Nonlinear optimal control: Numerical approximations via moments and LMI-relaxations

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Abstract— **We consider the class of nonlinear optimal control problems with all data (differential equation, state and control constraints, cost) being polynomials. We provide a simple hierarchy of LMI-relaxations whose optimal values form a nondecreasing sequence of lower bounds on the optimal value. Preliminary results show that good approximations are obtained with few moments.**

I. INTRODUCTION

In general, solving a general nonlinear optimal control problem (OCP) is a difficult challenge, despite powerfull theoretical tools are available, e.g. the maximum principle and Hamilton-Jacobi-Bellman optimality equation. However there exist many numerical methods to approximate the solution of a given optimal control problem. For instance, *Multiple shooting* techniques which solve two-point boundary value problems as decribed in e.g. [17], [7], or *direct methods*, as in e.g. [18], [5], [6], which for instance, use descent algorithms, among others.

Contribution. In this paper, we consider the particular class of nonlinear OCPs for which all data describing the problem (dynamics, state and control constraints) are *polynomials*. We propose a completely different approach to provide a good approximation of (only) the optimal value of the OCP, via a sequence of increasing lower bounds. As such, it could be seen as a complement to the above shooting or direct methods which provide an upper bound, and when the sequence of lower bounds converges to the optimal value, a test of their efficiency.

We first adopt an infinite-dimensional linear programming (LP) formulation based on the Hamilton-Jacobi-Bellman equation, as developed in e.g. Hernandez-Hernandez et al. [11]. We then follow a numerical approximation scheme (a relaxation of the original LP) in the vein of the general framework developed in Hernandez and Lasserre [10] for infinite-dimensional linear programs. We here exploit the fact that all data are *polynomials* to provide a hierarchy of semidefinite programming (SDP) (or, LMI) relaxations, whose optimal values form a monotone nondecreasing sequence of lower bounds on the optimal value of the OCP. The first numerical experiments show that good approximations are obtained early in the hierarchy, i.e., with few moments, confirming the efficiency of SDP-relaxations of the same vein developed for other applications; see e.g. Lasserre [12], [13] in global optimization and probability, Henrion and Lasserre [8] for robust control problems, among others.

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II. GENERAL FRAMEWORK

Let $\mathbb{R}[x] = [x_1, \dots x_n]$ (resp. $\mathbb{R}[t,x,u] = \mathbb{R}[t, x_1, \dots x_n, u_1, \dots, u_m]$) denote the ring of polynomials in the variables x (resp. in the variables t, x, u). Next, with $T > 0$, let:

- $X, X_T \subset \mathbb{R}^n$ and $U \subset \mathbb{R}^m$ be semi-algebraic sets.
- $\mathcal{U}$ be the set of measurable functions $\mathbf{u} : [0,T] \to U$.
- $h \in \mathbb{R}[t,x,u]$, $H \in \mathbb{R}[x]$
- $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ a polynomial map, i.e. $f_k \in \mathbb{R}[t,x,u]$ for all $k = 1, \dots, n$.

Let $x_0 \in X$ and consider the following OCP

$$J^*(0, x_0) := \inf_{\mathbf{u} \in \mathcal{U}} J(0, x_0, \mathbf{u}), \tag{1}$$

where

$$\begin{array}{rcl} J(0, x_0, \mathbf{u}) & = & \int_0^T h(s, x(s), u(s))\, ds + H(x(T)) \\ \dot{x}(s) & = & f(s, x(s), u(s)), \quad s \in [0,T) \\ (x(s), u(s)) & \in & X \times U \quad s \in [0,T) \\ x(T) & \in & X_T, \end{array} \tag{2}$$

and with initial condition $x(0) = x_0 \in X$.

Note that a particular OCP is the minimal-time controllability from x_0 to X_T, by letting $h \equiv 1$, $H \equiv 0$, i.e.

$$T^* = \min_{\mathbf{u} \in \mathcal{U}} \{ T \mid \quad \text{under (2) and } x(0) = x_0 \in X. \} \tag{3}$$

The optimal value T^* is the first *hitting time* of the set X_T.

A. A linear programming formulation

With a stochastic or deterministic OCP, one may associate an abstract infinite-dimensional linear programming (LP) problem P together with a dual P^*; see for instance Hernandez-lerma and Lasserre [9] for discrete-time Markov control problems, and Hernandez et al. [11] for deterministic optimal control problems, as well as the many references therein.

