# Support maximization with linear programming in the cone of nonnegative measures

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**Abstract**

This is a short summary of a talk given at the workshop "Structured Function Systems and Applications" organized by Maria Charina, Jean-Bernard Lasserre, Mihai Putinar and Joachim Stöckler at the Mathematisches Forschungsinstitut Oberwolfach, Germany, from 24 February to 2 March 2013. It reports on joint work with Milan Korda, Jean-Bernard Lasserre and Carlo Savorgnan.

We address the problem of maximizing (the volume) of the support of a linearly constrained nonnegative measure. We show that this decision problem admits an infinite-dimensional convex linear programming (LP) formulation, which implies that it can be solved numerically with a converging hierarchy of finite-dimensional semidefinite programming (SDP) problems if the problem data are semialgebraic. We describe applications of these techniques to the computation of the moments of a semialgebraic set, and to the estimation of the region of attraction of a polynomial dynamical system.

In [1] an infinite-dimensional LP approach was introduced to compute the moments of a given compact set $X_0 \subset \mathbb{R}^n$. The basic idea was to notice that the Lebesgue measure on $X_0$ is the measure $\mu_0$ solving the LP

$$\begin{array}{ll} \sup & \int \mu_0 \\ \text{s.t.} & \mu_0 + \hat{\mu}_0 = \lambda \\ & \mu_0 \geq 0, \ \text{spt}\, \mu_0 \subset X_0 \\ & \hat{\mu}_0 \geq 0, \ \text{spt}\, \hat{\mu}_0 \subset X \end{array} \tag{1}$$

where the supremum is over nonnegative measures $\mu_0(dx)$, $\hat{\mu}_0(dx)$ respectively supported on $X_0$ and a given compact set $X$ (say, a ball) which contains $X_0$, and such that $\mu_0$ and $\hat{\mu}_0$ sum up to $\lambda$, the Lebesgue measure on $X$. The dual to this LP is as follows

$$\begin{array}{ll} \inf & \int v_0 \lambda \\ \text{s.t.} & v_0(x) \geq 1 \ \text{on} \ X_0 \\ & v_0(x) \geq 0 \ \text{on} \ X \end{array} \tag{2}$$

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where the infimum is over a continuous function $v_0(x)$ nonnegative on $X$, which can be interpreted as a dual Lagrange multiplier for the primal linear constraint $\mu_0 + \hat{\mu}_0 = \lambda$. Note that the supremum in problem (1) is attained by $\mu_0$ equal to the Lebesgue measure on $X_0$. In contrast, the infimum in problem (2) is not attained, but it can be shown that the (continuous) function $v_0$ converges almost uniformly to the (discontinuous) indicator function of $X_0$ (equal to one on $X_0$ and zero elsewhere).

In [1] it is explained that if set $X_0$ is basic semialgebraic and described by finitely many given polynomial inequalities, infinite-dimensional primal-dual LP problems (1-2) can be solved by a converging hierarchy of finite-dimensional semidefinite programming (SDP) problems (in turn solved numerically with powerful primal-dual interior-point algorithms). At a given relaxation order $d$, the primal SDP is a moment relaxation of LP (1), whereas the dual SDP is a polynomial sum-of-squares restriction of LP (2). From the solution of the primal SDP of order $d$, we obtain a vector approximating the moments of degree up to $2d$ of the Lebesgue measure on $X_0$.

In [2] we extended this approach to compute the region of attraction of a constrained dynamical system, defined as the set

$$X_0 := \{x_0 \in \mathbb{R}^n \; : \; \frac{dx(t)}{dt} = f(t, x(t)),\; x(0) = x_0,\; x(1) \in X_1,\; x(t) \in X,\; \forall t \in [0, 1]\}$$

where the smooth vector field $f$, the compact state constraint set $X$ (say, a ball) and the target constraint set $X_1 \subset X$ are given. The basic idea was to notice that the Lebesgue measure on $X_0$ is the solution to the LP

