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Finding largest small polygons with GloptiPoly

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Abstract A small polygon is a convex polygon of unit diameter. We are interested in small polygons which have the largest area for a given number of vertices n. Many instances are already solved in the literature, namely for all odd n, and for $n = 4, 6$ and 8. Thus, for even $n \geq 10$, instances of this problem remain open. Finding those largest small polygons can be formulated as nonconvex quadratic programming problems which can challenge state-of-the-art global optimization algorithms. We show that a recently developed technique for global polynomial optimization, based on a semidefinite programming approach to the generalized problem of moments and implemented in the public-domain Matlab package GloptiPoly, can successfully find largest small polygons for $n = 10$ and $n = 12$. Therefore this significantly improves existing results in the domain. When coupled with accurate convex conic solvers, GloptiPoly can provide numerical guarantees of global optimality, as well as rigorous guarantees relying on interval arithmetic.

Keywords: extremal convex polygons, global optimization, nonconvex quadratic programming, semidefinite programming

1. Introduction

The problem of finding the largest small polygons was first studied by Reinhardt in 1922 [13]. Reinhardt solved the problem by proving that the solution corresponds to the regular polygons but only when the number of vertices n is odd. He also solved the case $n = 4$ by proving that a square with diagonal length equal to 1 is a solution. However, it exists an infinity of other different solutions (it is just necessary that the two diagonals intersect with a right angle). The hexagonal case $n = 6$ was solved numerically by Graham in 1975 [6]. Indeed, Graham studied possible structures that the optimal solution must have. He introduced the diameter graph of a polygon which is defined by the vertices of the polygon and by edges with length one (if and only if the corresponding two vertices of the edge are at distance one). Using a result due to Woodall [14], he proved that the diameter graph of the largest small polygons must be connected, yielding 10 distinct possible configurations for $n = 6$. Discarding 9 of these 10 possibilities by using standard geometrical reasonings plus the fact that all the candidates must have an area greater than the regular small hexagon, he determined the only possible diameter graph configuration which can provide a better solution. He solved this last case numerically, yielding the largest small hexagon. Following the same principle, Audet et al. in 2002 found the largest small octagon [4]. The case $n = 8$ is much more complicated than the case $n = 6$ because it generates 31 possible configurations and just a few of them can be easily discarded by geometrical reasonings. Furthermore, for the remaining cases, Audet et al. had to solve difficult global optimization problems with 10 variables and about 20 constraints. These problems are formulated as quadratic programs with quadratic constraints [4]. Audet et al. used for that a global solver named QP [1]. Notice that optimal solutions for $n = 6$ and $n = 8$ are not the regular polygons [4, 6]. In 1975, Graham proposed a conjecture which is the following: when n is even and $n \geq 4$, the largest small polygon must have a diameter graph with a cycle with $n-1$ vertices and with an additional edge attached to a vertex of the cycle; this is true for $n = 4, 6$ and also $n = 8$, see Figure 1. Therefore, this yields only one possible diameter graph configuration that must have the optimal solution. In 2007, Foster and Szabo proved Graham's conjecture [5]. Thus to solve the following open cases $n \geq 10$, it is just necessary to solve one global optimization problem defined by the configuration of the diameter graph with a cycle with $n-1$ vertices and an additional pending edge. In order to have an overview of these subjects, refer to [2, 3].

2. Nonconvex quadratic programming

As mentioned above, for an even $n \geq 4$, finding the largest small polygon with n vertices amounts to solving only one global optimization problem. All these problems depending on n can be formulated as nonconvex quadratic programs under quadratic constraints [4]. For illustration, here is the problem corresponding to the case $n = 8$ (the definitions of the variables are given in Figure 1):

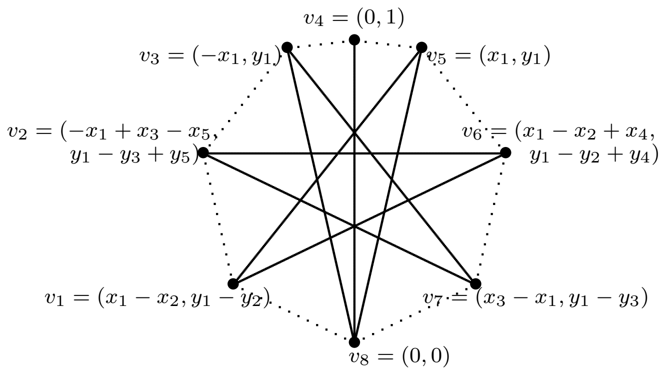

Figure 1. Case of $n = 8$ vertices. Definition of variables following Graham's conjecture.

$$\left\{ \begin{array}{lll} \max\limits_{x,y} & \frac{1}{2}\{(x_2+x_3-4x_1)y_1 + (3x_1-2x_3+x_5)y_2 + (3x_1-2x_2+x_4)y_3 & \\ & \qquad +(x_3-2x_1)y_4 + (x_2-2x_1)y_5\} + x_1 & \\ \text{s.t.} & \|v_i - v_j\| \leq 1, \forall (i,j) \in \{1,\cdots,8\}, i \neq j & \\ & \|v_2 - v_6\| = 1 & \\ & x_i^2 + y_i^2 = 1 & i = 1,2,3,4,5 \\ & x_2 - x_3 \geq 0 & y \geq 0 \\ & 0 \leq x_1 \leq 0.5 & 0 \leq x_i \leq 1, \quad i = 2,3,4,5. \end{array} \right. \tag{1}$$

Without loss of generality we can insert the additional constraint $x_2 \geq x_3$ which eliminates a symmetry axis. In program (1), all the constraints are quadratic. The quadratic objective function corresponds to the computation of the area of the octagon following Graham's diameter graph configuration. This formulation is easy to extend to the cases $n \geq 10$ with n even.