Typically, the primal problem P is related with Hamilton-Jacobi-Bellman optimality conditions whereas its dual P^* is defined in terms of *occupation measures* (and their associated invariance conditions for infinite horizon problems). One always has $\sup \mathrm{P} \le \inf \mathrm{P}^* \le J^*(0, x_0)$, where $J^*(0, x_0)$ is the optimal value of the OCP. Under suitable assumptions one may sometimes prove that $\sup \mathrm{P} = \min \mathrm{P}^* = J^*(0, x_0)$, i.e., both optimal values of P and P^* coincide with the optimal value $J^*(0, x_0)$ of the OCP, and P^* has an optimal solution; see e.g. the OCP analyzed in [11]. However, in their full generality, both linear programs P and P^* are rather abstract as they are not directly solvable as finite dimensional LPs. Then, for a numerical approximation of $\inf \mathrm{P}^*$ or $\sup \mathrm{P}$, one may invoke approximation schemes as defined in e.g. Hernandez-Lerma and Lasserre [10].

More precisely, let us follow Hernandez et al. [11] (where $X = X_T = \mathbb{R}^n$). With $\mathcal{X}$ a metric space, and $C(\mathcal{X})$ the space of real-valued bounded continuous functions on $\mathcal{X}$, let $b : \mathcal{X} \to \mathbb{R}$ be a continuous function such that $b \geq 1$. Then let $C_b(\mathcal{X})$ be the Banach space of continuous functions on $\mathcal{X}$, with norm $\|f\|_b := \sup_{x\in\mathcal{X}} |f(x)|/b(x)$, with topological dual $C_b(\mathcal{X})^*$. On the other hand, let $M_b(\mathcal{X})$ be the Banach space of finite signed Borel measures μ on $\mathcal{X}$, with finite norm $\|\mu\|_b := \int b\, d|\mu|$, where $|\mu|$ denotes the total variation of μ. Obviously, $M_b(\mathcal{X}) \subset C_b(\mathcal{X})^*$.

Let $U \subset \mathbb{R}^m$ be the control set, and let $\Sigma := [0, T] \times X$, $S := \Sigma \times U$. The set of controls $\mathcal{U}$ is the set of measurable functions $\mathbf{u} : [0, T] \to U$. Let $C_b^1(\Sigma)$ be the Banach space of functions $\varphi \in C_b(\Sigma)$ with partial derivatives $\partial\varphi/\partial x_j$ in $C_b(\Sigma)$ for all $j = 1, \ldots, n$.

With $u \in U$, let $A : C_b^1(\Sigma) \to C_b(\Sigma)$ be the mapping

$$\varphi \mapsto A\varphi(t, x, u) := \frac{\partial\varphi}{\partial t}(t, x) + \langle f(t, x, u), \nabla_x\varphi(t, x)\rangle, \tag{4}$$

and let $\mathcal{L} : C_b^1(\Sigma) \to C_b(S) \times C_b(\mathbb{R}^n)$ be the mapping

$$\varphi \mapsto \mathcal{L}\varphi := (-A\varphi, \varphi_T),$$

where $\varphi_T(x) := \varphi(T, x)$, for all $x \in X$. (Here we assume that the bounding function b is such that $\langle\nabla_x\varphi, f\rangle \in C_b(S)$ for all $\varphi \in C_b^1(\Sigma)$.)

Similarly, let $\mathcal{L}^* : C_b(S)^* \times C_b(\mathbb{R}^n)^* \to C_b^1(\Sigma)^*$ is the adjoint mapping of $\mathcal{L}$, defined by

$$\langle(\mu, \nu), \mathcal{L}\varphi\rangle = \langle\mathcal{L}^*(\mu, \nu), \varphi\rangle,$$

for all $((\mu, \nu), \varphi) \in C_b(S)^* \times C_b(\mathbb{R}^n)^* \times C_b^1(\Sigma)$. A function $\varphi : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, is a solution of the Hamilton-Jacobi-Bellman optimality equation if

$$\inf_{u\in U}\{A\varphi(s, x, u) + h(s, x, u)\} = 0, \quad (s, x) \in [0, T) \times X, \tag{5}$$

with boundary condition $\varphi_T(x)\ (= \varphi(T, x)) = H(x)$, for all $x \in X_T$. And, a function $\varphi \in C_b^1(\Sigma)$ is said to be a smooth *subsolution* of the Hamilton-Jacobi-Bellman equation (5) if

$$A\varphi + h \geq 0 \quad \text{on } [0, T) \times X \times U,$$

and $\varphi(T, x) \leq H(x)$, for all $x \in X_T$. Then, consider the infinite-dimensional linear program P

$$\mathrm{P}: \quad \sup_{\varphi\in C_b^1(\Sigma)} \{\langle\delta_{(0,x_0)}, \varphi\rangle \mid \quad \mathcal{L}\varphi \leq (h, H)\}, \tag{6}$$

and its dual

$$\mathrm{P}^*: \quad \inf_{0\leq(\mu,\nu)\in\Delta} \{\langle(\mu, \nu), (h, H)\rangle \mid \quad \mathcal{L}^*(\mu, \nu) = \delta_{(0,x_0)}\}. \tag{7}$$

(where $\Delta := C_b(S)^* \times C_b(X_T)^*$).

