$\mathcal{D}$-Stability of Polynomial Matrices[1]

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Abstract

Necessary and sufficient conditions are formulated for the zeros of an arbitrary polynomial matrix to belong to a given region $\mathcal{D}$ of the complex plane. The conditions stem from a general optimization methodology mixing quadratic and semidefinite programming, LFRs and rank-one LMIs. They are expressed as an LMI feasibility problem that can be tackled with widespread powerful interior-point methods. Most importantly, the $\mathcal{D}$-stability conditions can be combined with other LMI conditions arising in robust stability analysis.

Keywords

Polynomial Matrix, $\mathcal{D}$-stability, LMI.

1 Introduction

Polynomial matrices play a central role in modern systems theory. Algebraic methods such as the polynomial approach [25] or the behavioral approach [37] heavily rely upon polynomial matrices. Unsurprisingly, fundamental system features are captured by properties of polynomial matrices. For example, the zeros of the denominator polynomial matrix in

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a matrix fraction description [23] characterize system dynamics and performance. Satisfactory transient time response can be ensured as soon as the zeros are located in some specific region of the complex plane [1]. The present paper is precisely concerned with checking whether the zeros of a given polynomial matrix belong to a given region $\mathcal{D}$. Following a terminology introduced in [7], this problem is henceforth referred to as the $\mathcal{D}$-stability problem.

In the special case of polynomials, i.e. scalar polynomial matrices, assessing stability is a long-standing problem that has motivated a lot of work since the last century. In order to check $\mathcal{D}$-stability in the left-half plane (for continuous-time systems), in the unit disk (for discrete-time systems) or in more general regions (for ensuring system performance), two approaches may be pursued. One can alternatively

- Compute the zeros of the polynomial, see [30] for a recent survey and [10] for promising new developments. A standard way to proceed consists in computing the eigenvalues of the companion matrix associated with the polynomial, using for example the numerically stable Schur decomposition [16];
- Use indirect methods such as the determinantal Routh-Hurwitz or Schur-Cohn criteria or the quadratic Hermite-Fujiwara criterion, see [22].

The case of polynomial matrices is far more involved. We can basically distinguish between two approaches to $\mathcal{D}$-stability. One can alternatively

- First, compute the determinant of the polynomial matrix [32], which may be numerically difficult although significant progress has been reported recently [21, 19]. Second, use one of the two methods depicted above on the resulting scalar polynomial;
- Compute the zeros of the polynomial matrix [32]. Here also, a standard technique is to compute the eigenvalues of the pencil matrix associated with the polynomial matrix [26], using for example the numerically stable QZ decomposition [16]. As pointed out in [10], some numerical difficulties may be expected with high multiplicity zeros.

Note that, up to the authors' knowledge, there is no satisfactory extension of the scalar indirect methods to the matrix case. Some attempts were reported to extend the Hermite-Fujiwara criterion [3] or the Routh-Hurwitz criterion [24] to polynomial matrices, but no unified approach has appeared so far. Finally, let us point out that the problem of checking stability of polynomial matrices is difficult enough to motivate the development of potentially conservative stability conditions that do not require the computation of the determinant [2, 29]. Note also that these conditions are only valid for checking stability in the unit disk.

In view of this unsatisfactory state of the art, this paper is an attempt to overcome the lack of sufficiently general methods for studying stability of polynomial matrices without performing tedious determinant computations or numerically hazardous evaluations of the zeros. The approach we pursue is based on an idea that was suggested by [8] and that can be traced back to [31] and [27]. The stability problem for polynomial matrices is first expressed as an optimization problem. Then, several techniques are used to come up with a standard formulation of the problem. The basic concepts we hinge upon are Linear Fractional Representations (LFRs, see [39]), rank-constrained Linear Matrix Inequalities (LMIs, see [9]), quadratic and semidefinite programming [35]. The strengths of our method are enumerated below

- Polynomial matrices with real or complex coefficients can be handled;
- Covered stability regions include half-planes, disks, parabolas or their non-convex complements or possibly non-connected unions, not necessarily symmetric with respect to the real axis;
- The stability conditions are expressed as an LMI, a linear feasibility problem over the cone of positive definite matrices [35] that can be solved through powerful interior-point methods [28];
- Most importantly, the stability conditions can be combined with other conditions of similar type arising in robust stability analysis problems, when some elements in the polynomial matrix are uncertain. This extension will be covered elsewhere.

The outline of the paper is as follows. In Section 2, we state the $\mathcal{D}$-stability problem to be solved. The stability regions covered by our approach are described in Section 3. A convex LMI formulation of the stability conditions is derived in Section 4. This is our main result. In Section 5, several examples are provided to illustrate the method. Finally, the paper winds up with some concluding remarks and directions for future research.

Notations: $\mathbb{R}$ and $\mathbb{C}$ are the sets of real and complex numbers, respectively. $s^\star = x - jy$ is the complex conjugate of the complex number $s = x + jy$. $A^\star(s)$ is the transpose conjugate of the complex polynomial matrix $A(s)$ in the indeterminate s. I_n denotes the identity matrix of dimension n. Trace$A$ is the trace of matrix A. Rank$A$ is the rank of matrix A. The matrix inequalities $A \succ B$ and $A \succeq B$ mean that matrix $A - B$ is positive definite and positive semidefinite, respectively. $A \otimes B$ stands for the Kronecker product of two matrices A and B.

2 Problem Statement

Suppose we are given a non-singular $n \times n$ complex polynomial matrix

$$A(s) = A_0 + A_1 s + \cdots + A_d s^d \in \mathbb{C}^{n\times n}[s]$$

of degree d together with a region

$$\mathcal{D} \subset \mathbb{C}$$

of the complex plane.

A zero of $A(s)$ is usually defined [23] as a complex value $z \in \mathbb{C}$ for which the rank of $A(s)$ drops below its normal value, i.e.

$$\mathrm{Rank}\, A(z) < \mathrm{Rank}\, A(s).$$

Without computing the determinant or the zeros of $A(s)$, we aim at finding necessary and sufficient conditions for $\mathcal{D}$-stability of $A(s)$, i.e. for the zeros of $A(s)$ to belong to $\mathcal{D}$.

3 Stability Regions

First we describe the class of regions $\mathcal{D}$ we will consider throughout the paper.

