques then amounts to solving parametrized LMI, see [1] for an overview and [14] for current achievements in the area. Significant progress have been achieved recently by using results on real algebraic geometry such as the decomposition of positive polynomials as sums-of-squares, see [11], [19], [22].

In this paper, we aim at applying LMI techniques to assess robust stability, thus further filling up the gap between exact (but NP-hard) robust stability analysis and conservative (but convex, hence polynomial-time) quadratic stability analysis. Our contribution is twofold. First, we show in section III that an explicit bound on the degree of the polynomial dependence on the parameters of the quadratic-in-the-state Lyapunov function can be derived easily. Second, in section IV we describe a systematic and general method to build up a hierarchy of LMI relaxations allowing to gradually control the computational burden when assessing robust stability.

II. Problem statement

Consider the continuous-time linear system

$$\dot{x} = A(q)x, \quad A(q) = A_0 + \sum_{i=1}^{m} q_i A_i \qquad (1)$$

of order n depending affinely on m real scalar parameters q_i belonging to some compact polytope $\mathcal{Q}$. We are interested in checking *robust stability* of system (1), i.e. whether all eigenvalues of matrix $A(q)$ lie in the open left-half plane for any admissible uncertainty q in $\mathcal{Q}$. Parameters q_i are not time-varying, i.e. we are testing stability of $A(q)$ for all possible frozen values of q.

III. Polynomial Lyapunov matrix of bounded degree

In virtue of Lyapunov's stability theorem – see e.g. section 4 in [18] – robust asymptotic stability of system (1) is equivalent to the existence, for each $q \in \mathcal{Q}$, of a Lyapunov function

$$V(x,q) = x^T P(q) x \qquad (2)$$

quadratic in state x, parametrized in q, such that

- $V(x,q) > 0$, and
- $\dot{V}(x,q) \leq 0$, not identically zero on any system trajectory except $x = 0$.

D. Henrion, D. Arzelier, D. Peaucelle and J. B. Lasserre are with LAAS-CNRS, 7 Avenue du Colonel Roche, 31077 Toulouse, France. E-mail: {henrion,arzelier,peaucelle,lasserre}@laas.fr

D. Henrion is also with the Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 18208 Prague, Czech Republic and with the Faculty of Electrical Enginerering, Czech Technical University in Prague, Technická 2, 16627 Prague, Czech Republic.

Matrix $P(q)$ in (2) is referred to as a parameter-dependent Lyapunov matrix.

Let

$$h(s,q) = \det\left(sI - A(q)\right) = \sum_{i=0}^{n} h_i(q)s^i \qquad (3)$$

denote the characteristic polynomial of system (1), where s is the Laplace variable and coefficients $h_i(q)$ are multivariate polynomials of degree at most nm in q. By considering the companion form of $A(q)$, it was shown in [17] that the *Hermite matrix* $H(q)$ of polynomial $h(s,q)$ is a valid Lyapunov matrix $P(q)$ ensuring asymptotic stability of system (1). Hermite matrix $H(q)$ has dimension n and is quadratic in coefficients $h_i(q)$, hence of degree $2nm$ in parameters q. See [15] for a modern definition of the Hermite matrix and its relationship with Bezoutians and polynomial resultants.

Lemma 1: Linear system (1) of order n affected by m uncertain real parameters q belonging to polytope $\mathcal{Q}$ is robustly stable if and only if

$$H(q) \succ 0 \quad \forall q \in \mathcal{Q}$$

where $H(q)$ is the Hermite matrix of the characteristic polynomial of the system and $\succ 0$ means positive definite. Matrix $H(q)$ is a polynomial parameter-dependent Lyapunov matrix of degree at most $2nm$ in q.

In other words, when checking robust stability of a linear system of order n affected by m uncertain parameters living in a polytope, it is not necessary to seek quadratic-in-the-state polynomial parameter-dependent Lyapunov functions of degree greater than $2nm$.

It must be noticed also that any rank-deficiency of uncertainty matrices A_i in (1) results in lower degree coefficients $h_i(q)$ of characteristic polynomial $h(s,q)$, and hence in a tighter estimate on the degree of the parameter-dependent Lyapunov matrix, the value $2nm$ being only the worst-case generic estimate. For example, in the case of interval matrices $A(q)$ (when $\mathcal{Q}$ is a hyper-rectangle) uncertainty matrices A_i have all rank-one, and the degree estimate becomes $2m$, independently of the system order.

