Low-order Robust Controller Synthesis for Interval Plants

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Abstract

This paper deals with robust controller synthesis for SISO linear plants subject to interval parametric uncertainty, a longstanding open problem of control theory. Based on Hermite-Fujiwara matrices and the Generalized Kharitonov's Theorem, a sufficient condition is derived for the existence of a robustly stabilizing controller of order up to three. This condition is formulated as a non-convex rank-one LMI feasibility problem in the controller parameters. This optimization problem is addressed with two standard heuristics relying upon semidefinite programming. In spite of the potentially conservative nature of the stabilizability condition and the lack of convergence of the proposed algorithms, several numerical examples bear out the usefulness of our approach for designing robust controllers of small order at low computational cost.

1 Introduction

We focus on the problem of robust stabilization of an uncertain single-input single-output plant whose parameters belong to given real intervals. Even though significant progress has been made recently in the realm of parametric robust control - see textbooks [3] and [5] for a good overview - this is still an open problem for which very few results are

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available. Indeed, Bhattacharyya and co-workers [5, p. xii] point out that a significant deficiency of control theory at the present time is the lack of non-conservative synthesis methods to achieve robustness under parameter uncertainty. Generally speaking, it turns out that most of the existing results in the area of parametric robust control are analysis results.

An early attempt to fill the gap between analysis and design results is described by Bhattacharyya *et al.* [4]. Structured parametric uncertainties are over-bounded by unstructured uncertainties and standard results from the $\mathcal{H}_\infty$ control theory are applied to derive a robustly stabilizing controller. This technique was recently revisited and improved by Datta and Patel [8]. Still, the simplicity of the approach sharply contrasts with the inherent conservatism induced by the uncertainty over-bounding. Another synthesis result is described by Barmish in Chapter 11 of his textbook [3], based on extreme point results. Unfortunately, the approach – essentially graphical – is restricted to two-parameter controller design. The author concedes that there is no nice extension of these results to higher order controllers. A different approach is pursued by Abdallah *et al.* [1], based on analytic-real positive functions. There also, obtained results remain narrow in scope since limited to plants affected by only one uncertain parameter. Further endeavor for parametric robust controller design is reported by Bhattacharyya *et al.* in [5, §5.6], but only in the very special case of state-feedback control of a single-input system whose state-space matrices A and B are in controllable companion form and the coefficients of the characteristic polynomial of A are subject to bounded perturbations. Finally, we conclude our little survey by pointing out the recent and interesting synthesis results proposed by Ho *et al.* [19]. In this reference, the authors use the Generalized Kharitonov's Theorem [5, Chapter 7] and the Hermite-Biehler Theorem [13] in order to describe the whole set of stabilizing proportional (P), proportional-integral (PI) and proportional-integral-derivative (PID) controllers for a given interval plant. Yet, it can be argued that their approach suffers from inherent shortcomings. On the one hand, it is restricted to P, PI and PID controllers and no extension to higher order controllers is available. On the other hand, the method involves a sweeping over the proportional gain for PI controller design, and a double sweeping over the proportional gain and a vertex parameter for PID controller design. However, the approach seems to be promising and can be considered as a significant step towards the solution to the general robust synthesis problem. To quote [5, p. 579], "so little is known about the solution that every step is of interest".

Motivated by the above developments and by the lack of synthesis results in parametric robust control, we propose in this paper a new alternative to stabilization of interval plants. The approach is similar to that of references [4, 8] in that we only come up with a sufficient stabilizability condition that may prove conservative. On the other hand, no parameter sweeping is required as in [3, 19]. As an outgrowth of our previous work [17, 18], this condition is formulated as a Linear Matrix Inequality (LMI) feasibility problem with a rank-one constraint, a non-convex optimization problem frequently arising in robust control problems. This problem can be addressed with standard tractable heuristics based on LMIs. Even though no convergence guarantee is available, several numerical examples coming from the literature illustrate that these algorithms often prove satisfactory for designing low-order robust controllers and improving already published results.