Define

$$\mathcal{D}^C = \{s \in \mathbb{C} : s \notin \mathcal{D}\} \subset \mathbb{C}$$

as the complement of region $\mathcal{D}$ in $\mathbb{C}$. In this paper, we restrict our attention to regions $\mathcal{D}$ whose complement reads

$$\mathcal{D}^C = \{s \in \mathbb{C} : B^{00} + B^{10}s + B^{01}s^\star + B^{11}ss^\star \succeq 0\} \tag{1}$$

where matrices $B^{gh} \in \mathbb{C}^{k \times k}$ are such that non-singular matrix

$$B = B^\star = \begin{bmatrix} B^{00} & B^{01} \\ B^{10} & B^{11} \end{bmatrix} \in \mathbb{C}^{2k \times 2k} \tag{2}$$

has at least one negative eigenvalue.

3.1 First Order Regions

Let $k = 1$. We can define three different regions in this case.

- Half-plane $\mathcal{D} = \{x + jy \in \mathbb{C} : ax + by + c < 0\}$ with $a, b, c \in \mathbb{R}$ and
$$B = \begin{bmatrix} 2c & a + jb \\ a - jb & 0 \end{bmatrix}.$$
- Disk interior $\mathcal{D} = \{s \in \mathbb{C} : |s - s_0| < r\}$ with $s_0 \in \mathbb{C}$, $r > 0 \in \mathbb{R}$ and
$$B = \begin{bmatrix} -r^2 + s_0 s_0^\star & -s_0 \\ -s_0^\star & 1 \end{bmatrix}.$$
- Disk exterior $\mathcal{D} = \{s \in \mathbb{C} : |s - s_0| > r\}$ with $s_0 \in \mathbb{C}$, $r > 0 \in \mathbb{R}$ and
$$B = \begin{bmatrix} r^2 - s_0 s_0^\star & s_0 \\ s_0^\star & -1 \end{bmatrix}.$$

3.2 Second Order Regions

When $k = 2$, we can describe the following useful regions.

- Ellipsoid $\mathcal{D} = \{x + jy \in \mathbb{C} : a^2(x - x_0)^2 + b^2(y - y_0)^2 < 1\}$ with $a, b, x_0, y_0 \in \mathbb{R}$ and

$$B = \left[\begin{array}{cc|cc} 4(a^2x_0^2 + b^2y_0^2 - 1) & 0 & -4(a^2x_0 + jb^2y_0) & 1 \\ 0 & 1/(b^2 - a^2) & 1 & 0 \\ \hline -4(a^2x_0 - jb^2y_0) & 1 & 4b^2 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \quad \text{when } a^2 < b^2,$$

$$B = \left[\begin{array}{cc|cc} 4(a^2x_0^2 + b^2y_0^2 - 1) & 0 & -4(a^2x_0 + jb^2y_0) & j \\ 0 & 1/(a^2 - b^2) & j & 0 \\ \hline -4(a^2x_0 - jb^2y_0) & -j & 4a^2 & 0 \\ -j & 0 & 0 & 0 \end{array}\right] \quad \text{when } a^2 > b^2.$$

- Parabola $\mathcal{D} = \{x + jy \in \mathbb{C} : x + x_0 + a^2y^2 < 0\}$ with $a, x_0, y_0 \in \mathbb{R}$ and

$$B = \left[\begin{array}{cc|cc} 4x_0 & 0 & 2 & a \\ 0 & 1 & a & 0 \\ \hline 2 & a & 4a^2 & 0 \\ a & 0 & 0 & 0 \end{array}\right].$$

A wide variety of other regions can be described when $k = 2$, but they are generally irrelevant as far as zero clustering is concerned.

3.3 Union of Regions

The regions mentioned above are connected. Such regions shall cover many practical problems but some specific investigations require more sophisticated regions such as unions of possibly disjoint subregions. This is particularly true when one wants to test the transient performances of a multi-time scale system. It would be quite inappropriate to check stability in a convex region containing the whole of dynamics. For instance, if a system has two separate dynamics, it is more judicious to choose $\mathcal{D}$ as a union of two disjoint subregions, one for the fast poles and the other one for the slow poles. As another example, the damping factor induced by a couple of dominant complex eigenvalues may be handled with a couple of disjoint subregions (such as disks or ellipses) symmetric to each other with respect to the real axis.

Actually, our formulation can easily handle union of regions, as in [6]. To see this, note that the intersection of two complementary regions

$$\mathcal{D}_i^C = \{s \in \mathbb{C} : B_i^{00} + B_i^{10}s + B_i^{01}s^\star + B_i^{11}ss^\star \succeq 0\}$$

for $i = 1, 2$ is a complementary region

$$\mathcal{D}^C = \{s \in \mathbb{C} : \begin{bmatrix} B_1^{00} & 0 \\ 0 & B_2^{00} \end{bmatrix} + \begin{bmatrix} B_1^{10} & 0 \\ 0 & B_2^{10} \end{bmatrix} s + \begin{bmatrix} B_1^{01} & 0 \\ 0 & B_2^{01} \end{bmatrix} s^\star + \begin{bmatrix} B_1^{11} & 0 \\ 0 & B_2^{11} \end{bmatrix} ss^\star \succeq 0\}.$$

Therefore, $s \in \mathcal{D}_1$ or $s \in \mathcal{D}_2$ if and only if $s \in \mathcal{D}$, where $\mathcal{D}$ is the union of $\mathcal{D}_1$ and $\mathcal{D}_2$. This property holds for any arbitrary number of regions. Since any region of the complex plane can be represented by the union of half-planes and disks at any desired accuracy, we conclude that the regions described in this paper are dense in the set of regions of the complex plane. These regions may be open, non-convex or even non-connected.

Note however that our formulation cannot easily handle intersection of regions. This is believed to be the major hurdle of our approach. For example, we were not able to find a suitable expression for describing a conic sector in the left half-plane, a typical clustering region.

3.4 Zeros at Infinity

Special care must be taken when studying stability of polynomial matrices in open regions such as disk complements or half-planes. This is because polynomial matrices generally feature zeros at infinity, i.e. when $ss^\star \to \infty$. The reader is referred to [36] for an accurate definition and a comprehensive treatment of zeros at infinity.

When studying stability in open regions, it may thus be necessary to exclude infinite zeros of polynomial matrices. For instance, when studying stability in the left half-plane $\mathcal{D} = \{s \in \mathbb{C} : s + s^\star < 0\}$, the complementary right half-plane $\mathcal{D}^C = \{s \in \mathbb{C} : s + s^\star \geq 0\}$ may be replaced by the half disk $\mathcal{D}_r^C = \{s \in \mathbb{C} : s + s^\star \geq 0,\ ss^\star \leq r^2\}$ for a sufficiently large $r \in \mathbb{R}$. Infinite zeros may belong to open regions $\mathcal{D}$ and $\mathcal{D}^C$, but definitely do not belong to bounded region $\mathcal{D}_r^C$ for finite r.