IV. Assessing robust stability

Coefficients of Hermite matrix $H(q)$ can be derived from coefficients $h_i(q)$ of polynomial $h(s,q)$ via the procedure outlined e.g. in [15]. It basically consists in computing a resultant, or equivalently solving a linear system of equations. In turn, coefficients $h_i(q)$ can be obtained from original system matrices A_i using straightforward symbolic or numerical linear algebra.

Assuming that for some nominal value $q_0 \in \mathcal{Q}$ matrix $A(q)$ is stable, hence $H(q_0) \succ 0$, then the polynomial

$$p(q) = \det H(q)$$

plays the role of a *guardian map* [3, Chapter 17] for robust stability of matrix $A(q)$. It means that the sign of $p(q)$ does not change when $q \in \mathcal{Q}$ if and only if matrix $A(q)$ is robustly stable.

Lemma 2: Solve the optimization problem

$$\begin{array}{rcll} p^\star & = & \min_q & p(q) \\ & & \text{s.t.} & q \in \mathcal{Q}. \end{array} \qquad (4)$$

Then system (1) is robustly stable if and only if $p^\star > 0$.

Hierarchy of LMI relaxations

Problem (4), minimizing a multivariate polynomial over a polyhedron, is a difficult NP-hard non-convex optimization problem in general. However we can use the methodology of [13] to build up a whole hierarchy of *convex LMI relaxations* with guarantee of asymptotic convergence to the global optimum.

Since $\mathcal{Q}$ is a compact polytope, we can build a sequence of LMI problems

$$\begin{array}{rcll} \underline{p}_k & = & \min_y & f(y) \\ & & \text{s.t.} & F_k(y) \succeq 0 \end{array} \qquad (5)$$

for $k = 1, 2, \ldots$ where $f(y)$ is a linear function of the vector y of decision variables, and $F_k(y)$ is an LMI constraint made of moment matrices and localization matrices of appropriate orders, see [13] for details. LMI problem (5) is referred to as the *LMI relaxation of order* k of problem (4). Valid relaxation orders are $k = k_0, k_0+1, k_0+2, \ldots$ where k_0 is the minimal relaxation order. If $d = \deg p(q)$ denotes the overall degree in q of polynomial p, then $k_0 = d/2$ if d is even and $k_0 = (d+1)/2$ if d is odd.

Computational complexity

Denote LMI (5) by

$$\begin{array}{rl} \max_y & b^T y \\ \text{s.t.} & c - A^T y \in \mathcal{K}, \end{array}$$

where y is a vector of M_k decision variables and $\mathcal{K}$ is the N_k-dimensional cone of positive semidefinite matrices. The dual LMI is given by

$$\begin{array}{rl} \min_x & c^T x \\ \text{s.t.} & Ax = b \\ & x \in \mathcal{K} \end{array}$$

where x is a vector of N_k decision variables. Let n_q be the number of linear constraints used to define polytope $\mathcal{Q}$. For example, $n_q = 2m$ if $\mathcal{Q}$ is a hyper-rectangle.

Then

$$M_k = \left(\begin{array}{c} m+2k \\ 2k \end{array}\right) - 1$$

and

$$N_k = \left(\begin{array}{c} m+k \\ k \end{array}\right) + n_q \left(\begin{array}{c} m+k-1 \\ k-1 \end{array}\right)$$

where

$$\left(\begin{array}{c} x \\ y \end{array}\right) = \frac{x!}{(x-y)!\,y!}$$

is a binomial coefficient. It follows that both the number M_k of decision variables and the size of the matrices N_k in

LMI (5) grow *polynomially* as functions of the relaxation order k. In particular, M_k grows in $\mathcal{O}(k^m)$ whereas N_k grows in $\mathcal{O}(n_q k^m)$.

Convergence

Solving LMI (5) for increasing orders $k = k_0, k_0+1, \ldots$ yields a monotically increasing sequence of lower bounds $\underline{p}_{k_0} \leq \underline{p}_{k_0+1} \leq \cdots \leq p^\star$ such that $\lim_{k\to\infty} \underline{p}_k = p^\star$. Clearly if $\underline{p}_k > 0$ at some order k, then we can stop, concluding that $p^\star > 0$.

