Piecewise-Linear Robust Control of Systems with Input Constraints*

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Abstract

Robust stabilization of an uncertain system subject to input constraints is addressed without making open-loop stability assumptions. A local approach is taken to find a robust control law and a set of initial conditions that can be stabilized. A piecewise-linear control law is described. It is generated by a parametrized algebraic Riccati equation or a parametrized linear matrix inequality determinant maximization problem. Once system trajectories are sufficiently close to the origin, a certain performance level is ensured through a guaranteed cost control. Simple and tractable design algorithms are proposed and illustrated by a numerical example.

1 Introduction

Recently, there has been a surge of interest concerning stabilization of systems with saturating controls. This phenomenon is indeed a common feature of every physical realizable control system [2, 20]. Several significant results have lately emerged that pertain to the problem of global [18, 19], semi-global [1, 9, 13] or local stabilization [6, 7, 17, 22]. These techniques differ in the following points. In global stabilization, the set of stabilizable initial conditions is the whole state space, whereas it is an a priori given arbitrary large

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bounded set in the semi-global framework. However, both approaches inherently require stability of the open-loop system. Relaxing this stability assumption, the local approach rather aims at characterizing a sufficiently large set of initial conditions for which a stabilizing control can be applied. Although restrictive, the local approach is realistic since physical system models are usually only valid locally.

In practical applications, it is also of paramount importance that control laws are designed so as to achieve acceptable performance under system uncertainty. Powerful techniques have been developed addressing the issue of robustness in the presence of norm-bounded uncertainties – see for instance [5, 24]. In general, these design procedures do not take directly into account control saturation and input constraints are usually met through outright conservatism. Since the full extent of the actuators range is not used, this often results in a serious performance sacrifice. Moreover, in practical problems uncertainty usually affects both dynamic and input matrices. However, a very few publications deal simultaneously with saturation and this type of uncertainty. In references [1, 9, 17, 18, 22], uncertainties are not considered, [13] only tackles matched uncertainties entering the saturation function, in [6] uncertain elements only appear in the dynamic matrix, and up to the authors knowledge, only [7] handles unmatched uncertainty directly entering the input matrix.

The objective of this paper is therefore to fill a need for easily implementable robust control laws that explicitly handle input saturation and uncertainty in the dynamic and input matrices, while ensuring a certain performance level. As a continuation of the research efforts initiated in [6] and [7], we focus on a local stabilization approach. Our main contributions are two simple algorithms that build a piecewise-linear control (PLC) law by solving a parametrized algebraic Riccati equation (ARE) similar to that arising in $\mathcal{H}_\infty$ control theory, or alternatively, a parametrized linear matrix inequality (LMI) determinant maximization problem. A family of nested ellipsoids is obtained in the system state space. Once initialized in the outermost ellipsoid of the family, system trajectories are steered to the origin by applying increasing piecewise-linear gains without causing input saturation. In the innermost ellipsoid, a guaranteed cost feedback [11] is eventually applied to ensure performance beyond stabilization. The algorithms presented here combine several key ideas found in [18, 22] and can be considered as an extension of these works to the case of unmatched uncertainties. Furthermore, our algorithms are practical implementations of the complex nonlinear implicit control law described in [6]. During the evaluation of the conference version of our paper, an anonymous reviewer pointed out that several results presented here were independently obtained in [17] without considering uncertainties. From this standpoint, our methods can also be viewed as a generalization of [17] to the uncertain case.

The paper is organized as follows. In the next section, we first describe the system under study and the control problem we are to solve. In Section 3, a first design algorithm is proposed. It is based on parametrized AREs, Lyapunov ellipsoid nesting and scaled robust stabilizing linear gains. In Section 4, a second design algorithm is described, this time based upon parametrized LMIs. Both algorithms are illustrated on a numerical example in Section 5.

The notations are standard. The identity matrix of dimension n is denoted by I_n, the

$n \times m$ matrix with zero entries is written $0_{n\times m}$, A' stands for the transpose of A, the matrix inequality $A > B$ is equivalent to the positive definite matrix inequality $A - B > 0$ and $\mathbb{R}$ is the field of real numbers.

2 Problem Statement

Throughout the paper, we consider the uncertain continuous-time system

$$\dot{x} = (A + \delta A)x + (B + \delta B)u \tag{1}$$

subject to the input constraints

$$|u_i| \leq \overline{u}_i, \quad i = 1, \ldots, m \tag{2}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^m$. Uncertainty matrices $\delta A, \delta B$ are norm-bounded, viz. $\delta A = DFE_1, \delta B = DFE_2$ where D, E_1, E_2 are given constant matrices and uncertainty matrix $F \in \mathbb{R}^{q\times r}$ verifies $\|F\|_2 \leq 1$. Moreover, pair (A, B) is assumed to be stabilizable.

The control problem we are to solve is as follows.

Constrained Robust Control Problem *Find a state-feedback control u for system (1) such that, for any initial condition x_0 in a given set, the state x is asymptotically steered to the origin while the input u verifies constraints (2).*

3 ARE Approach

We propose a first approach for addressing our constrained robust control problem. It is based on a parametrized ARE from the $\mathcal{H}_\infty$ control theory. It results in a piecewise-linear state-feedback control which stabilizes system (1) in a given ellipsoid while verifying input constraints (2).