3.5 Relation to Standard Stability Regions

3.5.1 Ω-Regions

Ω-regions, proposed in [17], are described by the so-called polynomial formulation

$$\mathcal{D} = \{s \in \mathbb{C} : \sum_{e,f} B^{ef} s^e (s^\star)^f < 0\}$$

where e, f are integers such that $e \geq 0$,$f \geq 0$ and $e + f \leq d$. Integer d is called the order of the region. In most practical problems we encounter first order regions ($d = 1$) such as half-planes or second order regions ($d = 2$) such as disks, ellipses, hyperbolas, parabolas, as well as vertical and horizontal strips.

It must be noticed that our complementary description (1) of stability regions can always be written as a finite set of Ω-regions. It suffices to enumerate the set of determinantal minors of the matrix expression in (1) to see that non-negativity of each minor can be expressed as an Ω-region.

3.5.2 LMI Regions

Ω-regions are not always convenient because of possible nonlinearities or even nonconvexities. That is why a more recent formulation of regions has been recently introduced in [11, 12] and widely used afterwards. The formulation of an LMI region is as follows

$$\mathcal{D} = \{s \in \mathbb{C} : B^{00} + B^{10}s + B^{01}s^{\star} \prec 0\}$$

where $B^{00} = (B^{00})^{\star} \in \mathbb{R}^{d\times d}$ and $B^{10} = (B^{01})^{\star} \in \mathbb{R}^{d\times d}$ are given matrices. Integer d is called the order of the LMI region. LMI regions are symmetric with respect to the real axis. The set of first order LMI regions corresponds to vertical half-planes while second order LMI regions include disks, ellipses, classical and hyperbolic sectors, vertical and horizontal strips.

The set of LMI regions is different from the set of Ω-regions since an Ω-region is not necessarily convex. Actually, none of these sets includes the other one and both can be of great practical interest. The LMI formulation can sometimes be obtained from the polynomial formulation by applying Schur complement. This can be seen as a way to reduce the degree of the scalar inequality by turning it into a matrix inequality. Note finally that many unions of possibly disjoint and non-symmetric convex LMI subregions can be formulated as generalized LMI regions, an extension of LMI regions [11, 5].

Our complementary region $\mathcal{D}^C$ defined in equation (1) reduces to an LMI region, provided $B^{11} = 0$. In this case, region $\mathcal{D}$ is a (possibly non-convex) complementary LMI region.

4 LMI Formulation of the Problem

Following these preliminaries, we can now derive an LMI formulation of a necessary and sufficient condition for $\mathcal{D}$-stability of a polynomial matrix $A(s)$. In the sequel, it is assumed that $A(s)$ has full column-rank. If the input matrix has full row-rank, then we work on its transpose.

The main steps of our approach can be sketched as follows. In Section 4.1, we show that checking $\mathcal{D}$-stability amounts to solving a quadratic optimization problem. Using LFRs, we derive a rank-one LMI formulation of this problem in Section 4.2. In Section 4.3 we show that the non-convex rank constraint can be dropped, leading to a tractable and general LMI formulation of $\mathcal{D}$-stability conditions. Finally, in Section 4.4 we propose an equivalent, but more compact, dual LMI $\mathcal{D}$-stability condition.

4.1 Quadratic Optimization Problem

If $z \in \mathbb{C}$ is a zero of $A(s)$, then there exists a non-zero vector $v \in \mathbb{C}^n$ such that

$$A(z)v = 0.$$

Consequently, all the zeros of $A(s)$ belong to $\mathcal{D}$ if and only if the optimal value μ of the quadratic optimization problem

$$\begin{array}{rcll} \mu & = & \min & v^\star A^\star(s)A(s)v \\ & & \text{s.t.} & s \in \mathcal{D}^C \end{array} \tag{3}$$

is strictly positive. To avoid triviality and for the above feasible set to be compact, we will henceforth assume that $v^\star v = 1$ in problem (3).

4.2 Rank-one LMI Problem

Now we show that solving quadratic optimization problem (3) amounts to solving a rank-one LMI optimization problem. We shall use the following important result. Let $r = dn$.

Lemma 1 *Polynomial matrix $A(s)$ has the following Linear Fractional Representation (LFR)*

$$A(s) = A_0 + L\Delta(I_r - D\Delta)^{-1}R.$$

In the above equation $\Delta = sI_r$ and

$$\left[\begin{array}{c|c} A_0 & L \\ \hline R & D \end{array}\right] = \left[\begin{array}{c|cccc} A_0 & A_1 & \cdots & A_{d-1} & A_d \\ \hline I_n & 0 & & 0 & 0 \\ 0 & I_n & & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & & 0 & 0 \\ 0 & 0 & & I_n & 0 \end{array}\right].$$

Proof of Lemma 1 See [39]. □

In relation to the above LFR, we define the following matrices

$$\left[\begin{array}{c} \mathcal{A} \\ \hline \mathcal{Q} \end{array}\right] = \left[\begin{array}{c|c} \mathcal{B} & \mathcal{R} \end{array}\right] = \left[\begin{array}{c|c} A_0 & L \\ \hline R & D \end{array}\right] \tag{4}$$

where $\mathcal{A} \in \mathbb{C}^{n\times(r+n)}$ and $\mathcal{Q} \in \mathbb{R}^{r\times(r+n)}$. In order to avoid redundancy in the representation, we may perform a series of reductions as described in the Appendix. This preliminary step is not necessary, but it can significantly reduce problem dimensions.

We can define vectors $p, q \in \mathbb{C}^r$ such that

$$\begin{array}{rcl} A(s)v & = & A_0 v + Lp \\ q & = & Rv + Dp \\ p & = & \Delta q \end{array} \tag{5}$$

Lemma 2 *Let B a Hermitian matrix and $\mathcal{D}^C$ the corresponding region as defined in equation (1). Then there exist vectors $p, q \in \mathbb{C}^r$ such that*

$$\begin{array}{rcl} Q & = & [I_k \otimes q \quad I_k \otimes p]B[I_k \otimes q \quad I_k \otimes p]^\star \\ & = & B^{00} \otimes qq^\star + B^{10} \otimes pq^\star + B^{01} \otimes qp^\star + B^{11} \otimes pp^\star \quad \succeq \quad 0 \end{array} \tag{6}$$

if and only if $p = sq$ and $s \in \mathcal{D}^C$.

