Overcoming non-convexity in polynomial robust control design

Didier Henrion[1,2,3,4] Michael Šebek[4]

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Abstract

When developing efficient and reliable computer-aided control system design (CACSD) tools for low-order robust control systems analysis and synthesis, the main issue faced by theoreticians and practitioners is the non-convexity of the stability domain in the space of polynomial coefficients, or equivalently, in the space of design parameters. In this paper, we survey some of the recently developed techniques to overcome this non-convexity, underlining their respective pros and cons. We also enumerate some related open research problems which, in our opinion, deserve particular attention.

Keywords: Computer-aided control system design (CACSD), Robust control, Low-order controller design, Polynomials, Convex optimization, Linear matrix inequalities (LMI)

1 Introduction

When developing efficient and reliable computer-aided control system design (CACSD) tools for low-order robust control systems analysis and synthesis, the main issue resides in the fundamental algebraic property that the stability domain in the space of coefficients of a polynomial is a non-convex set in general. Similarly, the set of square matrices with eigenvalues within a given region of the complex plane (such as the left half-plane or the unit disk) is generally non-convex.

As a result, several very basic control problems such as multivariable static output feedback or simultaneous stabilization of three (or more) plants are still open, meaning that

[1]Corresponding author. FAX: +33 5 61 33 69 69. E-mail: henrion@laas.fr

[2]LAAS-CNRS, 7 Avenue du Colonel Roche, 31 077 Toulouse, France.

[3]Institute of Information Theory and Automation, Academy of Sciences of the Czech Republic, Pod vodárenskou věží 4, 182 08 Prague, Czech Republic.

[4]Department of Control Engineering, Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 2, 166 27 Prague, Czech Republic.

there have been no efficient (polynomial-time) algorithms proposed so far to solve them. Most of the textbooks on the parametric and polynomial approaches to linear systems control focus on robust stability [1, 2, 4], and a very few low-order controller design results are available.

In this work, we survey some recent techniques that have been developed to overcome non-convexity in the parameter space. This survey is not meant to be comprehensive or exhaustive. However, in our opinion, focusing on non-convexity allows to cover in a unified way several topical issues in linear systems control. These issues may seem unrelated at first sight, but all of them are correlated in some way. These issues deserve attention and may open up further (sometimes new) research directions lying between control engineering, mathematical programming and numerical analysis.

2 Different approaches to overcoming non-convexity

2.1 Approximations of stability domain

Since the stability domain in the space of polynomial coefficients or design parameters is generally non-convex, it makes sense to approximate this domain with a convex set. We can resort either to (a) outer approximations, yielding necessary stability conditions, or constructive sufficient conditions for instability (analysis) or non-existence of a stabilizing controller (design); or to (b) inner approximations, yielding constructive sufficient conditions for stability (analysis) or existence of a stabilizing controller (design).

Concerning inner convex approximations, we can distinguish basically between

- polytopic approximations, where the polytope is either described by its vertices or its faces; most of the results then follow from variations around Kharitonov's theorem; typically, a valid polytope is built within the stability region around a stable nominal point, and it is scaled up until stability is lost, see [3] and references therein;

- spherical approximations, where a sphere is built around a stable nominal point, and its radius is increased until stability is lost [4]; the advantage of spherical approximations is the compactness of their representation (only one scalar quadratic inequality), in contrast with polytopic approximations (typically characterized by many inequalities);

- ellipsoidal approximations, allowing to optimize over the ellipsoid shape, contrary to the spherical approximation, see [12] for a recent overview;

- LMI approximations, which are more flexible representations of convex sets (including polytopes, spheres and ellipsoids); an LMI approximation based on strict positive realness (SPR) and polynomial positivity is proposed in [11] for fixed-order robust controller design.

2.2 LMI relaxations

As shown in [6], control systems analysis and synthesis systematically boil down to solving a set of multivariate polynomial inequalities. These inequalities are generally highly non-convex in the parameter space. In order to overcome non-convexity, one can resort to lift and project techniques, originally used in the context of combinatorial optimization. The idea is to lift the original non-convex problem into a higher dimensional optimization space, with the help of additional variables and additional constraints. Non-convexity of the lifted problem is then relaxed, yielding a convex optimization problem which is solved. Then the solution is projected back onto the lower dimensional space of original variables.

Sum-of-squares decompositions of positive multivariate polynomials and other results of real algebraic geometry can be invoked to build systematically a hierarchy of convex LMI relaxations, see [22] and [18]. Under mild assumptions, we can guarantee asymptotic convergence of the solutions of the convex LMI relaxations to the solution of the original non-convex problem. The hierarchy allows to gradually removing conservatism, at the price of additional variables and constraints, hence increased computational burden.

Two software implementations of the hierarchy of convex LMI relaxations for non-convex multivariate polynomial optimization problems are available:

- SOStools, based on the sum-of-squares formalism [24];
- GloptiPoly, based on the dual theory of polynomial moments [14].

See [25] for applications of SOStools in solving non-linear control problems, and [15] for applications of GloptiPoly in solving various other control problems.

2.3 Non-convex optimization

With the help of the Hermite stability criterion (a symmetrized version of the Routh-Hurwitz criterion), the set of stable polynomials can be described via a bilinear matrix inequality (BMI) in the polynomial coefficient space. Contrary to LMI problems, BMI problems are difficult, non-convex global optimization problems [8]. For almost one decade, this has precluded the development of efficient BMI solvers, except for very special, dedicated problem instances. This trend is changing now in our opinion, and the interest in developing general-purpose BMI solvers is growing.