3.1 Parametrized ARE

We build the parametrized ARE

$$\mathcal{A}'P(\alpha) + P(\alpha)\mathcal{A} - P(\alpha)\mathcal{B}P(\alpha) + \mathcal{C} + \alpha\mathcal{Q} = 0, \tag{3}$$

where $\alpha \in [0, 1]$ is a given scalar parameter, and

$$\begin{array}{ll} \mathcal{A} = A - B(\mathcal{R} + E_2'E_2)^{-1}E_2'E_1, & \mathcal{B} = B(\mathcal{R} + E_2'E_2)^{-1}B' - DD', \\ \mathcal{C} = E_1'(I_r - E_2(\mathcal{R} + E_2'E_2)^{-1}E_2')E_1, & \mathcal{R} = \epsilon R, \quad \mathcal{Q} = \epsilon Q \end{array}$$

for a positive scalar ϵ and arbitrary symmetric positive definite weighting matrices R, Q. Although not reflected by the notation, ARE solution $P(\alpha)$ depends on scalar ϵ. We shall drop this dependence on ϵ later on. According to standard results on $\mathcal{H}_\infty$ control theory [14], if ARE (3) has a symmetric positive definite solution $P(\alpha)$ for a sufficiently small scalar $\epsilon \in [0, \overline{\epsilon}]$, a robust control for uncertain system (1) without control constraints reads $u = K(\alpha)x$ where

$$K(\alpha) = -(\mathcal{R} + E_2'E_2)^{-1}(B'P(\alpha) + E_2'E_1). \quad (4)$$

Moreover, suppose that initial condition x_0 is a zero mean random variable satisfying $\mathbf{E}[x_0x_0'] = I_n$, where $\mathbf{E}$ denotes the expectation. Associated with system (1) is the quadratic performance energy criterion

$$J = \int_0^\infty (x'Qx + u'Ru)dt. \quad (5)$$

For a given ϵ, we define $X(\epsilon) = \epsilon^{-1}P(1)$, where $P(1)$ is a symmetric positive definite solution to ARE (3) when $\alpha = 1$. It is shown in [11] that $\mathbf{E}[J] \leq \text{trace } X(\epsilon)$. A certain performance level is therefore ensured if this upper bound on $\mathbf{E}[J]$ is minimized, namely if ϵ equals

$$\begin{array}{rll} \epsilon_* = \arg & \min\limits_{\epsilon} & \text{trace } X(\epsilon) \\ & \text{s.t.} & \epsilon \in [0, \overline{\epsilon}]. \end{array} \quad (6)$$

We can easily solve for ϵ this convex optimization problem and derive the so-called guaranteed cost feedback $K(1)$ corresponding to the solution to ARE (3) when $\epsilon = \epsilon_*$ and $\alpha = 1$. For the sake of clarity, it is assumed in the sequel that $\epsilon = \epsilon_*$ in ARE (3), so that we can drop any dependence on ϵ hereafter.

Since input u is subject to magnitude constraints (2), guaranteed cost performance cannot be ensured for any initial condition x_0. This is captured by the following lemma.

Lemma 1 *Let $k_i(\alpha)$ stand for the i-th row of matrix $K(\alpha)$. We define*

$$c(\alpha) = \min_{i=1,\ldots,m} \frac{\overline{u}_i^2}{k_i(\alpha)P^{-1}(\alpha)k_i'(\alpha)} \quad (7)$$

and

$$\Pi(\alpha) = c^{-1}(\alpha)P(\alpha).$$

Then the Lyapunov level set

$$\mathcal{E}(\alpha) = \{x : x'\Pi(\alpha)x \leq 1\}$$

is the maximum volume ellipsoid shaped by $P(\alpha)$ such that input $u = K(\alpha)x$ verifies constraints (2).

Proof See [18]. Lyapunov level $c(\alpha)$ is solution to the optimization problem

$$\begin{array}{lll} \min\limits_{i=1,\ldots,m} & \max\limits_{x} & x'P(\alpha)x \\ & \text{s.t.} & |k_i(\alpha)x| \leq \overline{u}_i \\ & & |k_i(\alpha)x| = \overline{u}_i. \end{array}$$

■

Remark 1 *One may only consider the control u_i for which the minimum is attained in (7). Hence multi-input results can readily be derived from the single-input case. For the sake of simplicity and without loss of generality, we will assume $m = 1$ in this section.*

Because $P(\alpha)$ solves (3), it is well-known that ellipsoid $\mathcal{E}(\alpha)$ is both positively invariant and contractive [4, 20] for system (1) with robust control $u = K(\alpha)x$. To put it another way, for any initial condition x_0 in $\mathcal{E}(\alpha)$, closed-loop trajectory stays in $\mathcal{E}(\alpha)$ while converging asymptotically to the origin. In particular, in view of Lemma 1, when $\alpha = 1$ guaranteed cost is only ensured within $\mathcal{E}(1)$. Outside of $\mathcal{E}(1)$ even stability cannot be guaranteed since controls are expected to saturate. This has motivated the parametrization of ARE (3) in α, as underlined in the following lemma.

Lemma 2 *Suppose that ARE (3) has a symmetric positive definite solution $P(1)$ for $\alpha = 1$. Then, ARE (3) has a symmetric positive definite solution $P(\alpha)$ for any $\alpha \in [0, 1]$. Furthermore, $P(\alpha_1) < P(\alpha_2)$ for any scalars α_1, α_2 such that $0 \leq \alpha_1 < \alpha_2 \leq 1$.*

Proof The statement follows from standard monotonicity properties of AREs, see [12]. ∎

As a result, the smaller is α, the smaller is the gain $K(\alpha)$. Consequently one might expect that input magnitude constraints are respected farther from the origin as α approaches 0. The following related result has been proved in [6] when uncertainty only enters matrix A (that is, when $E_2 = 0$).

Lemma 3 *$\mathcal{E}(\alpha_2) \subset \mathcal{E}(\alpha_1)$ for any scalars α_1, α_2 such that $0 \leq \alpha_1 < \alpha_2 \leq 1$.*

Proof Since $E_2 = 0$, $c(\alpha) = (\overline{u}\mathcal{R})^2(B'P(\alpha)B)^{-1}$. From Lemma 2, $P(\alpha_1) < P(\alpha_2)$ and $c(\alpha_1) > c(\alpha_2)$. Hence $\Pi(\alpha_1) < \Pi(\alpha_2)$, which is equivalent to $\mathcal{E}(\alpha_2) \subset \mathcal{E}(\alpha_1)$. ∎

The key idea in [6], originally proposed in [1, 18], was then to let α, and thus gain $K(\alpha)$, increase from 0 to 1 as a function of x when x approaches the origin. Therefore, any initial condition chosen in the outermost ellipsoid $\mathcal{E}(0)$ can be stabilized by a low gain $K(0)$. When α eventually equals 1, x has reached the innermost ellipsoid $\mathcal{E}(1)$ where guaranteed cost feedback $K(1)$ is applied.