$$\begin{array}{ll} \sup & \int \mu_0 \\ \text{s.t.} & \mu_0 + \hat{\mu}_0 = \lambda \\ & \frac{\partial \mu}{\partial t} + \operatorname{div}(f\mu) = \delta_0 \mu_0 - \delta_1 \mu_1 \\ & \mu_0 \geq 0,\; \operatorname{spt} \mu_0 \subset X \\ & \hat{\mu}_0 \geq 0,\; \operatorname{spt} \hat{\mu}_0 \subset X \\ & \mu_1 \geq 0,\; \operatorname{spt} \mu_1 \subset X_1 \\ & \mu \geq 0,\; \operatorname{spt} \mu \subset [0, 1] \times X \end{array} \tag{3}$$

where the supremum is over nonnegative measures $\mu_0(dx)$, $\hat{\mu}_0(dx)$, $\mu_1(dx)$, $\mu(dt, dx)$ respectively supported on $X$, $X$, $X_1$ and $[0, 1] \times X$, such that $\mu_0$ and $\hat{\mu}_0$ sum up to $\lambda$, the Lebesgue measure on $X$. In problem (3), the linear constraint $\frac{\partial \mu}{\partial t} + \operatorname{div}(f\mu) = \delta_0\mu_0 - \delta_1\mu_1$ is called Liouville's equation, or the advection equation, or the equation of conservation of mass in fluid dynamics, statistical physics or kinetic theory. It should be understood in the sense of distributions, i.e.

$$\int_0^1 \int_X \left( \frac{\partial v(t, x)}{\partial t} + \operatorname{grad} v(t, x) \cdot f(t, x) \right) \mu(dt, dx) = \int_{X_1} v(1, x) d\mu_1(dx) - \int_X v(0, x) d\mu_0(dx)$$

for all sufficiently smooth test functions $v(t, x)$ supported on $[0, 1] \times X$. The dual to LP (3) is as follows

$$\begin{array}{ll} \inf & \int v_0 \lambda \\ \text{s.t.} & v_0(x) \geq 1 \;\text{ on }\; X \\ & v_0(x) \geq 1 + v(0, x) \;\text{ on }\; X \\ & v(1, x) \geq 0 \;\text{ on }\; X_1 \\ & -\frac{\partial v(t,x)}{\partial t} - \operatorname{grad} v(t, x) \cdot f(t, x) \geq 0 \;\text{ on }\; [0, 1] \times X \end{array} \tag{4}$$

where the infimum is over continuous function $v_0(x)$ supported on $X$, interpreted as a dual Lagrange multiplier for the primal linear constraint $\mu_0+\hat{\mu}_0 = \lambda$, and over continuous function $v(t,x)$ supported on $[0,1] \times X$, interpreted as a dual Lagrange multiplier for the primal Liouville equation. Note that the supremum in problem (3) is attained by $\mu_0$ equal to the Lebesgue measure on $X_0$. In contrast, the infimum in problem (4) is not attained, but it can be shown that the (continuous) function $v_0$ converges almost uniformly to the (discontinuous) indicator function of $X_0$.

In [2] it is explained that if $f$ is a given polynomial vector field and $X$, $X_1$ are basic semi-algebraic sets described by finitely many given polynomial inequalities, infinite-dimensional primal-dual LP problems (3-4) can be solved by a converging hierarchy of finite-dimensional semidefinite programming (SDP) problems. At a given relaxation order $d$, the primal SDP is a moment relaxation of LP (3), whereas the dual SDP is a polynomial sum-of-squares restriction of LP (4). From the solution of the dual SDP of order $d$, we obtain a polynomial $v_0^d(x)$ of degree $2d$ which is an approximation of $v_0(x)$ such that the semi-algebraic set $X_0^d := \{x \in X \ : \ v_0^d(x) \geq 1\}$ is a valid outer approximation of $X_0$, i.e. $X_0 \subset X_0^d$. Moreover, the approximation converges in Lebesgue measure, i.e. $\lim_{d\to\infty} \lambda(X_0^d) = \lambda(X_0)$.

# References


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