In particular if $p^\star > 0$ in problem (4), then necessarily $\underline{p}_k > 0$ for all $k \geq \underline{k}$ and some finite relaxation order $\underline{k}$. Note however that the value of $\underline{k}$ is a priori unknown. The problem of deriving tight upper bounds on $\underline{k}$ based on the a priori knowledge of polynomial $p(q)$ remains open.

Let y_k be the vector of first-order moments in the optimal solution y of LMI problem (5). By construction of the moment matrices, and since we assumed that $\mathcal{Q}$ is a polytope, then necessarily $y_k \in \mathcal{Q}$. It follows that the polynomial $p(q)$ evaluated at $q = y_k$ returns an upper bound on $p^\star$. Among all the upper bounds obtained at the previous relaxations, we keep the less pessimistic, defining $\overline{p}_k = \min\{\overline{p}_{k_0}, \overline{p}_{k_0+1}, \ldots, \overline{p}_{k-1}, p(y_k)\}$ as a monotonically decreasing sequence of upper bounds $\overline{p}_{k_0} \geq \overline{p}_{k_0+1} \geq \cdots \geq p^\star$. Clearly if $\overline{p}_k \leq 0$ at some order k, then we can stop, concluding that $p^\star \leq 0$.

It can be shown [23] that if problem (4) has a unique global minimizer (which is a generic property) then $\lim_{k\to\infty} \overline{p}_k = p^\star$. In particular if $p^\star < 0$, then necessarily $\overline{p}_k < 0$ for all $k \geq \overline{k}$ and some finite relaxation order $\overline{k}$.

Note finally that the case $p^\star = 0$ corresponds to marginal robust stability, and is not a generic case. Therefore, in almost all cases, the hierarchy of LMI relaxations allows to conclude in *finitely many steps*.

This hierarchy of LMI relaxations was implemented in the general purpose Matlab freeware GloptiPoly [10]. A numerical linear algebra algorithm is available in GloptiPoly to certify global optimality and extract globally optimal solutions. The numerical examples below indicate that it is generally not necessary to resort to high order LMI relaxations for solving robustness analysis problems.

Problem reduction

Since continuous-time polynomial $h(s,q)$ has real coefficients, it can be shown that the Hermite matrix $H(q)$, after permutation of even and odd rows and columns, can be split into two diagonal blocks $H_1(q)$ and $H_2(q)$ of respective size $n/2$ and $n/2$ (if n is even) or $(n+1)/2$ and $(n-1)/2$ (if n is odd), i.e.

$$\Pi^T H(q) \Pi = \left[\begin{array}{cc} H_1(q) & 0 \\ 0 & H_2(q) \end{array}\right]$$

where Π is a suitable permutation matrix. This result is generally referred to as the Liénard-Chipart stability criterion, see [9]. Obviously, each polynomial

$$p_1(q) = \det H_1(q), \quad p_2(q) = \det H_2(q)$$

independently plays the role of a guardian map for robust stability of matrix $A(q)$. Moreover, the degree of polynomials $p_i(q)$ is bounded by $n^2 m$, meaning that we can significantly reduce the computational burden when solving optimization problem (4).

Corollary 1: Solve the optimization problems

$$\begin{array}{rlcrl} p_1^\star = \min_q & p_1(q) & \quad & p_2^\star = \min_q & p_2(q) \\ \text{s.t.} & q \in \mathcal{Q} & & \text{s.t.} & q \in \mathcal{Q}. \end{array} \quad (6)$$

Then system (1) is robustly stable if and only if $p_1^\star > 0$ and $p_2^\star > 0$.

Example 1

Consider the uncertain matrix

$$A(q) = \left[\begin{array}{ccc} -2+q_1 & 0 & -1+q_1 \\ 0 & -3+q_2 & 0 \\ -1+q_1 & -1+q_2 & -4+q_1 \end{array}\right]$$

proposed in [24] and also studied in [1].