Unfortunately, one can check by simple inspection that Lemma 3 does not necessarily hold when $E_2 \neq 0$. Counter-examples can be given so that inequality $c(\alpha_1) > c(\alpha_2)$ is no longer verified since the denominator of $c(\alpha)$ features both $P(\alpha)$ and $P^{-1}(\alpha)$ – see equations (4) and (7). Without any assumption on input uncertainty matrix E_2, extension of the result in [6] does not seem to be possible. An alternative approach is required.

3.2 ARE PLC Law

3.2.1 Robust Feedback Gain Scaling

The following lemma will be instrumental to the generation of the ARE-based control law.

Lemma 4 *Given $\alpha \in [0,1]$, suppose that ARE (3) has a symmetric positive definite solution $P(\alpha)$ and therefore that the feedback gain $K(\alpha)$ given in (4) stabilizes system (1). Define respectively $\underline{\rho}$ and $\overline{\rho}$ as the greatest negative and smallest positive real zeros of quadratic polynomial matrix*

$$S(\rho) = S_0 + S_1\rho + S_2\rho^2,$$

where

$$\begin{aligned} S_0 &= -\alpha\mathcal{Q} - K'(\alpha)\mathcal{R}K(\alpha) \\ S_1 &= -2K'(\alpha)\mathcal{R}K(\alpha) \\ S_2 &= K'(\alpha)E_2'E_2K(\alpha). \end{aligned}$$

Then the scaled feedback gain $(\rho+1)K(\alpha)$ also robustly stabilizes system (1) for any real scalar ρ in the interval $[\underline{\rho}, \overline{\rho}]$.

Proof ARE (3) can also be written

$$A'P(\alpha) + P(\alpha)A + P(\alpha)DD'P(\alpha) + E_1'E_1 - K'(\alpha)\mathcal{R}K(\alpha) + \alpha\mathcal{Q} = 0. \qquad (8)$$

By Lyapunov arguments, robust stabilization of closed-loop system (1) with feedback $(\rho+1)K(\alpha)$ is equivalent to the quadratic inequality

$$2x'P[A + DFE_1 + (\rho+1)(B + DFE_2)K(\alpha)]x < 0 \qquad (9)$$

for any admissible uncertainty matrix F and any vector $x \in \mathbb{R}^n$. Standard upper bounds on uncertain terms [10] yield

$$\begin{aligned} &2x'P(\alpha)[A + (\rho+1)BK(\alpha)]x + 2x'P(\alpha)\{DF[E_1 + (\rho+1)E_2K(\alpha)]\} \leq \\ &2x'P(\alpha)[A + (\rho+1)BK(\alpha)]x + x'P(\alpha)DD'P(\alpha)x + \\ &\quad x'[E_1 + (\rho+1)E_2K(\alpha)]'[E_1 + (\rho+1)E_2K(\alpha)]x = \\ &2x'P(\alpha)Ax + x'P(\alpha)DD'P(\alpha)x + x'E_1'E_1x + 2(\rho+1)x'[P(\alpha)B + E_1'E_2]K(\alpha)x + \\ &\quad (\rho+1)^2x'K'(\alpha)E_2'E_2K(\alpha)x. \end{aligned}$$

For inequality (9) to hold, the latter expression must be negative definite. Using equation (8), one can check that it is equivalent to the requirement $S(\rho) < 0$. Since $S(0) = S_0 = -\alpha\mathcal{Q} - K'(\alpha)\mathcal{R}K(\alpha) < 0$, polynomial matrix $S(\rho)$ remains negative definite as long as ρ lies between the zeros of $S(\rho)$ nearest to the origin. ■

Remark 2 *Bounds $\underline{\rho}$ and $\overline{\rho}$ can be determined numerically using the eigenvalue method described in [16] and implemented in [8].*

Remark 3 *When $E_2 = 0$, one can easily check that $\overline{\rho} \to \infty$. In that case, ρ only admits a lower bound.*

Remark 4 *When no uncertainty affects A and B and A has no eigenvalues with positive real part, ρ is then referred to as a high-gain parameter. It is used in [13] to ensure performance beyond stabilization. In the same reference, α is referred to as a low-gain parameter and is used to stabilize the system arbitrarily far from the origin.*

When applying feedback gain $(\rho+1)K(\alpha)$, the Lyapunov level set defined in Lemma 1 is replaced by its homothetic defined as

$$(\rho+1)^{-1}\mathcal{E}(\alpha) = \{x : x'\Pi(\alpha)x \leq (\rho+1)^{-2}\}.$$

Corollary 1 *Suppose that we are given two scalars α_1, α_2 verifying $0 \leq \alpha_1 < \alpha_2 \leq 1$ and that the inclusion $\mathcal{E}(\alpha_2) \subset \mathcal{E}(\alpha_1)$ does not hold. Moreover, suppose that scalar ρ_2 is given and define $\varrho = \sqrt{(\rho_2+1)^2\underline{\lambda}(\Pi(\alpha_2)\Pi^{-1}(\alpha_1))}$, where $\underline{\lambda}(.)$ denotes the minimal eigenvalue of a matrix. Then the inclusion $(\rho_2+1)^{-1}\mathcal{E}(\alpha_2) \subset (\rho_1+1)^{-1}\mathcal{E}(\alpha_1)$ holds if ρ_1 lies in the intersection of intervals $[\underline{\rho}_1, \overline{\rho}_1]$ and $[-\varrho-1, \varrho-1]$.*

Proof Ellipsoid $(\rho_2+1)^{-1}\mathcal{E}(\alpha_2)$ is included in $(\rho_1+1)^{-1}\mathcal{E}(\alpha_1)$ if and only if $(\rho_1+1)^2\Pi(\alpha_1) < (\rho_2+1)^2\Pi(\alpha_2)$. Since ρ_2 is given, a sufficient condition for this matrix equation to hold is that ρ_1 satisfies the scalar quadratic inequality $(\rho_1+1)^2 < (\rho_2+1)^2\underline{\lambda}(\Pi(\alpha_2)\Pi^{-1}(\alpha_1))$. ■

3.2.2 ARE PLC Generation Algorithm

Input Linear system (1) with input constraints (2).