The outline of the paper is as follows. In Section 1, we state the problem to be solved and impart the necessary material. Hermite-Fujiwara matrices and the Generalized Kharitonov's Theorem are the backbone of our approach. In Section 2, a sufficient condition for robust stabilization of an interval plant is derived as an LMI feasibility problem with a non-convex rank-one constraint. We briefly underline the relevance of this key optimization problem in control. Section 3 recalls two standard heuristics that are aimed at solving the rank-one LMI feasibility problem. Both methods rely upon semidefinite programming, an efficient optimization technique widely spread in control. Applicability of our results is then illustrated in Section 4 on several numerical examples found in the literature. This is believed to be the main contribution of the paper. It is shown that our approach, albeit conservative, stands up satisfactorily to the challenge of designing robust controllers of small orders at low computational cost. Finally, we conclude the paper with some remarks.

The notations in this paper are fairly standard. $\mathrm{Tr}X$ and $\mathrm{rank}X$ respectively stand for the trace and the rank of matrix X, the notations $X > 0$ and $X \geq 0$ respectively mean that matrix X is positive definite and positive semidefinite. Finally, $p(\mathtt{s})$ represents a polynomial p in the indeterminate $\mathtt{s}$.

2 Preliminaries

2.1 Problem Statement

An interval plant provides a simple and general way to model parametric uncertainty and is described by a ratio of interval polynomials

$$P(\mathtt{s}) = \frac{b(\mathtt{s})}{a(\mathtt{s})} \tag{1}$$

where

$$\begin{aligned} a(\mathtt{s}) &= a_0 + a_1\mathtt{s} + a_2\mathtt{s}^2 + \cdots \\ b(\mathtt{s}) &= b_0 + b_1\mathtt{s} + b_2\mathtt{s}^2 + \cdots \end{aligned}$$

and the coefficients of $a(\mathtt{s})$ and $b(\mathtt{s})$ vary independently in given intervals, i.e.

$$a_i \in [\underline{a}_i, \overline{a}_i], \quad b_i \in [\underline{b}_i, \overline{b}_i], \quad i = 0, 1, 2, \ldots$$

We consider a controller

$$C(\mathtt{s}) = \frac{y(\mathtt{s})}{x(\mathtt{s})}$$

where

$$\begin{aligned} x(\mathtt{s}) &= x_0 + x_1\mathtt{s} + x_2\mathtt{s}^2 + \cdots + x_{d_x}\mathtt{s}^{d_x} \\ y(\mathtt{s}) &= y_0 + y_1\mathtt{s} + y_2\mathtt{s}^2 + \cdots + y_{d_y}\mathtt{s}^{d_y}. \end{aligned}$$

When controller $C(\mathtt{s})$ is placed in series with plant $P(\mathtt{s})$ and closed under unity feedback, the closed-loop characteristic polynomial of the system reads

$$\delta(\mathtt{s}) = a(\mathtt{s})x(\mathtt{s}) + b(\mathtt{s})y(\mathtt{s}).$$

It is well seen that characteristic polynomial $\delta(\mathtt{s})$ is an interval polynomial.

Given interval polynomials $a(\mathtt{s})$ and $b(\mathtt{s})$, the problem of finding polynomials $x(\mathtt{s})$ and $y(\mathtt{s})$ such that interval polynomial $\delta(\mathtt{s})$ is stable is referred to as the problem of robust stabilization of an interval plant. This is the problem we are to study in this paper. For this purpose, we need to introduce two components that will prove instrumental to our approach, namely Hermite-Fujiwara matrices and the Generalized Kharitonov's Theorem.

2.2 Hermite-Fujiwara Matrix

Definition 1 *Given a pair of polynomials $p(\mathtt{s}) = p_0 + p_1\mathtt{s} + \cdots + p_m\mathtt{s}^m$ and $q(\mathtt{s}) = q_0 + q_1\mathtt{s} + \cdots + q_m\mathtt{s}^m$, the Bézoutian matrix of $p(\mathtt{s})$ and $q(\mathtt{s})$ is the m-by-m symmetric matrix $B(p(\mathtt{s}), q(\mathtt{s}))$ with entries*

$$b_{jk} = \sum_{l=j+k-m-1}^{j-1} (p_{j+k-l-1}q_l - q_{j+k-l-1}p_l)$$

where it is assumed that $p_l = q_l = 0$ if $l < 0$ or $l > m$.