First let

$$q = [q_1,\ q_2] \in \mathcal{Q}_1 = [-1,\ 1] \times [-1,\ 1].$$

The polynomial $h(s,q)$ in (3) is given by

$$\begin{array}{rcl} h(s,q) & = & \sum_{i=0}^{3} h_i(q) s^i \\ & = & (21 - 12q_1 - 7q_2 + 4q_1q_2) \\ & & +(25 - 10q_1 - 6q_2 + 2q_1q_2)s \\ & & +(9 - 2q_1 - q_2)s^2 + s^3 \end{array}$$

and the corresponding Hermite matrix is

$$H = \left[\begin{array}{ccc} h_0h_1 & 0 & h_0h_3 \\ 0 & h_1h_2 - h_0h_3 & 0 \\ h_0h_3 & 0 & h_2h_3 \end{array}\right]$$

where we removed the dependence on q for notational simplicity. After permutations of even and odd rows and columns, we obtain polynomials

$$p_1 = h_0(h_1h_2 - h_0h_3), \quad p_2 = h_1h_2 - h_0h_3$$

in problem (6). Robust stability is therefore guarded by polynomial p_1 only, and polynomial p_2 can be discarded here. Note that polynomial $p_1(q)$ has overall degree 5 in parameter q. Let us denote $p(q) = p_1(q)$ for convenience.

In order to globally minimize polynomial $p(q)$ on polytope $\mathcal{Q}_1$ we use GloptiPoly [10]. After less than 0.5 secs of CPU time on a SunBlade 150 workstation, we obtain at the LMI relaxation of order $k = k_0 = 3$ the certified global optimum $p^\star = \underline{p}_3 = 360$ achieved at $q^\star = [1,\ 1]$. Hence robust stability is guaranteed over $\mathcal{Q}_1$.

Now let

$$q \in \mathcal{Q}_2 = [-2,\ 2] \times [-2,\ 2].$$

After approximately 0.6 secs of CPU time, the LMI relaxation of order $k = k_0 = 3$ in GloptiPoly yields the global optimum $p^\star = \underline{p}_3 = -340$ achieved at $q^\star = [2,\ -2]$. Therefore, the system is not robustly stable over $\mathcal{Q}_2$.

These results are consistent with the actual robust stability domain which is $\mathcal{R} = \{q_1 < 7/4,\ q_2 < 3\}$. This domain can be found by factorizing polynomial $p_1(q) = 2(3-q_1)(7-4q_1)(3-q_2)(34-10q_1-12q_2+q_2^2+2q_1q_2)$. With the help of GloptiPoly, we can check that the factor $34-10q_1-12q_2+q_2^2+2q_1q_2$ is non-negative on $\mathcal{R}$, vanishing at $q = [7/4,\ 3]$.

Example 2

Consider the interval matrix of [2]

$$A(q) = \begin{bmatrix} q_1 & 0 & 0 \\ 0 & q_2 & q_3 \\ 0 & -0.7115 & q_4 \end{bmatrix}$$

where

$$\begin{array}{rcl} q \in \mathcal{Q} & = & [-2.4780,\ -1.4471] \times [-0.0518,\ -0.0194] \\ & & \times [2.0000,\ 3.4347] \times [-0.0026,\ -0.0012]. \end{array}$$

As in Example 1, we consider only polynomial $p(q) = p_1(q)$ which has here overall degree 6 in the 4 indeterminates q_i. The LMI relaxation of order $k = k_0 = 3$ solved with GloptiPoly (CPU time = 4.5 secs) is unbounded. Enforcing a feasibility radius of 10^3, we obtain the lower bound $\underline{p}_3 = -170.8477$ and the upper bound $\overline{p}_3 = 0.15046$, so we cannot conclude. The next LMI relaxation (order k = 4) solved with GloptiPoly (CPU time = 30 secs) yields the certified global optimum $p^\star = \underline{p}_4 = 0.15046$ attained at $q^\star = [-1.4471,\ -0.0194,\ 2.0000,\ -0.0012]$, hence matrix $A(q)$ is robustly stable over $\mathcal{Q}$.

Example 3

Consider the interval matrix of [2]

$$A(q) = \begin{bmatrix} q_1 & -12.06 & -0.06 & 0 \\ -0.25 & -0.03 & 1.00 & 0.5 \\ 0.25 & -4.00 & -1.03 & 0 \\ 0 & 0.50 & 0 & q_2 \end{bmatrix}$$

where

$$q \in \mathcal{Q} = [-1.50,\ -0.50] \times [-4.00,\ -1.00].$$

The polynomial $h(s, q)$ in (3) is given by

$$\begin{array}{rcl} h(s,q) & = & (1.853 + 3.773q_1 + 1.985q_2 + 4.032q_1q_2) \\ & & +(3.164 + 4.841q_1 + 1.561q_2 + 1.06q_1q_2)s \\ & & +(2.871 + 2.06q_1 + 1.561q_2 + q_1q_2)s^2 \\ & & +(2.56 + q_1 + q_2)s^3 + s^4 \end{array}$$

and the Hermite matrix, after row and column permutations, is given by

$$\begin{bmatrix} h_0h_1 & h_0h_3 & 0 & 0 \\ h_0h_3 & h_2h_3 - h_1h_4 & 0 & 0 \\ 0 & 0 & h_1h_2 - h_0h_3 & h_1h_4 \\ 0 & 0 & h_1h_4 & h_3h_4 \end{bmatrix}.$$