Output A PLC law that robustly stabilizes system (1) in a given set of initial conditions $\mathcal{E}(0)$. The PLC law consists in a family of $N+1$ nested ellipsoids $\mathbb{E} = \{\mathcal{E}_0 \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_N\}$ and corresponding feedback gains $\mathbb{K} = \{K_0, K_1, \ldots, K_N\}$.

Step 0 Build an increasing sequence $\alpha_0 = 0 > \alpha_1 > \ldots > \alpha_M = 1$. Given weighting matrices Q, R and $\alpha = \alpha_M = 1$, solve the guaranteed cost optimization problem (6) and let $\epsilon = \epsilon_*$. Solve ARE (3) for $P(\alpha_0), P(\alpha_1), \ldots, P(\alpha_M)$. Compute the corresponding robust feedback gains $K(\alpha_0), K(\alpha_1), \ldots, K(\alpha_M)$ given in (4). Build guaranteed cost ellipsoid $\underline{\mathcal{E}} = \mathcal{E}(\alpha_M)$ according to Lemma 1 and let $\rho_M = 0$. Finally, let $\mathbb{E} = \{\underline{\mathcal{E}}\}$ and $\mathbb{K} = \{K(\alpha_M)\}$.

Step $\mathbf{i = 1, .., M}$ Using Corollary 1, select a scalar ρ_{M-i} such that ellipsoids $\underline{\mathcal{E}}$ and $\overline{\mathcal{E}} = (\rho_{M-i}+1)^{-1}\mathcal{E}(\alpha_{M-i})$ are nested, i.e. $\underline{\mathcal{E}} \subset \overline{\mathcal{E}}$. If such a scalar does not exist, then skip ellipsoid $\overline{\mathcal{E}}$ and go to Step $i+1$. Otherwise, let $\underline{\mathcal{E}} = \overline{\mathcal{E}}$, $\mathbb{E} = \{\underline{\mathcal{E}}, \mathbb{E}\}$, $\mathbb{K} = \{(\rho_{M-i}+1)K(\alpha_{M-i}), \mathbb{K}\}$ and go to Step $i+1$.

3.2.3 ARE PLC Implementation

Step 0 System (1) is initialized in the outermost ellipsoid $\mathcal{E}_0$ and is controlled by low gain linear feedback K_0.

Step $\mathbf{i = 1, .., N}$ We continue to apply gain K_{i-1} until state trajectory encounters the next inner nested ellipsoid $\mathcal{E}_i$ at which time we switch to the next higher gain K_i and proceed to Step $i+1$.

Step $\mathbf{N+1}$ In the innermost ellipsoid $\mathcal{E}_N$, system (1) is controlled by the guaranteed cost feedback K_N and is stabilized with a certain performance level.

Since ellipsoids $\mathcal{E}_i$ are positively invariant, contractive and belong to a nested family, convergence of the above control law readily follows.

Remark 5 *Since two ellipsoids corresponding to two consecutive values α_i and α_{i-1} are not necessarily nested, the output ellipsoid sequence $\mathbb{E}$ is composed of a number N of nested ellipsoids that is generally less than M.*

Remark 6 *The crucial point of the ARE PLC generation algorithm is the choice of a sequence of parameters α_i in Step 0. By experience, a logarithmic sequence produces satisfying results, namely*

$$\alpha_i = \frac{\delta^{\frac{i}{M}} - 1}{\delta - 1} \tag{10}$$

for $i = 0, 1, \ldots, M$ and $\delta > 1$. When δ approaches 1, scalars α_i tend to be linearly equally spaced. To increase δ implies tighter ellipsoid nesting far from the origin. As pointed out in [22], to aid in the selection of an appropriate spacing factor δ, we can use a function which measures the percentage change in the major axes of two consecutive nested ellipsoids.

Remark 7 *A straightforward choice for scalars ρ_i is $\rho_0 = \rho_1 = \ldots = \rho_N = 0$. A more natural choice is a decreasing sequence $\rho_0 = 0 > \rho_1 > \cdots > \rho_N$ since ellipsoid $\mathcal{E}_i$ shrinks when ρ_i increases. In particular, the outermost ellipsoid $\mathcal{E}_0$ should be as large as possible, which entails minimizing ρ_0.*

4 LMI Approach

In this section, we describe a second approach for addressing our constrained robust control problem. It is based on a parametrized LMI determinant maximization problem. As for the ARE PLC law exposed in the previous section, it results in a piecewise-linear state-feedback control which stabilizes system (1) under control constraints (2). However, we shall see that the LMI formulation provides more flexibility and namely results in greater domains of stabilizable initial conditions.

4.1 Parametrized LMI

We have seen in Section 3 that there are two conflicting design objectives in our control problem, namely the stabilization of the greatest domain of initial conditions and the best achievable performance around the origin.

When there are hard constraints on the control input as in (2), it is known that the whole state-space cannot be stabilized, unless open-loop system (1) is assumed to be stable. This is captured by the following standard lemma, which aims at maximizing the volume of the stabilizable initial condition set while ensuring robust stability under control constraints.

Note that in the sequel, for notational conciseness upper blocks in symmetric LMI matrices are not repeated and are denoted by stars.