Definition 2 *Given a polynomial $p(\mathtt{s}) = p_{\mathrm{even}}(\mathtt{s}^2) + \mathtt{s}p_{\mathrm{odd}}(\mathtt{s}^2)$ of degree $\leq m$ (some leading coefficients of $p(\mathtt{s})$ may be zero), define $p^\star(\mathtt{s}) = p_{\mathrm{even}}(\mathtt{s}^2) - \mathtt{s}p_{\mathrm{odd}}(\mathtt{s}^2)$. Denote by b_{jk} the entries of $B(p(\mathtt{s}), p^\star(\mathtt{s}))$, the Bézoutian matrix of $p(\mathtt{s})$ and $p^\star(\mathtt{s})$. The Hermite-Fujiwara matrix of $p(\mathtt{s})$ of order m is the m-by-m matrix $H_m(p(\mathtt{s}))$ with entries*

$$h_{jk} = (-1)^j b_{jk}, \quad j, k = 1, \ldots, m.$$

A fundamental property of the above matrix is captured by the following theorem, proved in [20, §3.1].

Theorem 1 *A polynomial $p(\mathtt{s})$ of degree m is Hurwitz stable if and only if the Hermite-Fujiwara matrix $H_m(p(\mathtt{s}))$ is positive definite.*

The following property, easily proved by inspection [17], characterizes the Hermite-Fujiwara matrix of a linear combination of polynomials.

Lemma 1 *Given a real vector $z = [z_0 \quad z_1 \quad \cdots \quad z_n]'$ and $n+1$ polynomials $p_i(\mathtt{s}) = p_{i\mathrm{even}}(\mathtt{s}^2) + \mathtt{s}p_{i\mathrm{odd}}(\mathtt{s}^2)$, $i = 0, 1, \ldots, n$ of degrees $\leq m$, it holds*

$$H_m(\sum_{i=0}^{n} z_i p_i(\mathtt{s})) = \sum_{j=0}^{n}\sum_{k=0}^{n} z_j z_k H_m(p_{j_{\mathrm{even}}}(\mathtt{s}^2) + \mathtt{s}p_{k_{\mathrm{odd}}}(\mathtt{s})).$$

2.3 Generalized Kharitonov's Theorem

We now recall a generalization of the famous Kharitonov's Theorem. It characterizes stability of a family of polynomials which consists of a linear combination, with fixed polynomial coefficients, of interval polynomials. For a comprehensive treatment, the reader is referred to [5, Chapter 7].

Consider an interval polynomial

$$r(\mathsf{s}) = r_0 + r_1\mathsf{s} + r_2\mathsf{s}^2 + \cdots$$

where

$$r_i \in [r_i^-, r_i^+], \quad i = 0, 1, 2, \ldots$$

The four Kharitonov polynomials corresponding to $p(\mathsf{s})$ are defined as follows

$$\begin{aligned}
k^1(\mathsf{s}) &= r_0^- + r_1^-\mathsf{s} + r_2^+\mathsf{s}^2 + r_3^+\mathsf{s}^3 + r_4^-\mathsf{s}^4 + r_5^-\mathsf{s}^5 + r_6^+\mathsf{s}^6 + \cdots \\
k^2(\mathsf{s}) &= r_0^- + r_1^+\mathsf{s} + r_2^+\mathsf{s}^2 + r_3^-\mathsf{s}^3 + r_4^-\mathsf{s}^4 + r_5^+\mathsf{s}^5 + r_6^+\mathsf{s}^6 + \cdots \\
k^3(\mathsf{s}) &= r_0^+ + r_1^-\mathsf{s} + r_2^-\mathsf{s}^2 + r_3^+\mathsf{s}^3 + r_4^+\mathsf{s}^4 + r_5^-\mathsf{s}^5 + r_6^-\mathsf{s}^6 + \cdots \\
k^4(\mathsf{s}) &= r_0^+ + r_1^+\mathsf{s} + r_2^-\mathsf{s}^2 + r_3^-\mathsf{s}^3 + r_4^+\mathsf{s}^4 + r_5^+\mathsf{s}^5 + r_6^-\mathsf{s}^6 + \cdots
\end{aligned}$$

Let t be an arbitrary integer and let $\{r_1(\mathsf{s}), \ldots, r_t(\mathsf{s})\}$ be a t-tuple of polynomials where each component $r_i(\mathsf{s})$, $i = 1, 2, \ldots, t$ is an interval polynomial. With each $r_i(\mathsf{s})$, we can associate its four Kharitonov polynomials $k_i^1(\mathsf{s})$, $k_i^2(\mathsf{s})$, $k_i^3(\mathsf{s})$ and $k_i^4(\mathsf{s})$.