Recently, the first publicly available BMI solver, called PENBMI, was released, based on an augmented Lagrangian and penalty function technique, an extension of a convex optimization algorithm implemented in the solver PENNON [17]. Convergence to a local minimizer satisfying first order optimality conditions is guaranteed, and numerical experiments on polynomial simultaneous stabilization problems are promising [13].

Independently, non-smooth eigenvalue optimization algorithms based on gradient sampling have been developed [5], and their application to polynomial stabilization problems and more generally BMI solving is currently under active study.

Note however that these solvers seem to be efficient only when optimizing over a parameter space of relatively low dimension. In our opinion, this is not restrictive since for implementation reasons controllers must generally have low order and low complexity. Almost 90% of the controllers in industry are PID controllers, with only three scalar parameters to be optimized.

What has been a major hurdle so far when applying non-convex optimization algorithms is the fact that two kind of decision variables generally appear in BMI control problems:

- a small number of controller variables, which are really the parameters over which the optimization is carried out;
- a large number of additional variables, typically a Lyapunov matrix instrumental to ensuring stability or performance.

Because of non-convexity, it is very unlikely that general-purpose BMI solvers can cope with a large number of decision variables, so Lyapunov-like matrices must be avoided as much as possible when formulating BMI control problems. In [13], simultaneous stabilization (a notoriously difficult robust control problem) was approached via the Hermite polynomial stability criterion precisely because this criterion does no feature any additional Lyapunov matrix variable. Therefore, the optimization can be performed directly on the controller parameters.

3 Open problems and directions for further research

3.1 Dedicated semidefinite programming algorithms

The convex inner LMI approximations of the stability domain described in section 2.1 yields LMI problems with a very specific structure reminiscent of the Kalman-Yakubovich-Popov lemma. LMI decision variables can be gathered into two categories:

- a small number of controller parameters, which are coefficients of the numerator and denominator controller polynomials;
- a large number of additional parameters, which are entries of a Lyapunov-like matrix proving positivity of the LMI, and hence acting as a closed-loop stability certificate.

Controller parameters and additional parameters appear in a decoupled fashion in the LMI.

The decoupling between controller and additional parameters can be exploited to reduce significantly the number of decision variables in the LMIs, see [28].

Relationships between this inner LMI approximation in the polynomial setting and the less conservative LMI robustness conditions of [21, 23] in the state-space setting were investigated in [10]. It turns out that the dedicated SDP algorithms mentioned above can also be applied to the state-space LMI formulations of [21, 23].

3.2 Better approximations of the stability domain

A question that naturally arises is whether we can build another LMI inner approximation of the stability domain which is systematically better (i.e. less conservative, or tighter) than the LMI inner approximation described in section 2.1. For example, can we use the hierarchy of LMI relaxations mentioned in section 2.2 in order to build a hierarchy of tighter LMI inner approximations ?

3.3 Hermite criterion for polynomial matrices

As pointed out in section 2.3, the Hermite stability criterion for scalar polynomials can be used to formulate some robust control problems, such as SISO simultaneous stabilization, directly as a BMI in the controller parameters, avoiding the introduction of a large number of Lyapunov variables.

A Lyapunov stability theory is available for MIMO linear systems described by polynomial matrices, see [9]. However, the stability conditions proposed there always involve an additional Lyapunov matrix. It is expected that we could remove this Lyapunov matrix in order to derive a BMI stability condition which would be the MIMO counterpart of the scalar Hermite stability criterion. Quadratic differential forms [30] may be instrumental to this derivation.

3.4 Efficient non-convex optimization algorithms

As noticed in section 2.3, the interest in developing BMI or eigenvalue optimization solvers is growing. Yet a very few software have been developed and tested so far, and a lot still remains to be done. Most of the developments have been of academic nature and besides PENBMI [13] no BMI solver is available.

A comprehensive database of more than one hundred notorious linear control problems (static output feedback, H_2 or H_∞) has been gathered in [19]. Developers of BMI solvers and eigenvalue optimization codes can test extensively their software on the examples from this database. This task has been made really easier since the recent release of a new version 3.0 of the parser YALMIP which provides all the necessary features to define and solve BMI problems in a user-friendly way [20].

3.5 Improving conditioning of polynomial problems

Although there has been so far no appropriate definition of conditioning for general LMI or BMI optimization problems, it is expected that numerical linear algebra techniques such as pre-conditioning can have a big impact on the performance of LMI or BMI solvers.

Without appropriate scaling or change of basis, it is well-known that linear algebra problems involving polynomials are generally ill-conditioned [29]. This has led researchers to resort systematically to state-space methods, as recalled in [16].

State-space problems can be ill-conditioned too however, and in our opinion there is no solid theoretical argument to justify why polynomials should be systematically avoided when solving control problems. In the control community, there is currently a lack of thorough studies on conditioning properties for polynomials. Achievements by numerical analysts deserve to be better studied and disseminated [27, 31]

4 Conclusion

We have surveyed some of the existing approaches to deal with the non-convexity inherent to the control problems traditionally deemed as difficult. By no means our survey is meant to be exhaustive, and there are several other techniques that look promising to deal with some of these problems, for example symbolic quantifier elimination [6] or probabilistic methods [26].

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