Lemma 5 *Define W_0 and Y_0 as the optimal solution to the LMI determinant maximization problem*

$$\begin{array}{ll} \min\limits_{W,Y,\epsilon} & \log\det W^{-1} \\ \text{s.t.} & \left[\begin{array}{cc} AW + WA' + BY + Y'B' + \epsilon DD' & \star \\ E_1 W + E_2 Y & -\epsilon I_r \end{array}\right] < 0 \\ & \left[\begin{array}{cc} W & Y_i' \\ Y_i & \bar{u}_i^2 \end{array}\right] \geq 0, \quad i = 1, 2, \ldots, m \end{array} \tag{11}$$

where Y_i stands for the i-th row in matrix Y. Then, the Lyapunov level set

$$\mathcal{E}_0 = \{x : x'W_0^{-1}x \leq 1\} \tag{12}$$

is the maximum volume ellipsoid such that for any initial condition x_0 in $\mathcal{E}_0$, the low-gain linear state-feedback control

$$u = K_0 x = Y_0 W_0^{-1} x, \tag{13}$$

asymptotically stabilizes system (1) and verifies constraints (2).

Proof See Section 7.2 in [3]. ■

Note that problem (11) is a well-known convex optimization problem referred to as the analytic centering problem. It was extensively studied in [21] and can be solved by efficient numerical tools.

Usually, applying feedback (13) results in poor performance, especially when the system state is close to the origin. The following lemma, which is an LMI counterpart to the development of the ARE guaranteed cost proposed in Section 3.1, is aimed at addressing the problem of achieving the best performance around the origin with respect to quadratic criterion (5).

Lemma 6 *Define W_1, Y_1 and γ_1 as the optimal solution to the LMI optimization problem*

$$\begin{array}{ll}
\min\limits_{W,Y,\gamma,\epsilon} & \gamma \\
\text{s.t.} & \begin{bmatrix} \gamma I_n & I_n \\ I_n & W \end{bmatrix} \geq 0 \\
& \begin{bmatrix} AW + WA' + BY + Y'B' + \epsilon DD' & \star & \star \\ E_1 W + E_2 Y & -\epsilon I_r & \star \\ \begin{bmatrix} 0_{m\times n} \\ Q^{\frac{1}{2}} \end{bmatrix} W + \begin{bmatrix} R^{\frac{1}{2}} \\ 0_{n\times m} \end{bmatrix} Y & 0_{(n+m)\times r} & -I_{m+n} \end{bmatrix} < 0 \\
& \begin{bmatrix} W & Y_i' \\ Y_i & \bar{u}_i^2 \end{bmatrix} \geq 0, \quad i = 1, 2, \ldots, m.
\end{array} \tag{14}$$

Then, for any initial condition x_0 in the Lyapunov level set

$$\mathcal{E}_1 = \{x : x' W_1^{-1} x \leq 1\}, \tag{15}$$

the high-gain linear state-feedback control

$$u = K_1 x = Y_1 W_1^{-1} x, \tag{16}$$

asymptotically stabilizes system (1), verifies constraints (2) and ensures a certain performance level.

Proof The guaranteed cost problem consists in minimizing the output energy

$$J = \int_0^\infty z'z \, dt$$

of uncertain system (1) endowed with the fictitious output

$$z = \begin{bmatrix} 0_{m\times n} \\ Q^{\frac{1}{2}} \end{bmatrix} x + \begin{bmatrix} R^{\frac{1}{2}} \\ 0_{n\times m} \end{bmatrix} u$$

where $Q^{\frac{1}{2}}$ and $R^{\frac{1}{2}}$ are the Cholesky factors of the weighting matrices in quadratic criterion (5). The LMI formulation of this problem is provided in [3, Section 7.4]. With the notations of Section 3.1, it holds

$$\mathbf{E}[J] \leq \text{trace} W^{-1} \leq n\gamma.$$

Hence minimizing γ entails minimizing an upper bound on $\mathbf{E}[J]$, which in turn ensures a certain performance level through guaranteed cost (16). ∎

A natural way to combine the design objectives of Lemmas 5 and 6 is to build the parametrized LMI optimization problem

$$
\begin{array}{ll}
\min\limits_{W,Y,\gamma,\epsilon} & (1-\alpha)\log\det W^{-1} + \alpha\gamma \\
& \left[\begin{array}{cc} \gamma I_n & I_n \\ I_n & W \end{array}\right] \geq 0 \\
& \left[\begin{array}{ccc} AW + WA' + BY + Y'B' + \epsilon DD' & \star & \star \\ E_1 W + E_2 Y & -\epsilon I_r & \star \\ \left[\begin{array}{c} 0_{m\times n} \\ \alpha Q^{\frac{1}{2}} \end{array}\right] W + \left[\begin{array}{c} \alpha R^{\frac{1}{2}} \\ 0_{n\times m} \end{array}\right] Y & 0_{(n+m)\times r} & -I_{m+n} \end{array}\right] < 0 \\
& \left[\begin{array}{cc} W & Y_i' \\ Y_i & \bar{u}_i^2 \end{array}\right] \geq 0, \quad i = 1, 2, \ldots, m.
\end{array}
\tag{17}
$$

One can check that when α tends to 0, problem (17) becomes problem (11). Similarly, when α tends to 1, problem (17) becomes problem (14). As in Section 3.1, the idea is then to let α increase from 0 to 1 as a function of x when x approaches the origin. Then, the α-parametrized gain

$$K_\alpha = Y_\alpha W_\alpha^{-1} \tag{18}$$

varies from low-gain K_0 to high-gain K_1. In order to ensure that ellipsoid

$$\mathcal{E}_\alpha = \{x : x' W_\alpha^{-1} x \leq 1\} \tag{19}$$

is included into ellipsoid $\mathcal{E}_\beta$ when $0 \leq \alpha < \beta \leq 1$, it is necessary to insert the LMI constraint

$$W_\alpha > W_\beta \tag{20}$$

into problem (17).

4.2 LMI PLC Generation Law

Input and **Output** See Section 3.2.

Step 0 Build an increasing sequence $\alpha_0 = 0 > \alpha_1 > \ldots > \alpha_N = 1$. Solve LMI problem (14) for W_1, Y_1. Using (15) and (16), let $\mathbb{E} = \{\mathcal{E}_1\}$ and $\mathbb{K} = \{K_1\}$. Let $\beta = \alpha_N$.