Proof of Lemma 2 The proof is adapted from Lemma C.1 in [13]. If $r = 1$, p and q are scalars and the result readily follows. Now suppose that $r > 1$. Assume that p and q are linearly independent vectors. Recall that rank $B = 2k$ by assumption. Then it follows from Sylvester's rank inequality that rank $Q = 2k$. Since matrix B has at least one negative eigenvalue by assumption, matrix Q has also at least one negative eigenvalue, which contradicts inequality (6). Equivalently, inequality (6) implies that p and q are linearly dependent vectors, i.e. there exists a scalar s such that $p = sq$. In this case, matrix

$$Q = (B^{00} + B^{10}s + B^{01}s^\star + B^{11}ss^\star) \otimes qq^\star$$

is positive semidefinite for any vector q. Therefore, scalar s belongs to region $\mathcal{D}^C$. □

Define the rank-one positive semidefinite matrix

$$X = xx^\star = \left[\begin{array}{c} v \\ p \end{array}\right]\left[\begin{array}{c} v \\ p \end{array}\right]^\star \succeq 0 \tag{7}$$

and the projection matrix

$$\mathcal{P} = [0 \quad I_r] = \left[\begin{array}{c|cccc} 0 & I_n & & & \\ 0 & & I_n & & \\ \vdots & & & \ddots & \\ 0 & & & & I_n \end{array}\right] \in \mathbb{R}^{r\times(r+n)}.$$

In virtue of equations (4) and (5), it holds

$$\begin{array}{rcl} A(s)v & = & \mathcal{A}x \\ q & = & \mathcal{Q}x \\ p & = & \mathcal{P}x. \end{array}$$

Inequality (6) is therefore an LMI in matrix X. Let us denote it by a linear map

$$F(X) = B^{00} \otimes (\mathcal{Q}X\mathcal{Q}^\star) + B^{10} \otimes (\mathcal{P}X\mathcal{Q}^\star) + B^{01} \otimes (\mathcal{Q}X\mathcal{P}^\star) + B^{11} \otimes (\mathcal{P}X\mathcal{P}^\star). \tag{8}$$

Using equations (5), (7) and (8), an alternative formulation of problem (3) can now be derived. It reads

$$\begin{array}{rcll} \mu & = & \min & \mathrm{Trace}\mathcal{A}^\star\mathcal{A}X \\ & & \text{s.t.} & F(X) \succeq 0 \\ & & & X = X^\star \succeq 0 \\ & & & \mathrm{Rank}X = 1. \end{array} \tag{9}$$

Problem (9) is an LMI optimization problem with a non-convex rank constraint. It must be pointed out that rank-constrained LMIs frequently arise in control problems

[14, 27, 20] but also in mathematical programming and combinatorial optimization. In [28], it is shown that all of the problems with polynomial objective and polynomial constraints (including quadratic programming, binary programming, integer programming, to mention just a few) may actually be formulated as rank-constrained LMI optimization problems.

4.3 LMI Formulation

In this section, we show that the non-convex rank constraint in problem (9) is irrelevant.

4.3.1 Preliminaries

Since polynomial matrix

$$A(s) = \mathcal{A} \begin{bmatrix} I_n \\ sI_n \\ \vdots \\ s^d I_n \end{bmatrix} \tag{10}$$

has full column-rank n, it follows that constant matrix $\mathcal{A}$ has full row-rank n. The dimension of its right null-space is therefore at most r. Let $\mathcal{N} \in \mathbb{C}^{(r+n)\times r}$ denote a matrix whose columns span this vector space, i.e.

$$\mathcal{A}\mathcal{N} = 0.$$

For simplicity and without loss of generality, it is assumed in the sequel that $A(s)$ has no repeated zeros. Repeated zeros can be handled at the price of heavier notations, see e.g. [4]. It is easy to check that $A(s)$ features exactly r zeros, some of which may be infinite [36].

Let $z_i \in \mathbb{C}$ be a zero of $A(s)$ and let $v_i \in \mathbb{C}^n$ be the correponding non-zero vector such that

$$A(z_i)v_i = 0.$$

In view of equation (10), the vector

$$x_i = \begin{bmatrix} v_i \\ z_i v_i \\ \vdots \\ z_i^d v_i \end{bmatrix}$$

belongs to the right null-space of $\mathcal{A}$. Let

$$V = [x_1 \quad \cdots \quad x_r]$$

denote a matrix built up from all the vectors $x_i \in \mathbb{C}^{n+r}$ associated with all the zeros z_i of $A(s)$. It follows from the above discussion that the columns of $\mathcal{N}$ and V span the same vector space.

Finally, partition vector x_i as

$$x_i = \begin{bmatrix} v_i \\ p_i \end{bmatrix} = \begin{bmatrix} q_i \\ z_i^d v_i \end{bmatrix} \tag{11}$$

where vectors $p_i, q_i \in \mathbb{C}^r$ are such that $p_i = z_i q_i$. By definition, vectors v_i and hence vectors q_i are linearly independent for $i = 1, \ldots, r$. Thus we can define vectors $\bar{q}_i \in \mathbb{C}^r$ such that

$$\begin{bmatrix} \bar{q}_1 & \cdots & \bar{q}_r \end{bmatrix}^\star \begin{bmatrix} q_1 & \cdots & q_r \end{bmatrix} = I_r, \tag{12}$$

i.e. $\bar{q}_i^\star q_j = 0$ when $i \neq j$ and $\bar{q}_i^\star q_i = 1$.

4.3.2 Rank Relaxation

Following these preliminaries, let us now define the following relaxation of rank-one LMI problem (9)

$$\begin{array}{rcll} \nu & = & \min & \mathrm{Trace}\mathcal{A}^\star\mathcal{A}X \\ & & \text{s.t.} & F(X) \succeq 0 \\ & & & X = X^\star \succeq 0 \end{array} \tag{13}$$

where the non-convex rank constraint has been dropped. Since the feasible set in problem (13) is the convex hull of the non-convex feasible set in problem (9), LMI optimization problem (13) is referred to as the convex relaxation of the non-convex rank-one LMI problem (9). Such relaxations are frequently used in combinatorial optimization and control, see [35, 20] and references therein.

In relation to the above problem, we can state the following central result.