Note that the feasible solutions φ of P are precisely smooth subsolutions of (5).

As already mentioned, and under some suitable assumptions, one may prove that $\sup \mathrm{P} = \min \mathrm{P}^* = J^*(0, x_0)$; see e.g. [11, Theor. 5.1] for the case $X = X_T = \mathbb{R}^n$.

In particular, the mapping $\mathcal{L}$ is continuous with respect to the weak topologies $\sigma(C_b(S) \times C_b(\mathbb{R}^n), C_b(S)^* \times C_b(\mathbb{R}^n)^*)$ and $\sigma(C_b^1(\Sigma), C_b^1(\Sigma)^*)$ if $\mathcal{L}^*(\Delta) \subset C_b^1(\Sigma)^*$.

Remark 2.1: The spaces $C_b^1(S)^*$ and $C_b(X_T)^*$ are useful, e.g. to use the celebrated Banach-Alaoglu theorem on the weak* compactness of their unit ball, and show $\sup \mathrm{P} = \min \mathrm{P}^*$ as in [11], but not very practical. Therefore, with $\Delta_0 := M_b(S) \times M_b(X_T)$, consider instead the stronger dual

$$\hat{\mathrm{P}}^*: \quad \inf_{0\leq(\mu,\nu)\in\Delta_0} \{\langle(\mu, \nu), (h, H)\rangle \mid \quad \mathcal{L}^*(\mu, \nu) = \delta_{(0,x_0)}\} \tag{8}$$

of P, obtained from P^* by replacing the Banach spaces $C_b^1(S)^*$ and $C_b(X_T)^*$, by $M_b(S) \subset C_b^1(S)^*$ and $M_b(X_T) \subset C_b(X_T)^*$, respectively. Obviously, we have

$$\sup \mathrm{P} \leq \inf \mathrm{P}^* \leq \inf \hat{\mathrm{P}}^* \leq J(0, x_0).$$

And so, $\inf \mathrm{P}^* = J(0, x_0)$, whenever $\sup \mathrm{P} = J(0, x_0)$.

The dual $\hat{\mathrm{P}}^*$ is in fact more practical to work with, as the elements of $M_b(S)$ (or $M_b(X_T)$) are objects with a physical meaning. In fact, as we shall next see, the dual $\hat{\mathrm{P}}^*$ has a nice and simple interpretation in terms of *occupation measures* of the trajectories $(s, x(s), u(s))$ of the OCP (1)-(2).

B. The linear pogram $\hat{\mathrm{P}}^*$ *and occupation measures*

Given an admissible control $\mathbf{u} = \{u(t),\ 0 \leq t < T\}$ for the OCP (1)-(2), introduce the probability measure $\nu^{\mathbf{u}}$ on $\mathbb{R}^n$, and the measure $\mu^{\mathbf{u}}$ on $[0, T] \times \mathbb{R}^n \times \mathbb{R}^m$, defined by

$$\nu^{\mathbf{u}}(B) := I_B[x(T)], \quad B \in \mathcal{B}_n \tag{9}$$

$$\mu^{\mathbf{u}}(A \times B) := \int_{[0,T]\cap A} I_B[(x(s), u(s))]\, ds, \tag{10}$$

for all $(A, B) \in \mathcal{A} \times \mathcal{B}_{nm}$, and where $\mathcal{B}_n$ (resp. $\mathcal{B}_{nm}$) denotes the usual Borel σ-algebra associated with $\mathbb{R}^n$ (resp. $\mathbb{R}^n \times \mathbb{R}^m$), and $\mathcal{A}$ the Borel σ-algebra of $[0, T]$, and $I_B(\bullet)$ the indicator function of the set B.)

The measure $\mu^{\mathbf{u}}$ is called the *occupation measure* of the state-action (deterministic) process $(s, x(s), u(s))$ up to time T, whereas $\nu^{\mathbf{u}}$ is the occupation measure of the state $x(T)$ at time T.