Step $\mathbf{i = 1, .., N}$ Let $\alpha = \alpha_{N-i}$. Solve LMI problem (17) and (20) for W_α, Y_α. Using (18) and (19), let $\mathbb{E} = \{\mathbb{E},\ \mathcal{E}_\alpha\}$, $\quad \mathbb{K} = \{\mathbb{K},\ K_\alpha\}, \beta = \alpha$ and go to Step $i+1$.

Remark 8 *Our ARE and LMI algorithms can be considered as practical implementations of the gain-scheduling control defined in Theorem 2 [18] (when $D = 0$) and Theorem 10 [6] (when $E_2 = 0$). Note that in these references one needs to solve a complex implicit nonlinear equation, while our algorithms only rely upon ARE or LMI resolution, for which powerful and reliable numerical tools are known to exist. A similar remark was made in the numerical example described in [6], but only in the case that $E_2 = 0$.*

Remark 9 *There are two main reasons for which our PLC laws differ from the one proposed in [22]. On the one hand, our domain of attraction is the innermost ellipsoid where guaranteed cost is ensured, whereas it is the origin in [22]. On the other hand, the PLC law exposed in [22] does not handle uncertainty.*

Remark 10 *All the computations that are required to implement our ARE or LMI PLC laws can be performed off-line. Only some on-line memory is needed to store nested ellipsoids* $\mathbb{E}$ *and piecewise-linear control* $\mathbb{K}$*.*

Remark 11 *Finally, it is still possible to stabilize initial conditions that do not belong to outermost ellipsoid* $\mathcal{E}_0$ *if controls are allowed to saturate. Using the technique exposed in [7], one can determine an homothetic ellipsoid including* $\mathcal{E}_0$ *such that robust stability is still preserved. This extension is not pursued here.*

5 Example

We consider the dynamics of a helicopter in a vertical plane studied in [15]. The corresponding linear model (1) has four state variables, two inputs and reads

$$A = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4556 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.2855 & -0.7070 & 1.3229 \\ 0 & 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0.4422 & 0.1761 \\ 3.0447 & -7.5922 \\ -5.5200 & 4.9900 \\ 0 & 0 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2.0673 \\ 0.2192 & 1.2031 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad E_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad E_2 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

Note that matrix A is unstable. Furthermore, we suppose that both inputs are bounded as follows

$$\begin{aligned} |u_1| &\leq 1 \\ |u_2| &\leq 1. \end{aligned}$$

5.1 ARE Approach

We chose ARE weighting matrices $R = I_m$ and $Q = I_n$. The ARE guaranteed cost control, obtained for $\epsilon_* = 0.7143$, is

$$u = \begin{bmatrix} -0.58 & 0.25 & 0.86 & 1.15 \\ 0.47 & 1.53 & -0.10 & -1.24 \end{bmatrix} x.$$

We implemented the ARE PLC generation algorithm for sequences of respective lengths $M = 4$ and $M = 50$.

In the first case ($M = 4$) we set all design parameters ρ_i to 0 and selected $\delta = 1000$ in formula (10) to generate the sequence $\alpha_0, \alpha_1, \ldots, \alpha_4$. The algorithm returned $N = 4$ switching surfaces whose projections on the state plane (x_1, x_2) are shown in Figure 1. The innermost ellipsoid $\mathcal{E}_4$ contains the states for which the guaranteed cost feedback can be applied without causing input saturation. Every initial condition that is chosen in the outermost ellipsoid $\mathcal{E}_0$ is steered to $\mathcal{E}_4$ while satisfying control constraints. Intermediate nested ellipsoids $\mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_3$ are represented in dashed lines. The nominal closed-loop system ($F = 0$) was then simulated for an initial condition $x_0 = [0.0210 \quad -0.9057 \quad 0.5953 \quad 0]'$ belonging to $\mathcal{E}_0$. In Figure 2, we compare the guaranteed cost control when saturation does not affect the input (in dotted line) and our piecewise-linear control that is designed to avoid saturation.

In the second case ($M = 50$), the sequence $\alpha_0, \alpha_1, \ldots, \alpha_{50}$ was obtained by letting $\delta = 10$ in relation (10). Moreover, we chose a decreasing sequence for scalars ρ_i so that ρ_0 is minimized. To stay within the admissible range of ρ_i for any value of α_i, we found graphically the linear expression $\rho_i = 0.3923(\alpha_i - 1)$. Figure 3 shows the sequence ρ_i as a function of α_i, together with lower and upper bounds on ρ_i. For this choice of scalars ρ_i, the PLC algorithm returned $N = 50$ nested ellipsoids. For comparison, the volume ratio of ellipsoids $(\rho_0 + 1)^{-1}\mathcal{E}_0$ and $\mathcal{E}_0$ is 2.7078. This illustrates the significant improvement obtained when exploiting the freedom one has in the choice of design parameters ρ_i. The nominal closed-loop system ($F = 0$) was then simulated for an initial condition $x_0 = [0.346 \quad -1.4901 \quad 0.9794 \quad 0]'$ belonging to $(\rho_0+1)^{-1}\mathcal{E}_0$. In Figure 4, we compare the guaranteed cost control when inputs are not constrained (in dotted line) and our piecewise-linear control that is designed to handle the constraints. Because of the important number of switching surfaces, the spiking observed in Figure 2 is drastically diminished.

5.2 LMI Approach

We used the software SDPSOL [23] for solving LMI determinant maximization problem (17). With $R = I_m$ and $Q = I_n$, the LMI guaranteed cost control, obtained for $\alpha = 1$, reads

$$u = \begin{bmatrix} -0.47 & 0.33 & 0.74 & 0.98 \\ 0.13 & 0.65 & 0.15 & -0.57 \end{bmatrix} x.$$

The volume ratio of the ellipsoids computed through the LMI approach and the ARE approach are approximately

- 106 for the innermost ellipsoids where performance is ensured, and
- 266 for the outermost ellipsoids of stabilizable initial conditions.

The projections of the outermost ellipsoids are shown in Figure 5. This clearly illustrates the greater flexibility of the LMI approach over the ARE approach. However, this is obtained at the price of a higher (off-line) computational load. Computation of 50 nested ellipsoids and linear feedback controllers, with MATLAB[1] 4.2 running on a Sun Sparc

[1]MATLAB is a trademark of The MathWorks, Inc.