Lemma 3 *$\mu > 0$ in rank-one LMI problem (9) if and only if $\nu > 0$ in LMI problem (13).*

Proof The inner product of positive semi-definite matrices $\mathcal{A}^\star\mathcal{A}$ and X is always non-negative, hence $\nu \geq 0$. Moreover, the fact that $\nu > 0$ implies $\mu > 0$ is trivial since the feasible set in problem (9) is a subset of the feasible set in problem (13), i.e. it holds $\mu \geq \nu$. Consequently, in order to show that $\mu > 0$ implies $\nu > 0$, the remainder of the proof will consist in proving that $\nu = 0$ implies $\mu = 0$. So suppose that X is a positive semi-definite matrix such that $\nu = 0$ in problem (13). Let W be a full column-rank matrix such that $X = WW^\star$. By putting matrix $\mathcal{A}^\star\mathcal{A}$ into Schur form, it can easily be shown that $\mathrm{Trace}\mathcal{A}^\star\mathcal{A}WW^\star = 0$ implies $\mathcal{A}^\star\mathcal{A}WW^\star = 0$. Consequently, the columns of W span a subspace that belongs to the right null-space of $\mathcal{A}$. In view of the above definition of matrix V, there exists a matrix M such that $W = VM$. Since matrix X cannot be zero by assumption, $MM^\star \in \mathbb{C}^{r\times r}$ is a non-zero positive semi-definite matrix whose entries are denoted by m_{ij}. Obviously, $m_{ii} \geq 0$ for $i = 1, \ldots, r$ and there necessarily exists one index l such that $m_{ll} > 0$ otherwise $MM^\star = 0$. Matrix X is feasible for problem (13) thus

$$F(X) = F(VMM^\star V^\star) = \sum_{i=1}^{r}\sum_{j=1}^{r} m_{ij} F(x_i x_j^\star) \succeq 0.$$

Recalling equations (11) and (12), it holds

$$\begin{array}{rcl}(I_k \otimes \bar{q}_i^\star)F(X)(I_k \otimes \bar{q}_i) & = & m_{ii}(I_k \otimes \bar{q}_i^\star)F(x_i x_i^\star)(I_k \otimes \bar{q}_i) \\ & = & m_{ii}(I_k \otimes \bar{q}_i^\star)[(B^{00} + B^{10}z_i + B^{01}z_i^\star + B^{11}z_i z_i^\star) \otimes q_i q_i^\star](I_k \otimes \bar{q}_i) \\ & = & m_{ii}(B^{00} + B^{10}z_i + B^{01}z_i^\star + B^{11}z_i z_i^\star) \succeq 0\end{array}$$

for all $i = 1, \ldots, r$. Since there exists at least one index l such that $m_{ll} > 0$, it follows that the matrix $B^{00} + B^{10}z_l + B^{01}z_l^\star + B^{11}z_l z_l^\star$ is positive semi-definite whence $z_l \in \mathcal{D}^C$. Vector x_l is such that $\mathrm{Trace}\mathcal{A}^\star\mathcal{A}x_l x_l^\star = 0$ and $F(x_l x_l^\star) \succeq 0$ in virtue of Lemma 2. Consequently, matrix $x_l x_l^\star$ is a solution to rank-one LMI problem (9) such that $\mu = 0$ and the lemma is proved. □

4.3.3 Main Result

Theorem 1 *The zeros of polynomial matrix $A(s)$ belong to region $\mathcal{D}$ if and only if $\nu > 0$ in LMI optimization problem (13).*

Remark 1 In Section 4.1 we have assumed that $v^\star v = 1$ in optimization problem (3). Equivalently we can enforce equality

$$\mathrm{Trace}\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} X = 1$$

in LMI optimization problem (13).

4.4 Dual LMI Formulation

Finally, we use standard semidefinite programming duality results [35] to come up with a more compact formulation of the $\mathcal{D}$-stability condition.

Let

$$P = P^\star = \begin{bmatrix} P_{11} & \cdots & P_{1k} \\ \vdots & & \vdots \\ P_{k1} & \cdots & P_{kk} \end{bmatrix}$$

where $P_{gh} \in \mathbb{C}^{r \times r}$. Similarly, let

$$B^{ef} = \begin{bmatrix} B_{11}^{ef} & \cdots & B_{1k}^{ef} \\ \vdots & & \vdots \\ B_{k1}^{ef} & \cdots & B_{kk}^{ef} \end{bmatrix}$$

where $B_{gh}^{ef} \in \mathbb{C}$ and define the linear map

$$F^D(P) = \sum_{g=1}^{k}\sum_{h=1}^{k}\left(B_{hg}^{00}\mathcal{Q}^\star P_{gh}\mathcal{Q} + B_{hg}^{10}\mathcal{P}^\star P_{gh}\mathcal{Q} + B_{hg}^{01}\mathcal{Q}^\star P_{gh}\mathcal{P} + B_{hg}^{11}\mathcal{P}^\star P_{gh}\mathcal{P}\right) \quad (14)$$

dual to the map introduced in (8) in the sense that

$$\mathrm{Trace} F^D(P)X = \mathrm{Trace} F(X)P.$$

Using standard duality arguments, we now show that the LMI feasibility problem

$$\begin{array}{l} \mathcal{A}^\star\mathcal{A} \succ F^D(P) \\ P = P^\star \succ 0. \end{array} \tag{15}$$

is dual to LMI optimization problem (13). To see this, build the Lagrangian

$$L(P,X,Y) = -\mathrm{Trace}(\mathcal{A}^\star\mathcal{A} - F^D(P))X - \mathrm{Trace} PY = -\mathrm{Trace}\mathcal{A}^\star\mathcal{A}X + \mathrm{Trace}(F(X) - Y)P$$

of problem (15) where $X = X^\star \succeq 0$ and $Y = Y^\star \succeq 0$ are Lagrange multiplier matrices. The dual function associated with the Lagrangian reads

$$g(X,Y) = \min_P L(P,X,Y) = \left\{ \begin{array}{ll} -\mathrm{Trace}\mathcal{A}^\star\mathcal{A}X & \text{if } F(X) = Y \succeq 0 \\ -\infty & \text{otherwise.} \end{array} \right.$$

The dual optimization problem, obtained by maximizing dual function $g(X,Y)$ is therefore LMI optimization problem (13). Since the inequalities in problem (15) are strict, Slater's qualification constraint holds and $\nu > 0$ in LMI optimization problem (13) if and only if LMI problem (15) is feasible.

Recall from Section 4.3 that $\mathcal{N}$ denotes a matrix whose columns span the right null-space of $\mathcal{A}$. Then it follows from the Elimination Lemma [9] that feasibility problem (15) can equivalently be written as

$$\begin{array}{l} \mathcal{N}^\star F^D(P)\mathcal{N} \prec 0 \\ P = P^\star \succ 0. \end{array} \tag{16}$$

We have shown the following corollary to Theorem 1.

Corollary 1 *The zeros of polynomial matrix $A(s)$ belong to region $\mathcal{D}$ if and only if there is a matrix P solution to LMI feasibility problem (16).*

Both problems (13) and (16) are linear problems over the cone of positive definite matrices. They can be solved very efficiently with the interior-point methods described in [28]. When an LMI features complex coefficients, it can be transformed into an LMI over reals using the standard manipulations described in [11].