Denote by $\mathcal{K}$ the set of vertex t-tuples obtained by setting $r_i(\mathsf{s}) = k_i^j(\mathsf{s})$ for some $j = 1, 2, 3, 4$ and every fixed integer i between 1 and t. There are at most 4^t distinct elements in $\mathcal{K}$. Moreover, let $\{\alpha_1(\mathsf{s}), \ldots, \alpha_t(\mathsf{s})\}$ be a t-tuple of fixed polynomials of the form $\alpha_i(\mathsf{s}) = \mathsf{s}^{f_i}(g_i\mathsf{s} + h_i)$, where $f_i \geq 0$ is an arbitrary integer and g_i, h_i are arbitrary real numbers. Consider the interval polynomial

$$\alpha_1(\mathsf{s})r_1(\mathsf{s}) + \cdots + \alpha_t(\mathsf{s})r_t(\mathsf{s}) \tag{2}$$

With these notations, the following result can be considered as a particular case of the Generalized Kharitonov's Theorem described in [5].

Theorem 2 *If interval polynomials $r_i(\mathsf{s})$ perturb independently of each other, a necessary and sufficient condition for interval polynomial (2) to be robustly Hurwitz stable is that all vertex t-tuples in $\mathcal{K}$ are Hurwitz stable.*

Now if we relax the assumption that coefficients of interval polynomials $r_i(\mathsf{s})$ are independent of each other, we resort to a standard interval overbounding of uncertainty [3, §5.11] and come up with a sufficient robust stabilizability condition.

Corollary 1 *If interval polynomials $r_i(\mathsf{s})$ do not perturb independently of each other, a sufficient condition for interval polynomial (2) to be robustly Hurwitz stable is that all vertex t-tuples in $\mathcal{K}$ are Hurwitz stable.*

3 Rank-one LMI Formulation

3.1 BMI Formulation

Let $n = d_x + d_y + 1$ and collect all the design parameters into a column vector

$$z = [z_0 \quad z_1 \quad \cdots \quad z_n]' = [x_0 \quad x_1 \quad \cdots \quad x_{d_x} \quad y_0 \quad y_1 \quad \cdots \quad y_{d_y}]'.$$

For different controller structures define polynomials $\alpha_i(\mathtt{s})$ and $r_i(\mathtt{s})$ as in Tables 1a and 1b so that the characteristic polynomial of the closed-loop system is the interval polynomial

$$\delta(\mathtt{s}) = \alpha_1(\mathtt{s})r_1(\mathtt{s}) + \cdots + \alpha_t(\mathtt{s})r_t(\mathtt{s}).$$

Numerator polynomial $x(\mathtt{s})$ is supposed to be monic, i.e $z_{d_x} = 1$.

Type	Controller struct.	Characteristic polynomial $\delta_i(\mathtt{s})$	Constr.
P	Proportional or static	$\underbrace{x_0}_{\alpha_1}\underbrace{a}_{r_1} + \underbrace{y_0}_{\alpha_2}\underbrace{b}_{r_2}$ $= z_0 \underbrace{a}_{p_{i0}} + z_1 \underbrace{b}_{p_{i1}}$	$z_0 = 1$
PI	Prop.-integral	$\underbrace{\mathtt{s}}_{\alpha_1}\underbrace{a}_{r_1} + \underbrace{(y_0 + y_1\mathtt{s})}_{\alpha_2}\underbrace{b}_{r_2}$ $= z_0 \underbrace{a}_{p_{i0}} + z_1 \underbrace{\mathtt{s}a}_{p_{i1}} + z_2 \underbrace{b}_{p_{i2}} + z_3 \underbrace{\mathtt{s}b}_{p_{i3}}$	$z_0 = 0$ $z_1 = 1$
C1	First order	$\underbrace{(x_0 + x_1\mathtt{s})}_{\alpha_1}\underbrace{a}_{r_1} + \underbrace{(y_0 + y_1\mathtt{s})}_{\alpha_2}\underbrace{b}_{r_2}$ $= z_0 \underbrace{a}_{p_{i0}} + z_1 \underbrace{\mathtt{s}a}_{p_{i1}} + z_2 \underbrace{b}_{p_{i2}} + z_3 \underbrace{\mathtt{s}b}_{p_{i3}}$	$z_1 = 1$

Table 1a. Zero- and first-order controllers.