Polynomial $p_1(q) = h_0h_1(h_2h_3 - h_1h_4) - h_0^2h_3^2$ has degree 7 in the first optimization problem in (6). After less than 1 second of CPU time, the LMI relaxation of order $k = k_0 = 4$ in GloptiPoly returns the certified global optimum $p_1^\star = -6.3220$ attained at $q = [0.48000,\ 1.2500]$. Hence interval matrix $A(q)$ is not robustly stable.

For completeness, we also solved the second optimization problem in (6), with a polynomial $p_2(q) = (h_1h_2 - h_0h_3)h_3h_4 - h_1^2h_4^2$ of degree 5. After about 0.6 secs of CPU time, the LMI relaxation of order $k = k_0 = 3$ yields $p_2^\star = -0.76469$ as a global optimum attained at $q = [0.46603,\ 1.1195]$.

V. Discrete-time systems

We can directly extend our results to discrete-time robust stability. Indeed, following [15], the Hermite matrix can be defined for any stability region that can be described by disks or half-planes in complex plane, or intersection thereof.

Polynomial $p(q) = \det H(q)$ plays the role of a guardian map for the discrete-time Hermite matrix $H(q)$ of polynomial $h(z, q) = \det\,(zI - A(q))$, where z is now the shift variable. Hence, globally minimizing of $p(q)$ over polytope $\mathcal{Q}$ as in (4) allows to conclude about robust stability of discrete-time matrix $A(q)$.

Note however that it is not clear for us how to split up the global minimization problem as in (6). Indeed, in contrast with the continuous-time Hermite matrix, the discrete-time Hermite matrix cannot be decomposed into two block submatrices. It is however expected that the particular structure of the Hermite matrix may be exploited [15]. This is left as a subject for further research.

Example 4

Consider [7, Ex. 5] where

$$\begin{array}{l} A(q) = \begin{bmatrix} -0.7 & 0.7 & 0 \\ -0.1 & -0.3 & -0.3 \\ -0.1 & 0.3 & 0.3 \end{bmatrix} + q_1 \begin{bmatrix} -0.7 & -0.3 & 0.4 \\ 0.7 & 0.7 & -0.5 \\ -1.5 & 0.1 & 0.7 \end{bmatrix} \\ \qquad +q_2 \begin{bmatrix} -1 & -1 & 0.6 \\ 0.4 & 0.9 & 0.1 \\ -2.7 & -1.2 & -0.6 \end{bmatrix} \end{array}$$

and

$$q \in \mathcal{Q}_\rho = [-\rho,\ \rho] \times [-\rho,\ \rho].$$

The Hermite matrix for the discrete-time polynomial $h(z, q) = \det\,(zI - A(q))$ of degree 3 in z is given by

$$H = \begin{bmatrix} h_3^2 - h_0^2 & h_2h_3 - h_0h_1 & h_1h_3 - h_0h_2 \\ h_2h_3 - h_0h_1 & h_2^2 + h_3^2 - h_0^2 - h_1^2 & h_2h_3 - h_0h_1 \\ h_1h_3 - h_0h_2 & h_2h_3 - h_0h_1 & h_3^2 - h_0^2 \end{bmatrix}.$$

Polynomial $p(q) = \det H(q)$ has degree 18 in q.

When $\rho = 0.3$, global minimization of polynomial $p(q)$ over polytope $\mathcal{Q}_{0.3}$ is carried out with GloptiPoly. The LMI relaxation of order $k = k_0 = 9$ is unbounded. Enforcing a feasibility radius of 10^3 we obtain (CPU time = 11 secs) the lower bound $\underline{p}_9 = -1.0402 \cdot 10^6$ and the upper bound $\overline{p}_9 = 0.5120$ so we cannot conclude. The next LMI relaxation (order k = 10) returns (CPU time = 17 secs) the global optimum $p^\star = 0.21196$ attained at

$q = [-0.30000,\ -0.30000]$. Matrix $A(q)$ is then robustly stable over $\mathcal{Q}_{0.3}$ in the discrete-time sense.