Station 30 approximately required

- 1.4 seconds for the ARE approach, and
- 35 seconds for the LMI approach.

6 Conclusion

In this paper we have addressed the problem of robust stabilization of an uncertain system subject to input constraints. We have focused our research efforts on a local stabilization approach. No open-loop stability assumptions were made, thus we have simultaneously determined a robust control law and a set of initial conditions that can be stabilized. Two design algorithms were exposed that build piecewise-linear feedbacks by solving a parametrized ARE of the $\mathcal{H}_\infty$ control theory and a parametrized LMI determinant maximization problem, respectively. The switching boundaries were defined as a nested family of positively invariant and contractive ellipsoids. The system state must be initialized in the outermost ellipsoid of the family. Our control law then ensures that state trajectories will converge to the innermost ellipsoid, where a certain performance is guaranteed beyond stabilization. Some refinements are also proposed that aim at easing ellipsoid nesting and, most notably, at extending the domain of safe initial conditions.

Main advantages of our design algorithm are believed to be its simplicity and the fact that it could readily be implemented. All calculations are performed off-line and only rely on ARE or LMI resolution, for which powerful numerical tools are available. For these reasons, we hope that our algorithms will fill a need for easily applicable robust control laws that handle input saturation. We are currently investigating the output feedback case in order to tackle a wider range of practical problems.

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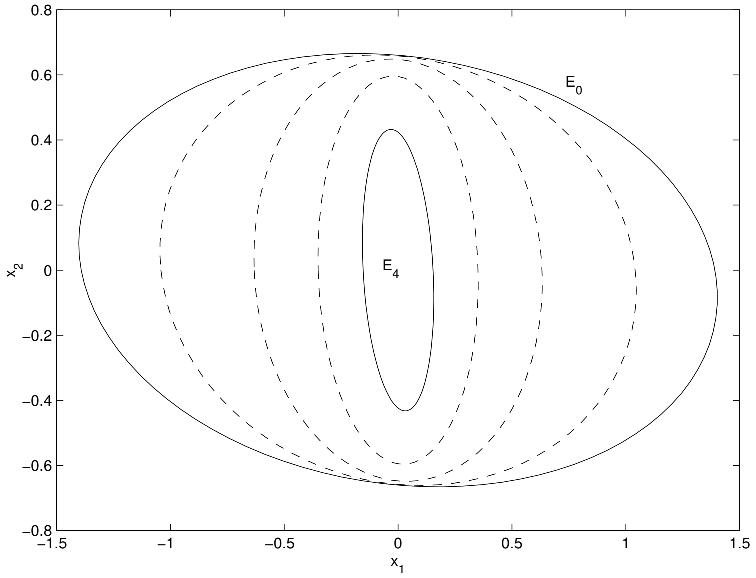

Figure 1: Case $M = N = 4$. Projection on the state plane (x_1, x_2). Switching surfaces are nested ellipsoids $\mathcal{E}_0 \supset \mathcal{E}_1 \supset \cdots \supset \mathcal{E}_4$.

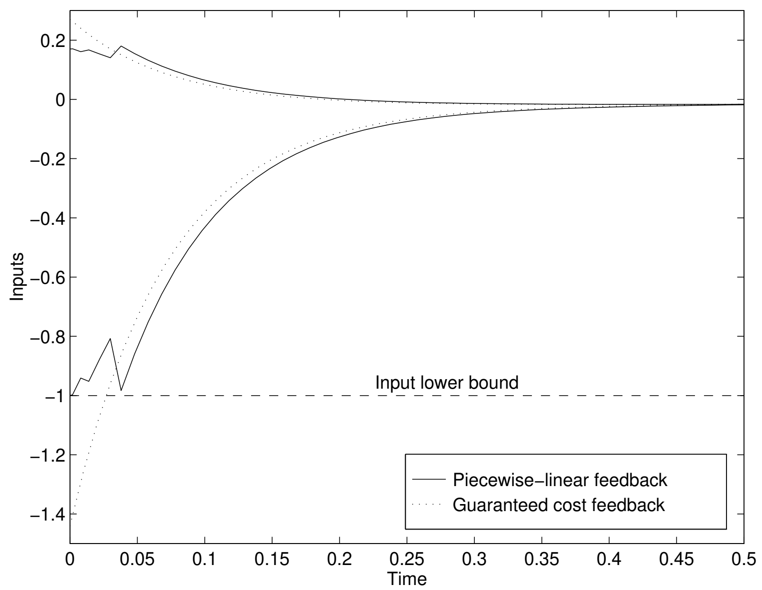

Figure 2: Case $M = N = 4$. Comparison between guaranteed cost inputs without saturation and PLC inputs.

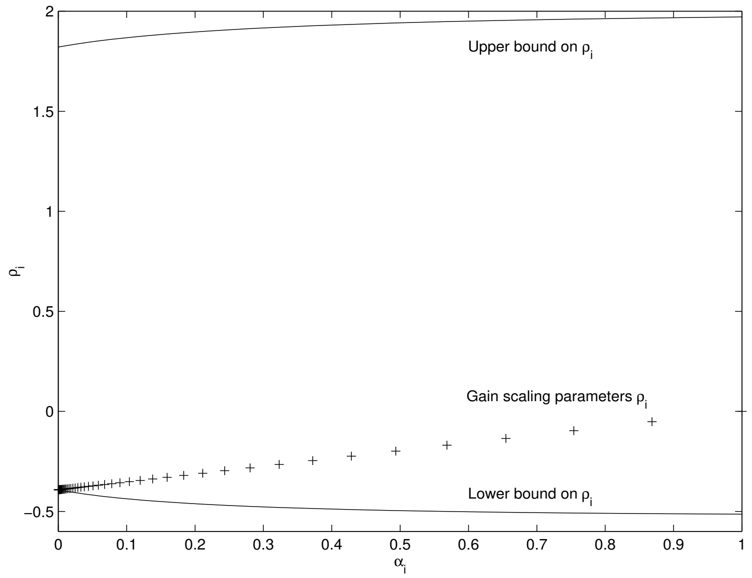

Figure 3: Case $M = N = 50$. Sequence of gain scalaing parameters ρ_i with lower and upper bounds $\underline{\rho}_i, \overline{\rho}_i$.