Type	Controller struct.	Characteristic polynomial $\delta_i(\mathtt{s})$	Constr.
PID	Prop.-integral-derivate	$\underbrace{\mathtt{s}}_{\alpha_1}\underbrace{a}_{r_1} + \underbrace{(y_0 + y_1\mathtt{s})}_{\alpha_2}\underbrace{b}_{r_2} + \underbrace{y_2\mathtt{s}^2}_{\alpha_3}\underbrace{b}_{r_3}$ $= z_0 \underbrace{a}_{p_{i0}} + z_1 \underbrace{\mathtt{s}a}_{p_{i1}} + z_2 \underbrace{b}_{p_{i2}} + z_3 \underbrace{\mathtt{s}b}_{p_{i3}} + z_4 \underbrace{\mathtt{s}^2 b}_{p_{i4}}$	$z_0 = 0$ $z_1 = 1$
C2	Strictly proper second order	$\underbrace{(x_0 + x_1\mathtt{s})}_{\alpha_1}\underbrace{a}_{r_1} + \underbrace{x_2\mathtt{s}^2}_{\alpha_2}\underbrace{a}_{r_2} + \underbrace{(y_0 + y_1\mathtt{s})}_{\alpha_3}\underbrace{b}_{r_3}$ $= z_0 \underbrace{a}_{p_{i0}} + z_1 \underbrace{\mathtt{s}a}_{p_{i1}} + z_2 \underbrace{\mathtt{s}^2 a}_{p_{i2}} + z_3 \underbrace{b}_{p_{i3}} + z_4 \underbrace{\mathtt{s}b}_{p_{i4}}$	$z_2 = 1$
C3	Third order with first degree num.	$\underbrace{(x_0 + x_1\mathtt{s})}_{\alpha_1}\underbrace{a}_{r_1} + \underbrace{(x_2 + x_3\mathtt{s})\mathtt{s}^2}_{\alpha_2}\underbrace{a}_{r_2} + \underbrace{(y_0 + y_1\mathtt{s})}_{\alpha_3}\underbrace{b}_{r_3}$ $= z_0 \underbrace{a}_{p_{i0}} + z_1 \underbrace{\mathtt{s}a}_{p_{i1}} + z_2 \underbrace{\mathtt{s}^2 a}_{p_{i2}} + z_3 \underbrace{\mathtt{s}^3 a}_{p_{i3}} + z_4 \underbrace{b}_{p_{i4}} + z_5 \underbrace{\mathtt{s}b}_{p_{i5}}$	$z_3 = 1$

Table 1b. Second- and third-order controllers.

For a given vertex number t and given t-tuples of polynomials $\{\alpha_1(\mathsf{s}),\ldots,\alpha_t(\mathsf{s})\}$, there are $N = 4^t$ characteristic polynomials $\delta_i(\mathsf{s})$, $i = 1,2,\ldots,N$ associated to the family $\mathcal{K}$ of vertex t-tuples introduced previously. Upon a suitable choice of basis polynomials $p_{ij}(\mathsf{s})$, see Tables 1a and 1b, these characteristic polynomials read

$$\delta_i(\mathsf{s}) = \sum_{j=0}^{n} z_j p_{ij}(\mathsf{s}).$$

For notational ease, let $m = \max_{i=1,2,\ldots,N} \deg\delta_i(\mathsf{s})$ and $H_{ijk} = H_m(p_{ij}(\mathsf{s}), p_{ik}(\mathsf{s}))$.

Theorem 3 *Any vector z solution to the Bilinear Matrix Inequalities (BMIs)*

$$\sum_{j=0}^{n}\sum_{k=0}^{n} z_j z_k H_{ijk} > 0, \quad i = 1,\ldots,N \tag{3}$$

parametrizes a pair of polynomials $x(\mathsf{s})$ and $y(\mathsf{s})$ giving rise to a robustly stabilizing controller for interval plant (1).

Proof: In the case of zero- and first-order controllers (Table 1a), one can check that interval polynomials $r_i(\mathsf{s})$ are independent of each other. In virtue of Theorem 2, interval polynomial $\delta(\mathsf{s})$ is robustly Hurwitz stable if and only if all vertex polynomials $\delta_i(\mathsf{s})$ are Hurwitz stable. Recalling Lemma 1, this is equivalently to the positive definiteness of Hermite-Fujiwara matrices

$$H_m(\delta_i(\mathsf{s})) = \sum_{j=0}^{n}\sum_{k=0}^{n} z_j z_k H_{ijk}$$

for $i = 1,\ldots,N$. BMIs (3) are therefore necessary and sufficient conditions of robust stabilizability and may be considered as a reformulation of the sixteen plant theorem of [3, Chapter 11]. In the case of second- and third-order controllers (Table 1b), interval polynomials $r_i(\mathsf{s})$ are not independent of each other. In virtue of Corollary 1, BMIs (3) become only sufficient conditions of robust stabilizability of the interval plant. $\square$

In Table 2, we report the number of BMIs that occur for different controller architectures, together with the number of decision variables, or design parameters.