Remark 2.2: As $\mathbf{u} \in \mathcal{U}$ is admissible, it follows that $\nu^{\mathbf{u}}$ is a probability measure supported on X_T, whereas $\mu^{\mathbf{u}}$ is supported on $[0, T] \times X \times U$. Conversely, let $\mathbf{u} \in \mathcal{U}$ be a control, and let $\mu^{\mathbf{u}}$ be defined as in (10), where $(x(s), u(s))$ satisfies the o.d.e. (2) (but not necessarily the state and control constraints). If $\mu^{\mathbf{u}}$ has its support on $[0, T] \times X \times U$, then $(x(s), u(s)) \in X \times U$ for almost all $s \in [0, T)$. In addition, if $\nu^{\mathbf{u}}$ has its support in X_T, then $x(T) \in X_T$, and so, $\mathbf{u}$ is an admissible control. Indeed, in view of (10),

$$T = \int_0^T I_{X\times U}[(x(s), u(s))]\, ds$$

$$\Rightarrow I_{X\times U}[(x(s), u(s))] = 1 \quad \text{for a.a. } s \in [0, T),$$

and so $(x(s), u(s)) \in X \times U$ for almost all $s \in [0, T)$.

Then, observe that the criterion in (1) now reads

$$J(0, x_0, \mathbf{u}) = \int H\, d\nu^{\mathbf{u}} + \int h\, d\mu^{\mathbf{u}} = \langle (\mu^{\mathbf{u}}, \nu^{\mathbf{u}}), (h, H) \rangle,$$

and, in addition, one may rewrite (2) as

$$\int g_T\, d\nu^{\mathbf{u}} - g(0, x_0) = \int \langle \nabla_x g, f \rangle\, d\mu^{\mathbf{u}} \qquad (11)$$

for all $g \in C_b^1(\Sigma)$ (and where $g_T(x) \equiv g(T, x)$ for all $x \in \mathbb{R}^n$). In other words,

$$\mathcal{L}^*(\mu^{\mathbf{u}}, \nu^{\mathbf{u}}) = \delta_{(0,x_0)},$$

that is, $(\mu^{\mathbf{u}}, \nu^{\mathbf{u}})$ is a feasible solution of $\hat{\mathrm{P}}^*$ defined in (8), with value $J(0, x_0, \mathbf{u})$.

Hence, if the linear program P is a rephrasing of the OCP (1)-(2) in terms of the Hamilton-Jacobi-Bellman equation, its LP dual $\hat{\mathrm{P}}^*$ is a rephrasing of the OCP in terms of *occupation measures* of its trajectories $(s, x(s), u(s))$. These two LPs are the deterministic analogues of the linear programs described in Hernandez and Lasserre [9, §6] for discrete time Markov control problems.

C. SDP-relaxations of $\hat{\mathrm{P}}^*$

The linear program $\hat{\mathrm{P}}^*$ is infinite-dimensional, and so, not tractable as it stands. Therefore, we first provide a relaxation scheme that provides a sequence of approximating LPs $\hat{\mathrm{P}}_r^*$, each with *finitely many* constraints.

Let $\mathcal{D}_1 \subset \mathcal{D}_2 \ldots \subset C_b^1(\Sigma)$ be an increasing sequence of finite spaces of functions. We next define the following infinite-dimensional linear programming problem

$$\hat{\mathrm{P}}_r^* : \left\{ \begin{array}{ll} \inf\limits_{\nu,\mu \geq 0} & \int H\, d\nu + \int h\, d\mu \\ \text{s.t.} & \int g_T\, d\nu - g(x_0) = \int \langle \nabla_x g, f \rangle\, d\mu, \quad \forall g \in \mathcal{D}_r \\ & (\mu, \nu) \in M_b(S) \times M_b(X_T) \end{array} \right. \qquad (12)$$

whose optimal value is denoted by $\inf \hat{\mathrm{P}}_r^*$.

Recall that the constraint

$$\int g_T\, d\nu - g(x_0) = \int \langle \nabla_x g, f \rangle\, d\mu, \qquad g \in \mathcal{D}_r$$

can be equivalently written

$$\langle g, \mathcal{L}^*(\mu, \nu) \rangle = \langle g, \delta_{\{0, x_0\}} \rangle, \qquad g \in \mathcal{D}_r.$$

Therefore, the linear program $\hat{\mathrm{P}}_r^*$ is a relaxation of $\hat{\mathrm{P}}^*$, and so $\inf \hat{\mathrm{P}}_r^* \uparrow$ as r increases, and $\hat{\mathrm{P}}_r^*$ is a relaxation of (1) for all $r \in \mathbb{N}$, that is, $\inf \hat{\mathrm{P}}_r^* \leq J^*(0, x_0)$ for all r.