When $\rho = 0.4$, the LMI relaxation of order $k = 9$ is unbounded. Enforcing a feasibility radius of 10^3, it yields (CPU time = 12 secs) the inconclusive bounds $\underline{p}_9 = -1.0402 \cdot 10^6$ and $\overline{p}_9 = 0.4482$. The second LMI relaxation returns (CPU time = 26 secs) the global optimum $p^\star = 4.5225 \cdot 10^{-9} \approx 0$ attained at $q = [-0.37963,\ -0.38689]$. As a result, matrix $A(q)$ is not robustly stable over $\mathcal{Q}_{0.4}$.

VI. Conclusion

The result in Lemma 1 on the degree of the Lyapunov matrix ensuring robust stability cannot be considered as original, since it follows in a straightforward way from the fact that the Hermite matrix can be chosen as a Lyapunov matrix [17], [18]. However, this result does not seem to be well-known in the control community. For example, a different degree estimate was proposed in [26] but only in the particular case of one uncertain parameter. The result in Lemma 1 can prove useful when following the LMI hierarchy approach proposed in [4], where robust stability is ensured only asymptotically, i.e. for a Lyapunov matrix of finite, yet arbitrary large degree. We showed that it is not necessary to seek a Lyapunov matrix of degree greater than $2nm$, where n is the system order and m the number of uncertain parameters. This degree estimate can be tightened if some information is available on the structure of the uncertainty matrices.

When assessing robust stability, the main difference between our approach and that of [1] or [4] is that we do not explicitly seek a Lyapunov matrix. As a result, we do not optimize over a (generally large) set of Lyapunov variables, but directly on the (generally small) set of uncertain parameters. This approach was already pursued in [7], where a related, yet distinct LMI relaxation technique was proposed to solve multivariate polynomial optimization problems obtained from the Hurwitz stability criterion. Our technique is similar in spirit since the Hermite criterion is a symmetrized version of the Hurwitz stability criterion. Note that however, global convergence of the hierarchy of LMI relaxations to the global optimum is not guaranteed with the approach of [7].

In [7] a comparative study on respective computational complexity of the methods proposed in [4] and [7] was proposed. The large number of different parameters renders such a study cumbersome, and no clear conclusion can be drawn. Moreover, computational complexity estimates are usually crude, pessimistic, and quite far from the actual achieved complexity. The behavior of LMI relaxation methods also strongly depends on the performance of the LMI solver. In our opinion, the best policy is to provide the user with several alternative numerical tools so that he can test them on his problems. This was our objective when developing the GloptiPoly software [10].

Our approach can be extended verbatim to other uncertainty sets $\mathcal{Q}$ than polytopes, as soon as $\mathcal{Q}$ is a compact semi-algebraic set, i.e. described by non-strict multivariable polynomial inequalities. However, in this case, the hierarchy of LMI relaxations described in [13] yields converging lower bounds only. Upper bounds cannot be guaranteed since a feasible point of an LMI relaxation is not necessarily feasible for the original problem when $\mathcal{Q}$ is not a polytope.

Finally, our approach can also be extended to state-space matrices depending polynomially, or even rationally on uncertain parameters. See e.g. [12] for converging hierarchies of LMI relaxations for global optimization over rational functions. In particular, it is not necessary to use linear fractional representations (LFR, see [27]) to cope with polynomial or rational uncertain parameters.

Acknowledgments

D. Henrion acknowledges support from grant No. 102/02/0709 of the Grant Agency of the Czech Republic and project No. ME 698/2003 of the Ministry of Education of the Czech Republic.