3. GloptiPoly

In 2000, Lasserre proposed to reformulate nonconvex polynomial optimization problems (POPs) as linear moment problems, in turn formulated as linear semidefinite programming (SDP) problems [10]. Using results on flat extensions of moment matrices and representations of polynomials positive on semialgebraic sets, it was shown that under some relatively mild assumptions, solving nonconvex POPs amounts to solving a sufficiently large linear hence convex SDP problem. In practice, a hierarchy of embedded SDP relaxations of increasing size are solved gradually. Convergence and hence global optimality can be guaranteed by examining a certain rank pattern in the moment matrix, a simple task of numerical linear algebra. A user-friendly Matlab interface called GloptiPoly was designed in 2002 to transform a given POP into an SDP relaxation of given size in the hierarchy, and then to call SeDuMi, a general-purpose conic solver [7]. A new version 3 was released in 2007 to address generalized problem of moments, including POPs but also many other decision problems. The interface was also extended to other public-domain conic solvers [8]. Almost a decade after the initial spark [10], Lasserre summarized the theoretical and practical sides of the approach in a monograph [11].

4. Numerical experiments

We applied GloptiPoly 3 and SeDuMi 1.1R3 to solve the quadratic problem in the cases $n = 8$ and 10. In order to obtain accurate solutions, we let SeDuMi minimize the duality gap as much as possible. We also tightened the tolerance parameters used by GloptiPoly to detect global optimality and extract globally optimal solutions. We used a 32 bit desktop personal computer with a standard configuration.

For the case $n = 8$ we obtain the solution (with 8 significant digits) $x_1 = 0.26214172$, $x_2 = 0.67123417$, $x_3 = 0.67123381$, $x_4 = 0.90909242$, $x_5 = 0.90909213$ whose global optimality is guaranteed numerically (the moment matrix has approximately rank one) at the second SDP relaxation in the hierarchy. This SDP problem is solved by SeDuMi in less than 5 seconds. The objective function of the SDP relaxation is equal to 0.72686849, and this is an upper bound on the exact global optimum. The quadratic objective function evaluated at the above solution is the same to 11 significant digits. Symmetry considerations indicate that $x_2 = x_3$ and $x_4 = x_5$ at the optimum, and we see that the above solution achieves this to 5 digits for x_2 and to 6 digits for x_4.

These results can be rigorously guaranteed by using Jansson's VSDP package which uses SDP jointly with interval arithmetic [9]. The solution of an SDP problem can be guaranteed at the price of solving a certain number of SDP problems of the same size. In our case, VSDP solved 8 instances of the second SDP relaxation to provide the guaranteed lower bound 0.72686845 and guaranteed upper bound 0.72686849 on the objective function, namely the area of the octagon.

In the case $n = 10$, we obtain the solution $x_1 = 0.21101191$, $x_2 = 0.54864468$, $x_3 = 0.54864311$, $x_4 = 0.78292524$, $x_5 = 0.78292347$, $x_6 = 0.94529290$, $x_7 = 0.94529183$ whose global optimality is guaranteed numerically at the second SDP relaxation. This SDP problem is solved by SeDuMi in less than 5 minutes. The objective function of the SDP relaxation, an upper bound on the exact global optimum, is equal to 0.74913736. The quadratic objective function evaluated at the above solution is the same to 10 significant digits.

For $n = 12$, we obtain the following solution without using the rigorous method of SDP: $x_1 = 0.17616131$, $x_2 = 0.46150224$, $x_3 = 0.46150519$, $x_4 = 0.67623091$, $x_5 = 0.67623301$, $x_6 = 0.85320300$, $x_7 = 0.85320328$, $x_8 = 0.96231370$, $x_9 = 0.96231344$. This SDP problem is solved within 1h06. The objective function of the SDP relaxation, an upper bound on the exact global optimum, is equal to 0.76072988. The solutions for the optimal decagon and dodecagon are drawn in Figure 2.

5. Conclusion

GloptiPoly can be efficiently used to find some largest small polygons with an even number of vertices. The case $n = 8$ is most efficiently solved than in [4]: (i) the accuracy on the value of the area is now 10^{-10} in place of 10^{-5} and (ii) the required CPU time is about 5 seconds in place of 100 hours. Furthermore, the next open instance for $n = 10$ is solved using GloptiPoly in only 5 minutes with always an accuracy of 10^{-10}. These two results are obtained with a

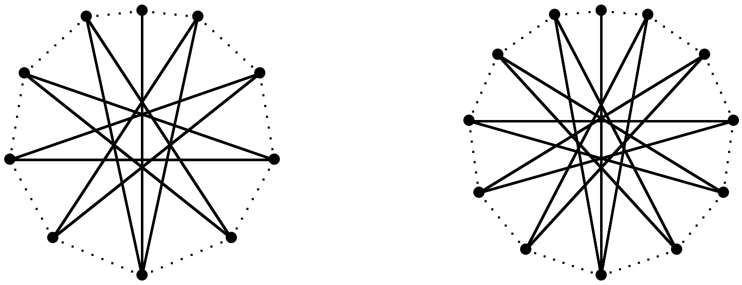

Figure 2. Largest Small Decagon and Dodecagon.

certified guarantee on 10 digits. For the case $n = 12$, GloptiPoly found the global solution, but for the moment without a certified guarantee. In future works, we have to certified and guarantee the solution obtained for the case $n = 12$. It seems to be also possible to solve the next open case $n = 14$. Note that all the found largest small polygons with an even number n of vertices (from $n = 4$ to 12) own a symmetry axis on the pending edge of their corresponding optimal diameter graph configurations.

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