However, if $\hat{\mathrm{P}}_r^*$ has now only finitely many (linear) constraints, it is still an infinite-dimensional LP. We next use the fact that all defining functions of the OCP (1) are polynomials, to provide a sequence of (finite dimensional) semidefinite programs (SDP), which are all relaxations of the linear program $\hat{\mathrm{P}}_r^*$. To do this, we will take for $\mathcal{D}_r$ a set of polynomials of total degree bounded by r

Recall that we asssume that all functions h, H, f in the description of the OCP (2) are polynomials, and the sets X, U, X_T are semi-algebraic sets.

Observe that when $g \in \mathbb{R}[t, x]$, then (11) defines countably many *linear* equalities linking the *moments* of $\mu^{\mathbf{u}}$ and $\nu^{\mathbf{u}}$, because if g is a polynomial then so are $\partial g/\partial t$ and $\partial g/\partial x_k$, for all k, and so is $\langle \nabla_x g, f \rangle$. Moreover, the criterion $J(0, x_0, \mathbf{u})$ is also a linear combination of the moments of $\mu^{\mathbf{u}}$ and $\nu^{\mathbf{u}}$.

So let $y = \{y_\alpha\}_{\alpha \in \mathbb{N}^n}$ (resp. $z = \{z_{p\alpha\beta}\}_{p \in \mathbb{N}, \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m}$) be a sequence indexed in the canonical basis $\{x^\alpha\}$ of $\mathbb{R}[x]$ (resp. $\{t^p x^\alpha u^\beta\}$ of $\mathbb{R}[t, x, u]$).

Next, let $L_y : \mathbb{R}[x] \to \mathbb{R}$ be the linear functional defined by

$$h (:= \sum_{\alpha \in \mathbb{N}^n} h_\alpha x^\alpha) \mapsto L_y(h) := \sum_{\alpha \in \mathbb{N}^n} h_\alpha y_\alpha,$$

and similarly, let $L_z : \mathbb{R}[t, x, u] \to \mathbb{R}$ be the linear functional be defined by

$$h \mapsto L_z(h) := \sum_{p \in \mathbb{N}, \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m} h_{p\alpha\beta}\, z_{p\alpha\beta},$$

whenever $h(t, x, u) = \sum_{p \in \mathbb{N}, \alpha \in \mathbb{N}^n, \beta \in \mathbb{N}^m} h_{p\alpha\beta}\, t^p x^\alpha u^\beta)$.

Then for $g := t^p x^\gamma$, $p \in \mathbb{N}$, $\gamma \in \mathbb{N}^n$, (11) reads

$$L_y(g_T) - L_z(\langle \nabla_x g, f \rangle) = \left\{ \begin{array}{ll} x_0^\gamma & \text{if } p = 0 \\ 0 & \text{if } p \geq 1 \end{array} \right. ,$$

or, equivalently,

$$T^p y_\gamma - \langle A_{p\gamma}, z \rangle = \left\{ \begin{array}{ll} x_0^\gamma & \text{if } p = 0 \\ 0 & \text{if } p \geq 1 \end{array} \right. ,$$

for some linear functional $A_{p\gamma}$ (identified with an infinite vector with finitely many non zero coefficients in the canonical basis $\{t^p x^\alpha u^\beta\}$).

Therefore, if we take for $\mathcal{D}_r$, the monomials of total degree less than r, in the canonical basis $\{t^p x^\gamma\}$, the linear program $\hat{\mathrm{P}}_r^*$ is entirely described only in terms of finitely many moments of the measures μ and ν.

Hence, with such a finite set $\mathcal{D}_r$, the linear program $\hat{\mathrm{P}}_r$ has only *finitely many* constraints and variables. However, it remains to express conditions on these variables y, z to be *moments* of measures with support in X_T and $[0, T] \times X \times U$, respectively. This is where we now invoke powerfull results in the theory of moments, on the so-called K-moment problem.

If $X_T \subset \mathbb{R}^n$ is a semi-algebraic set defined by (finitely many) polynomials inequalities $\theta_j(x) \geq 0$, $j \in J_T$ then, a sequence y has a representing measure ν supported on X, i.e.,

$$y_\alpha = \int_{X_T} x^\alpha d\nu, \quad \forall \alpha \in \mathbb{N}^n,$$

only if

$$L_y(h^2) \geq 0; \quad L_y(\theta_j h^2) \geq 0, \quad \forall h \in \mathbb{R}[x],\ j \in J_T.$$

Notice that, if $X_T = \{0\}$ (as in minimum time OCPs), then it simplifies to $y_0 = 1$ and $y_\alpha = 0$, for all $0 \neq \alpha \in \mathbb{N}^n$.