5 Illustration

In the following examples, we defined and solved the LMI problems with the user-friendly interface LMITOOL 2.0 for MATLAB[1] [15]. We used a tolerance of 0.001 to enforce strict definiteness of the LMIs. The computations involving polynomial matrices were performed with the Polynomial Toolbox 2.0 for MATLAB [33].

[1]MATLAB is a trademark of The MathWorks, Inc.

5.1 Lyapunov Theorem as a Particular Case

As a first illustration, we show that the well-known Lyapunov Theorem can be derived as a particular case of Theorem 1, as pointed out in [8]. Consider the polynomial matrix

$$A(s) = A_0 - sI_n.$$

Hurwitz stability of constant matrix A_0 is equivalent to polynomial matrix $A(s)$ having its zeros in the closed left half-plane

$$\mathcal{D} = \{x + jy \in \mathbb{C} : x < 0\}.$$

Associated with region $\mathcal{D}$ is the indefinite matrix

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

An obvious choice for a right null-space basis for matrix

$$\mathcal{A} = \begin{bmatrix} A_0 & -I_n \end{bmatrix}$$

is given by

$$\mathcal{N} = \begin{bmatrix} I_n \\ A_0 \end{bmatrix}.$$

Consequently, LMI problem (16) becomes

$$\mathcal{N}^\star F(P)\mathcal{N} = [\, I_n \quad A_0^\star \,]\,([\, 0 \quad I_n \,]^\star P[\, I_n \quad 0 \,] + [\, I_n \quad 0 \,]^\star P[\, 0 \quad I_n \,])\,[\, I_n \quad A_0^\star \,]^\star \prec 0$$

for a positive definite Hermitian matrix P. This can be written more compactly as the standard LMI stemming from the Lyapunov Theorem

$$\begin{array}{l} A_0^\star P + PA_0 \prec 0 \\ P = P^\star \succ 0. \end{array}$$

5.2 Stability Bounds

Consider as in [2] the second degree polynomial matrix

$$A(s) = \begin{bmatrix} s^2 + 1/2 & as \\ as & s^2 + 1/4 \end{bmatrix}.$$

arising from a closed-loop system with feedback gain $a \in \mathbb{R}$. We study stability of $A(s)$ with respect to the unit disk, i.e.

$$\mathcal{D} = \{s \in \mathbb{C} : ss^\star < 1\}.$$

Associated with region $\mathcal{D}$ is the indefinite matrix

$$B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The LFR matrices corresponding to $A(s)$ are as follows

$$\left[\begin{array}{c|c} A_0 & L \\ \hline R & D \end{array}\right] = \left[\begin{array}{c} \mathcal{A} \\ \hline \mathcal{Q} \end{array}\right] = \left[\ \mathcal{B} \mid \mathcal{R}\ \right] = \left[\begin{array}{cc|cccc} 1/2 & 0 & 0 & a & 1 & 0 \\ 0 & 1/4 & a & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

Then, we solve the LMI feasibility problem of Theorem 1 for different values of a.

Letting $a = 1.369$ we obtain a strictly feasible positive definite solution

$$P = \left[\begin{array}{cccc} 744.3 & 0 & 0 & 1062 \\ 0 & 387.1 & 653.8 & 0 \\ 0 & 653.8 & 1555 & 0 \\ 1062 & 0 & 0 & 1809 \end{array}\right] \succ 0.$$

When $a \geq 1.370$ the solver dectects unfeasibility of the LMIs. Similarly, letting $a = -1.369$ we obtain a strictly feasible positive definite solution

$$P = \left[\begin{array}{cccc} 499.1 & 0 & 0 & -721.4 \\ 0 & 264.1 & -449.0 & 0 \\ 0 & -449.0 & 1075 & 0 \\ -721.4 & 0 & 0 & 1242 \end{array}\right] \succ 0.$$

When $a \leq -1.370$ the solver detects unfeasibility of the LMIs.

After a tedious application of the Schur-Cohn criterion to the determinant of $A(s)$, it can be shown that $A(s)$ has all its zeros inside the unit circle if and only if $|a| \leq 1.369$, which is consistent with our experiments.

5.3 Zeros at Infinity

Consider the polynomial matrix

$$A(s) = \left[\begin{array}{cc} a+s & as+s^2 \\ 2s & 1+2s^2 \end{array}\right]$$

parametrized in $a \in \mathbb{R}$. We study stability of $A(s)$ with respect to the left half-plane, i.e.

$$\mathcal{D}^C = \{s \in \mathbb{C} : s + s^\star \geq 0\}.$$

Associated with region $\mathcal{D}^C$ is the indefinite matrix

$$B = \left[\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right].$$

We readily compute $\det A(s) = s + a$ hence the polynomial matrix is stable if and only if $a > 0$.

LFR matrices associated with $A(s)$ are given by

$$\left[\begin{array}{c|c} A_0 & L \\ \hline R & D \end{array}\right] = \left[\begin{array}{c} \mathcal{A} \\ \hline \mathcal{Q} \end{array}\right] = \left[\begin{array}{c|c} \mathcal{B} & \mathcal{R} \end{array}\right] = \left[\begin{array}{cc|cccc} a & 0 & 1 & a & 0 & 1 \\ 0 & 1 & 2 & 0 & 0 & 2 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right].$$

Following the reduction procedure described in the Appendix, we successively

- remove column 3 in $\mathcal{R}$ and row 3 in $\mathcal{Q}$
- cancel column 3 in $\mathcal{R}$ with matrix

$$V = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

  and remove column 3 in $\mathcal{R}$ and row 3 in $\mathcal{Q}$

to eventually come up with the reduced LFR

$$\left[\begin{array}{c|c} A_0 & L \\ \hline R & D \end{array}\right] = \left[\begin{array}{cc|cc} a & 0 & 1 & a \\ 0 & 1 & 2 & 0 \\ \hline 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{array}\right].$$

LMI optimization problem (13) reads

$$\begin{array}{rcl} \nu & = & \min \quad \mathrm{Trace} \begin{bmatrix} a^2 & 0 & a & a^2 \\ 0 & 1 & 2 & 0 \\ a & 2 & 5 & a \\ a^2 & 0 & a & a^2 \end{bmatrix} X \\ & & \text{s.t.} \quad \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^{\star} X \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}^{\star} X \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \succeq 0 \\ & & \quad\quad X = X^{\star} \succeq 0. \end{array}$$

Recalling Remark 1, we add the constraint

$$\mathrm{Trace} \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} X = 1$$

to the above LMI problem.