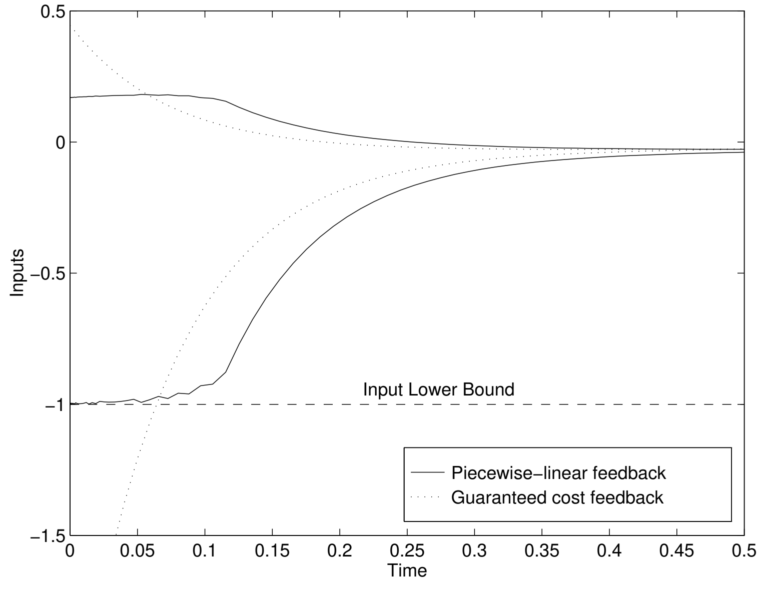

Figure 4: Case $M = N = 50$. Comparison between guaranteed cost inputs without saturation and PLC inputs.

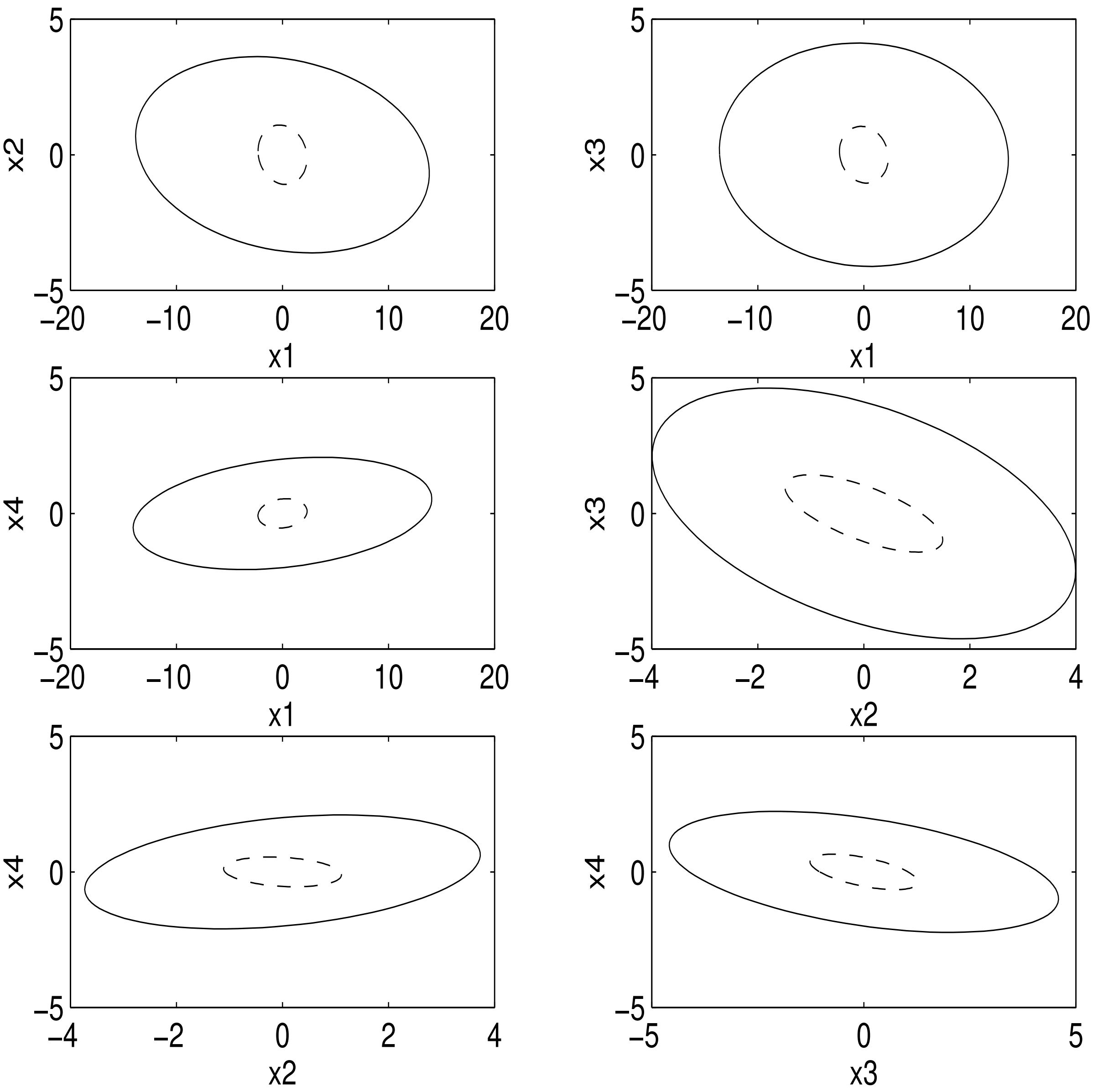

Figure 5: Projections of stabilizable initial condition sets $\mathcal{E}_0$ for the LMI approach (solid line) and the ARE approach (dashed line).