Similarly, let $X \subset \mathbb{R}^n$ (resp. $U \subset \mathbb{R}^m$) be a semi-algebraic set defined by polynomial inequalities $v_j(x) \geq 0$, $j \in J$ (resp. $w_k(u) \geq 0$, $k \in K$). Then the sequence $z = \{z_{p\alpha\beta}\}$

has a representing measure μ on $[0,T]\times X\times U$ only if $L_z(h^2)\geq 0$, and

$$L_z(v_jh^2)\geq 0;\quad L_z(w_kh^2)\geq 0;\quad L_z(t(T-t)h^2)\geq 0,$$

for all $h\in\mathbb{R}[t,x,u]$, and $j\in J_T, k\in K$. If X_T, and $X\times U$ are compact then under relatively weak assumptions the conditions are also sufficient (by using a result of Putinar in [14]). Moreover, if one already knows that z has a representing measure μ, then under the same compactness assumption, to ensure that μ has its support in $[0,T]\times X\times U$, it is enough to state conditions on the marginal $\{z_{\alpha 0}\}$ and $\{z_{0\beta}\}$ of $\{z_{\alpha\beta}\}$, that is $L_z(v_jh^2)\geq 0$, $\forall h\in\mathbb{R}[x]$,$j\in J_T$; $L_z(w_kh^2)\geq 0$, $\forall h\in\mathbb{R}[u]$, $k\in K$, and $L_z(t(T-t)h^2)\geq 0$, $\forall h\in\mathbb{R}[t]$.

The important property of all the above conditions, is that when stated for all polynomials h of degree less than say r, they translate into *LMI conditions* on the y and z, via *moment* and *localizing* matrices associated with y,z,θ_j,v_jw_k, and as defined in e.g. Lasserre [10].

To summarize, let $\mathcal{D}_r^\bullet\subset\mathbb{R}[\bullet]$ be the monomials of the canonical basis of the space of polynomials in the variables $\bullet$, and of total degree less than r. For a polynomial θ_j, and depending on its parity, define $\deg\theta_j=2r(\theta_j)$ or $2r(\theta_j)-1$; and similarly for the polynomials $\{v_j\}$ and $\{w_k\}$. Finally, let $\deg f=2r_0$ or $2r_0-1$. Then, we end up with the sequence of LMI-relaxations

$$\mathrm{Q}_r:\begin{cases}\inf\limits_{y,z}\quad L_z(h)+L_y(H)\\ L_y(g_T)-g(x_0)-L_z(\langle\nabla_xg,f\rangle)=0,\ \forall g\in\mathcal{D}_{r-r_0}^{tx}\\ L_z(h^2)\geq 0,\ \forall h\in\mathcal{D}_r^{txu};\quad L_y(h^2)\geq 0,\ \forall h\in\mathcal{D}_r^{x}\\ L_y(h^2\theta_j)\geq 0,\quad\forall h\in\mathcal{D}_{r-r(\theta_j)}^{x},\ j\in J_T\\ L_z(h^2v_j)\geq 0,\quad\forall h\in\mathcal{D}_{r-r(v_j)}^{x},\ j\in J\\ L_z(h^2w_k)\geq 0,\quad\forall h\in\mathcal{D}_{r-r(w_k)}^{u},\ k\in K\\ L_z(h^2t(T-t))\geq 0,\quad\forall h\in\mathcal{D}_{r-1}^{t}\\ z_0=T,\end{cases}\tag{13}$$

whose optimal value is denoted by $\inf\mathrm{Q}_r$. Obviously we have $\inf\mathrm{Q}_r\leq\inf\hat{\mathrm{P}}_r^*\leq J^*(0,x_0)$, for all $r\in\mathbb{N}$.

For the *minimum time* OCP (3), we need adapt the notation. T is now a variable and $X_T=\Omega\subset\mathbb{R}^n$ a fixed semi-algebraic set. That is, given a control $\mathbf{u}=u(t)\in\mathcal{U}$, let $T_{\mathbf{u}}:=\inf\{t\geq 0\mid\ x(t)\in\Omega\}$. Of course we also must have $X\subset\mathbb{R}^n\setminus\Omega$. Then the SDP-relaxation Q_r now reads

$$\mathrm{Q}_r:\begin{cases}\inf\limits_{y,z}\quad z_0\\ L_y(g_T)-g(x_0)-L_z(\langle\nabla_xg,f\rangle)=0,\ \forall g\in\mathcal{D}_{r-r_0}^{tx}\\ L_z(h^2)\geq 0,\ \forall h\in\mathcal{D}_r^{txu};\quad L_y(h^2)\geq 0,\ \forall h\in\mathcal{D}_r^{x}\\ L_y(h^2\theta_j)\geq 0,\quad\forall h\in\mathcal{D}_{r-r(\theta_j)}^{x},\ j\in J_T\\ L_z(h^2v_j)\geq 0,\quad\forall h\in\mathcal{D}_{r-r(v_j)}^{x},\ j\in J\\ L_z(h^2w_k)\geq 0,\quad\forall h\in\mathcal{D}_{r-r(w_k)}^{u},\ k\in K\\ L_z(h^2t(T-t))\geq 0,\quad\forall h\in\mathcal{D}_{r-1}^{t}\end{cases}\tag{14}$$