Let $a = 1$. The optimal solution to the above LMI problem is $\nu = 0$ for

$$X = \left[\begin{array}{c} v \\ \hline p \end{array}\right] \left[\begin{array}{c} v \\ \hline p \end{array}\right]^{\star} = \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ -1 \end{array}\right] \left[\begin{array}{c} 1 \\ 0 \\ \hline 0 \\ -1 \end{array}\right]^{\star} \succeq 0.$$

One can check that vector q is equal to zero in equations (5). Since $q = p/s$ and vector p is non-zero, it follows that $ss^\star$ tends to infinity. We can thus conclude that polynomial matrix $A(s)$ features a zero at infinity. This zero belongs to the unstable region $\mathcal{D}^C$, yet $A(s)$ is a stable matrix since its finite zero -1 does not belong to $\mathcal{D}^C$. That clearly illustrates the fact that we cannot conclude about stability of finite zeros of a polynomial matrix featuring infinite zeros when working in open regions such as the left half-plane.

As pointed out in Section 3.4, this problem can be overcome when considering half-disk

$$\mathcal{D}^C_r = \left\{ s \in \mathbb{C} : s + s^\star \geq 0,\ ss^\star \leq r^2 \right\}$$

as a bounded complementary stability region, for a sufficiently large $r \in \mathbb{R}$. Associated with region $\mathcal{D}^C_r$ is the indefinite matrix

$$B = \left[\begin{array}{cc|cc} 0 & 0 & 1 & 0 \\ 0 & r^2 & 0 & 0 \\ \hline 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{array} \right].$$

When $a = 1$, the optimal solution to LMI problem (13) is $\nu = 2 \cdot 10^{-3} > 0$ when $r = 10$ and $\nu = 2 \cdot 10^{-5} > 0$ when $r = 100$. This is consistent with the fact that $A(s)$ has a finite stable zero at -1.

When $a = -1$, the optimal solution to LMI problem (13) is $\nu = 1.5 \cdot 10^{-11} \approx 0$ when $r = 10$ and $\nu = 8.4 \cdot 10^{-11} \approx 0$ when $r = 100$. This is consistent with the fact that $A(s)$ has a finite unstable zero at 1.

5.4 Union of Regions

Consider the model of a VTOL helicopter proposed in [38]. A state-feedback is designed as in [34] to locate the closed-loop poles of the linearized system. The state-space system matrix is given by

$$\left[\begin{array}{c|c} A & B \\ \hline C & D \end{array} \right]$$

where

$$A = \begin{bmatrix} 6.8026 & 1.4906 & 1.8411 & -1.9942 \\ -142.7218 & -24.1727 & -13.4252 & 67.4017 \\ 61.5028 & 8.0195 & -2.6299 & -41.5893 \\ 0.0000 & 0.0000 & 1.0000 & 0.0000 \end{bmatrix} + a \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

for an uncertain parameter $a \in \mathbb{R}$ and

$$B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5922 \\ -5.5200 & 4.4900 \\ 0.0000 & 0.0000 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad D = 0.$$

We investigate pole clustering of the above system in a non-connected region $\mathcal{D}$ which is the union of ellipse

$$\{x+jy \in \mathbb{C} : 4(x+2)^2/9 + y^2 < 1\}$$

and half-plane

$$\{x+jy \in \mathbb{C} : x < -5\}.$$

As pointed out in Section 3.3, the ellipse corresponds to slow dynamics and the half-plane corresponds to fast dynamics. The indefinite matrix corresponding to the union of the above regions is given by

$$B = \left[\begin{array}{ccc|ccc} 28/9 & 0 & 0 & 32/9 & 1 & 0 \\ 0 & 9/5 & 0 & 1 & 0 & 0 \\ 0 & 0 & 10 & 0 & 0 & 1 \\ \hline 32/9 & 1 & 0 & 4 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right]$$

For a given value of parameter a, we build the right coprime matrix fraction description [25, 23]

$$N(s)D^{-1}(s) = C(sI - A)^{-1}B$$

of the input-output transfer function. We enforce denominator polynomial matrix $D(s)$ to be column-reduced [23]. As a result, $D(s)$ has no infinite zeros and it is not necessary to pursue the approach of Section 3.4.

Then we study stability of polynomial matrix $D(s)$ by solving LMI optimization problem (13). By virtue of Remark 1, we add the constraint

$$\text{Trace} \begin{bmatrix} I_2 & 0 \\ 0 & 0 \end{bmatrix} X = 1.$$

When $a = 0$, we get

$$N(s) = \begin{bmatrix} -2.5921 + 0.0799s & 1.2546 + 3.9112s \\ 23.3757 + 0.6407s & -2.0989 - 34.4643s \end{bmatrix}$$

and

$$D(s) = \begin{bmatrix} -25.0083 - 2.6667s + 0.1808s^2 & -4.5784 + 37.1725s + 5.9338s^2 \\ -46.2327 - 6.1021s & -18.4499 + 53.7205s + 7.3098s^2 \end{bmatrix}.$$

LMI optimization problem (13) yields

$$X = \begin{bmatrix} 0.9563 & -0.2044 & -3.3471 & 0.7153 & 11.7144 & -2.5035 \\ -0.2044 & 0.0437 & 0.7153 & -0.1529 & -2.5034 & 0.5350 \\ -3.3471 & 0.7153 & 11.7146 & -2.5035 & -41.0005 & 8.7621 \\ 0.7153 & -0.1529 & -2.5035 & 0.5350 & 8.7620 & -1.8725 \\ 11.7144 & -2.5034 & -41.0005 & 8.7620 & 143.5028 & -30.6671 \\ -2.5035 & 0.5350 & 8.7621 & -1.8725 & -30.6671 & 6.5538 \end{bmatrix}$$

and $\nu = 0.0552 > 0$. Hence all the zeros of polynomial matrix $D(s)$ are located in region $\mathcal{D}$.