References

[1] P. Apkarian, H. D. Tuan. Parametrized LMIs in control theory. SIAM J. Contr. Optim., Vol. 38, No. 4, pp. 1241–1264, 2000.
[2] B. R. Barmish, M. Fu, S. Saleh. Stability of polytope of matrices: counterexamples. IEEE Trans. Autom. Control, Vol. 33, pp. 569-572, 1988
[3] B. R. Barmish. New tools for robustness of linear systems. Macmillan, New York, NY, 1994.
[4] P. A. Bliman. Nonconservative LMI approach to robust stability for systems with uncertain scalar parameters. Proc. IEEE Conf. on Decision and Control, Las Vegas, NE, 2002.
[5] V. D. Blondel, J. N. Tsitsiklis. A survey of computational complexity results in systems and control. Automatica, Vol. 36, No. 9, pp. 1249–1274, 2000.
[6] S. Boyd, L. El Ghaoui, E. Feron, V. Balakrishnan. Linear matrix inequalities in system and control theory. SIAM, Philadelphia, PA, 1994.
[7] G. Chesi. Robust analysis of linear systems affected by time-invariant hybercubic parametric uncertainty. Proc. IEEE Conf. on Decision and Control, Maui, HI, 2003.
[8] E. Feron, P. Apkarian, P. Gahinet. Analysis and synthesis of robust control systems via parameter-dependent Lyapunov functions. IEEE Trans. Autom. Control, Vol. 41, No. 7, pp. 1041–1046, 1996.
[9] F. R. Gantmacher. The theory of matrices. Chelsea, New York, 1959.
[10] D. Henrion, J. B. Lasserre. GloptiPoly: global optimization over polynomials with Matlab and SeDuMi. ACM Trans. Math. Software, Vol. 29, No. 2, pp. 165–194, 2003.
[11] D. Henrion, J. B. Lasserre. Convergent LMI relaxations for nonconvex optimization over polynomials in control. To appear in the IEEE Control Systems Magazine, 2004.
[12] D. Jibetean, E. de Klerk. Global optimization of rational functions: a semidefinite programming approach. Research Report MAS-R0304, CWI Amsterdam, The Netherlands, 2003.
[13] J. B. Lasserre. Global optimization with polynomials and the problem of moments. SIAM J. Optim., Vol. 11, No. 3, pp. 796–817, 2001.
[14] V. J. S. Leite, P. L. D. Peres. An improved LMI condition for robust $\mathcal{D}$-stability of uncertain polytopic systems. IEEE Trans. Autom. Control, Vol. 48, No. 3, pp. 500–504, 2003.
[15] H. Lev-Ari, Y. Bistritz, T. Kailath. Generalized Bezoutians and families of efficient zero-location procedures. IEEE Trans. Circuits and Systems, Vol. 38, No. 2, pp. 170–186, 1991.
[16] M. C. de Oliveira, J. Bernussou, J. C. Geromel. A new discrete-time robust stability condition. Systems and Control Letters, Vol. 37, pp. 261–265, 1999.
[17] P. C. Parks. A new proof of the Routh-Hurwitz stability criterion using the second method of Lyapunov. Proc. Cambridge Phil. Soc., Vol. 58, Part. 4, pp. 694-702, 1962.

[18] P. C. Parks, V. Hahn. Stability theory. Prentice Hall, NY, 1993. Published in German in 1981.
[19] P. A. Parrilo, S. Lall. Semidefinite programming relaxations and algebraic optimization in control. European Journal of Control, Vol. 9, No. 2-3, 2003.
[20] D. Peaucelle, D. Arzelier, O. Bachelier, J. Bernussou. A new robust $\mathcal{D}$-stability condition for real convex polytopic uncertainty. Systems and Control Letters, Vol. 40, pp. 21–30, 2000.
[21] M. Putinar. Positive polynomials on compact semi-algebraic sets. Ind. Univ. Math. J., Vol. 42, pp. 969–984, 1993.
[22] C. W. Scherer. LMI relaxations in robust control: how to reduce conservatism ? Plenary talk at the IFAC Symp. on Robust Control Design, Milan, Italy, 2003.
[23] M. Schweighofer. Optimization of polynomials on compact semi-algebraic sets. Research Report, IRMAR, Université de Rennes, France, 2003.
[24] D. D. Šiljak. Parameter-space methods for robust control design: a guided tour. IEEE Trans. Autom. Control, Vol. 34, pp. 674–688, 1989.
[25] A. Trofino. Parameter dependent Lyapunov functions for a class of uncertain linear systems: an LMI approach. Proc. IEEE Conf. on Decision and Control, Phoenix, AZ, 1999.
[26] X. Zhang, P. Tsiotras, T. Iwasaki. Parameter-dependent Lyapunov functions for stability analysis of LTI parameter dependent systems. Proc. IEEE Conf. on Decision and Control, Maui, HI, 2003.
[27] K. Zhou, J. C. Doyle, K. Glover. Robust and optimal control. Prentice Hall, Upper Saddle River, NJ, 1996.