and similarly, $\inf\mathrm{Q}_r\leq\inf\hat{\mathrm{P}}_r^*\leq T^*$, for all $r\in\mathbb{N}$.

For a *time-homogeneous* OCP, f,h do not depend on t, and so, simplifications occur. The measure $\mu^{\mathbf{u}}$ is now supported on $X\times U$ instead of $[0,T]\times X\times U$, and the functions φ in the definition of the linear program P in (6) are now in $C_b^1(X)$ instead of $C_b^1([0,T]\times X)$. And for instance, the SDP-relaxation Q_r defined in (13) now reads

$$\mathrm{Q}_r:\begin{cases}\inf\limits_{y,z}\quad L_z(h)+L_y(H)\\ L_y(g)-g(x_0)-L_z(\langle\nabla_xg,f\rangle)=0,\ \forall g\in\mathcal{D}_{r-r_0}^{x}\\ L_z(h^2)\geq 0,\ \forall h\in\mathcal{D}_r^{xu};\quad L_y(h^2)\geq 0,\ \forall h\in\mathcal{D}_r^{x}\\ L_y(h^2\theta_j)\geq 0,\quad\forall h\in\mathcal{D}_{r-r(\theta_j)}^{x},\ j\in J_T\\ L_z(h^2v_j)\geq 0,\quad\forall h\in\mathcal{D}_{r-r(v_j)}^{x},\ j\in J\\ L_z(h^2w_k)\geq 0,\quad\forall h\in\mathcal{D}_{r-r(w_k)}^{u},\ k\in K\\ L_z(h^2t(T-t))\geq 0,\quad\forall h\in\mathcal{D}_{r-1}^{t}\\ z_0\ =T.\end{cases}\tag{15}$$

So, the above defined LMI-relaxations Q_r contain moments y and z, up to order $2r$ only. In the next section, we consider two applications of this approach to nonlinear (time-homogeneous) control problems in Sections III-A and III-B.

TABLE I

LMI-RELAXATIONS: $\inf\mathrm{Q}_r$

r	2	3	4	5	6	7
$\inf\mathrm{Q}_r$	1.0703	1.7100	2.5951	3.2026	3.3888	3.4350
% error	68 %	50,4%	24,7%	7,16%	1,76%	0,42%

III. ILLUSTRATIVE EXAMPLES

We here consider minimum time OCPs (3), that is, we want to approximate the *minimum time* to steer a given initial condition to the the origin. We have tested the above methodology on two test OCPs, the double and Brockett *integrators*, because the associated optimal value T^* can be calculated exactly.

A. The double integrator

Consider the double integrator system in $\mathbb{R}^2$:

$$\begin{array}{rcl}\dot{x_1}(t)&=&x_2(t)\\ \dot{x_2}(t)&=&u(t)\end{array}\tag{16}$$

where $x=(x_1,x_2)$ is the state and the control $\mathbf{u}=u(t)\in\mathcal{U}$, satisfies the constraint $|u(t)|\leq 1$, for all $t\geq 0$.

1) Exact computation: For this very simple system the Hamilton-Jacobi-Bellman equation can been solved explicitly, as in e.g. [3]. Denoting $T(x_1,x_2)$ the minimal time to reach the origin from (x_1,x_2), we have:

$$T(x_1,x_2)=\begin{cases}2\sqrt{x+\frac{y^2}{2}}+y\ \text{if}\ x\geq-\frac{y^2}{2}\text{sign}(y)\\ 2\sqrt{x+\frac{y^2}{2}}-y\ \text{if}\ x<-\frac{y^2}{2}\text{sign}(y)\end{cases}\tag{17}$$

In this case, and with the notation of Section II, we have $X=\mathbb{R}^2$, and $X_\tau=\{(0,0)\}$. As the dynamics is linear, the LMI-relaxation Q_r contains moments of order $2r$ only.

2) Numerical approximation: Table I displays the optimal values $\inf\mathrm{Q}_r$, $r=1,\ldots,7$, for the initial condition $x_0=(1,1)$. The optimal value is $T^*=1+\sqrt{6}\approx 3.4495$.