When $a = -1$, we get

$$N(s) = \begin{bmatrix} -0.4635 + 0.0801s & 1.0146 + 0.6363s \\ 3.8053 + 0.6418s & 0.0265 - 5.5520s \end{bmatrix}$$

and

$$D(s) = \begin{bmatrix} -4.4818 + 0.5287s + 0.1811s^2 & -6.8279 + 7.6310s + 0.9678s^2 \\ -14.3680 - 2.0357s & -22.6110 + 9.3277s + 1.1831s^2 \end{bmatrix}$$

LMI optimization problem (13) yields

$$X = \begin{bmatrix} 0.7301 & -0.4439 & -0.0591 & 0.0359 & 0.0048 & -0.0029 \\ -0.4439 & 0.2699 & 0.0359 & -0.0218 & -0.0029 & 0.0018 \\ -0.0591 & 0.0359 & 0.0048 & -0.0029 & -0.0005 & 0.0003 \\ 0.0359 & -0.0218 & -0.0029 & 0.0018 & 0.0003 & -0.0001 \\ 0.0048 & -0.0029 & -0.0005 & 0.0003 & 0.0005 & -0.0001 \\ -0.0029 & 0.0018 & 0.0003 & -0.0001 & -0.0001 & 0.0000 \end{bmatrix}$$

and $\nu = 9.6713 \cdot 10^{-8} \approx 0$. Therefore, some zeros of polynomial matrix $D(s)$ are located outside of region $\mathcal{D}$.

We can check these results by plotting the root locus of matrix A. As can be seen in Figure 1, all the eigenvalues belong to region $\mathcal{D}$ when $a = 0$, but not when $a = -1$.

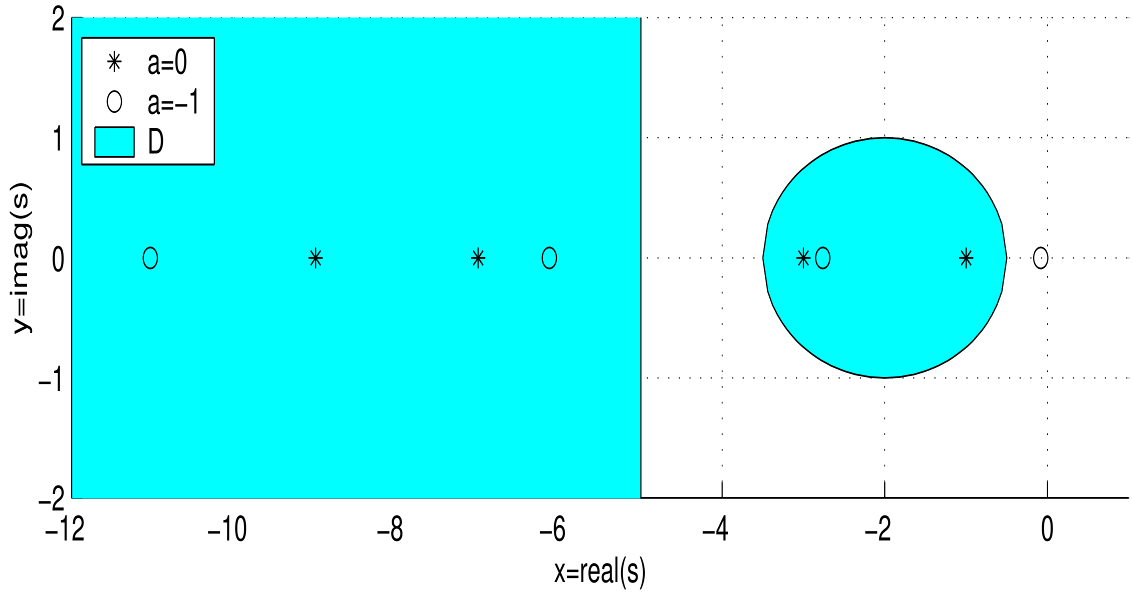

Figure 1. Root locus of matrix A when $a = 0$ and $a = -1$.

6 Conclusion

We have proposed a general LMI methodology for determining whether the zeros of a given polynomial matrix belong to a given region of the complex plane. Contrary to most of the existing $\mathcal{D}$-stability methods, no determinant or zero computation is required. As another advantage, we can easily handle polynomial matrices with complex coefficients. The class of regions covered by our approach is very large and most notably includes non-convex regions or non-connected unions thereof. Another strength of the method is its reliance upon semidefinite programming, for which powerful and efficient algorithms are widely available.

Our work hints at several directions for future research. The most promising one is probably the extension of our approach to uncertain polynomial matrices and robust stability analysis. Our $\mathcal{D}$-stability conditions can naturally be combined with other LMIs representing norm-bounded or interval uncertainties. However, the resulting optimization problem may lose its nice convexity properties and heuristic or global optimization algorithms must be devised. In the same vein, we can also handle 2-D or more general n-D polynomial matrices.

A point that deserves further attention is the numerical performance of our method compared to standard routines based on determinant computation and scalar stability tests. It is believed that the performance will strongly hinge upon the LMI solver in use and that some built-in parameter tunings will be required.

Another interesting extension would be to consider synthesis problems and Diophantine equations over polynomial matrices [25]. Here also, the resulting optimization problem becomes a non-convex bilinear matrix inequality problem.

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Appendix

Suppose we are a given a polynomial matrix $A(s)$ and its LFR as in Lemma 1. The following procedure reduces the LFR such that matrices $\mathcal{Q}$ and $\mathcal{R}$ have full rank.

Step 1. If $\mathcal{Q}$ is rank deficient, let U be the lower-left triangular matrix with ones along the diagonal such that the linearly dependent rows in $\mathcal{Q}$ are forced to zero in matrix $U\mathcal{Q}$. Let J be the set of indices of these rows. Replace matrices L,R and D by LU^{-1},UR and UDU^{-1}, respectively. Suppress rows J in L,D,Δ and suppress columns J in R,D,Δ. Update matrices $\mathcal{R}$ and $\mathcal{Q}$ accordingly.

Step 2. If $\mathcal{R}$ is rank deficient, let $V = U^{-1}$ be the upper-right triangular matrix with ones along the diagonal such that the linearly dependent columns in $\mathcal{R}$ are forced to zero in matrix $\mathcal{R}V$. Let J be the set of indices of these columns. Replace matrices L,R and D by LV,$V^{-1}R$ and $V^{-1}DV$, respectively. Suppress columns J in L,D,Δ and suppress rows J in R,D,Δ. Update matrices $\mathcal{R}$ and $\mathcal{Q}$ accordingly.

Step 3. Go to Step 1 until no reduction can be achieved.

The above procedure results in a reduced LFR of $A(s)$ with full rank matrices $\mathcal{Q}$ and $\mathcal{R}$. The proof follows by careful inspection. The key point is in noting that matrix Δ commutes with matrix $U = V^{-1}$, i.e. $\Delta U = U\Delta$. Therefore, Δ is not affected by the successive similarity transformations. Redundant components in p and q corresponding to zero rows in $\mathcal{Q}$ or zero columns in $\mathcal{R}$ can therefore be removed without altering the LFR of $A(s)$. Note that reduction matrices U and V can be computed using numerically stable operations [16].

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