A very good approximation of T^* with less than 2% relative error, is obtained with moments or order 12 only.

B. The Brockett integrator

Consider the so-called *Brockett system* in $\mathbb{R}^3$

$$\begin{array}{rcl} \dot{x_1}(t) & = & u_1(t) \\ \dot{x_2}(t) & = & u_2(t) \\ \dot{x_3}(t) & = & u_1(t)x_2(t) - u_2(t)x_1(t), \end{array} \tag{18}$$

where $x = (x_1, x_2, x_3)$, and the control $\mathbf{u} = (u_1(t), u_2(t)) \in \mathcal{U}$, satisfies the constraint

$$u_1(t)^2 + u_2(t)^2 \leq 1, \quad \forall t \geq 0. \tag{19}$$

So, in this case, we have $X = \mathbb{R}^3$, $X_\tau = \{(0,0,0)\}$.

1) Exact computation: Let $T(x)$ be the minimum time needed to steer an initial condition $x \in \mathbb{R}^3$ to the origin. We recall the following result of [1] (in fact given for equivalent (reachability) OCP of steering the origin to a given point x).

Proposition 3.1: Consider the minimum time OCP for the system (18) with control constraint (19). The minimum time $T(x)$ needed to steer the origin to a point $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ is given by

$$T(x_1, x_2, x_3) = \frac{\theta\sqrt{x_1^2 + x_2^2 + 2|x_3|}}{\sqrt{\theta + \sin^2\theta - \sin\theta\cos\theta}} \tag{20}$$

where $\theta = \theta(x_1, x_2, x_3)$ is the unique solution in $[0, \pi)$ of

$$\frac{\theta - \sin\theta\cos\theta}{\sin^2\theta}(x_1^2 + x_2^2) = 2|x_3|. \tag{21}$$

Moreover, the function T is continuous on $\mathbb{R}^3$, and is analytic outside the line $x_1 = x_2 = 0$.

Remark 3.2: Along the line $x_1 = x_2 = 0$, one has

$$T(0, 0, x_3) = \sqrt{2\pi|x_3|}.$$

The singular set of the function T, *i.e.* the set where T is not C^1, is the line $x_1 = x_2 = 0$ in $\mathbb{R}^3$. More precisely, the gradients $\partial T/\partial x_i$, $i = 1, 2$, are discontinuous at every point $(0, 0, x_3)$, $x_3 \neq 0$. For the interested reader, the level sets $\{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid T(x_1, x_2, x_3) = r\}$, with $r > 0$, are displayed in Prieur and Trélat [16].

2) Numerical approximation: Recall that $\inf \mathrm{Q}_r \uparrow$ as r increases, i.e., the more moments we consider, the closer to the exact value we get. For instance, with the initial condition $x_0 = (1, 1, 1)$, one has $T^* = 1.8257$, and the first four LMI-relaxations Q_r, $r = 1, 2, 3$, give the following results:

$$T_2 = 1.4462; \quad T_3 = 1.5892; \quad T_4 = 1.7476,$$

and so, with moments of order 8 only, the relative error is 4.2%. With the LMI solver that we used, we have encountered memory space problems at the fifth LMI-relaxation, and so we display results only for the first four LMI-relaxations.

In Table II we have displayed the relative error $1 - \inf \mathrm{Q_r}/T^*$, $r \leq 4$, for 16 different values of the initial state $x(0) = x_0$, in fact, all 16 combinations of $x_{01} = 0$, $x_{02} = 0, 2/3, 4/3, 2$, and $x_{03} = 0, 2/3, 4/3, 2$. So, the entry $(2, 3)$ of Table II for the second LMI-relaxation, is $1 - \inf \mathrm{Q}_2/T^*$ for the initial condition $x_0 = (0, 2/3, 4/3)$.

Notice that the upper triangular part (i.e., when both fist coordinates x_{02}, x_{03} of the initial condition are away from zero) displays very good approximations with very few moments. In addition, the further the coordinates from zero, the best.

The regularity property of the minimal-time function seems to be an important topic of further investigation.

TABLE II

LMI-RELAXATIONS: $1 - \inf \mathrm{Q}_r/T^*$

Second LMI-relaxation: r=2			
0%	0%	0%	0%
98.6%	56%	14%	3%
97.4%	69%	33%	11%
97.4%	75%	45%	21%
Third LMI-relaxation: r=3			
0%	0%	0%	0%
89%	47%	9%	1%
83%	58%	23%	5%
82%	62%	34%	11%
Fourth LMI-relaxation: r=4			
0%	0%	0%	0%
71%	33%	2%	0%
61%	40%	11%	1%
58%	42%	9%	2%

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