Controller type	Number of BMIs	Number of design parameters
P	16	1
PI	16	2
C1	16	3
PID	64	3
C2	64	4
C3	64	5

Table 2. Problem dimensions.

Recall that the dimension of each BMI equals m, thus it depends on the order of the plant. One can see that the number of BMIs quickly becomes very large as the order of the controller increases. Considering controllers of higher orders, or with more design parameters than those proposed in Table 1, would have led to systems of 256 BMIs or more that must be discarded for obvious practical reasons. This is a main hindrance to the design of high-order controllers, and helps to pinpoint the origin of the difficulty of robust stabilization of interval plants.

In [15], the authors showed that many of the problems considered in the robust control literature can be formulated as BMIs. In view of Theorem 3, this is also the case for the robust stabilization of interval plants. Unfortunately, BMIs are highly non-convex optimization problems and solving a general BMI was shown to be NP-hard [26].

3.2 LMI with Rank Constraint

BMI system (3) can be formulated as a special LMI problem where all the non-convexity is concentrated into a rank constraint. The motivation for doing so is that LMI problems are well-behaved convex problems for which many numerical methods have recently been developed, see [6] for a good overview.

Following [7], bilinear terms in inequalities (3) are collected into the entries

$$z_{jk} = z_j z_k, \quad j, k = 0, 1, \ldots, n$$

of a rank-one positive semidefinite matrix

$$Z = zz' \geq 0 \tag{4}$$

so that BMI system (3) becomes a rank-one constrained LMI system

$$\begin{array}{l} \sum_{j=0}^{n} \sum_{k=0}^{n} z_{jk} H_{ijk} > 0, \quad i = 1, \ldots, N \\ Z \geq 0, \quad \mathrm{rank} Z = 1. \end{array} \tag{5}$$

It must be underlined that the idea of formulating non-convex control problems as an LMI problem with a rank constraint is not new, see for instance [10]. Beyond the scope of systems control, rank constrained LMI problems also occur in mathematical programming and combinatorial optimization. In [25, 23, 7] it is shown that all of the problems with polynomial objective and polynomial constraints (including quadratic programming, binary programming, integer programming and job-shop scheduling, to mention just a few) may actually be formulated as rank-one LMI optimization problems.

4 Numerical Methods

Several heuristic and global optimization algorithms were devised in [14] for finding solutions to BMIs. However, these methods are usually far too general. In this section, we recall two standard heuristics for problem (5) that rely only on polynomial-time algorithms. No guarantee of convergence is available for these algorithms, but they may find solutions at low computational cost.

4.1 Potential Reduction

A first heuristic for tackling problem (5) hinges upon an extension of a potential reduction method for solving convex optimization problems, and was first proposed in [9]. Applications of the potential reduction method were recently described for low-order $\mathcal{H}_\infty$ control of flexible structures [27] and IQC synthesis with inertia constraints [16]. The principle lying behind the approach is as follows.

Instead of solving problem (5) for a matrix Z of rank one, we will try to solve the equivalent problem

$$\begin{array}{ll} \min & \operatorname{rank} Z \\ \text{s.t.} & \sum_{j=0}^{n}\sum_{k=0}^{n} z_{jk} H_{ijk} > 0, \quad i = 1, \ldots, N \\ & Z \geq 0 \end{array} \tag{6}$$

where a non-convex constraint must be minimized over a convex set. As soon as the above criterion equals one, a solution Z to problem (5) is found. It turns out that minimization of the rank of a matrix subject to convex LMI constraint is a long-standing problem in control theory. Its most important application resides in the design of reduced order controllers and static output feedback stabilization [10, 11, 22].

In [9], the authors proposed an extension of a potential reduction method – an interior point method for convex optimization – for solving rank minimization problems. The approach for solving (6) is similar to the Newton algorithm proposed by Nesterov and Nemirovski [23], with the notable exception that the logarithmic barrier function is not convex and must be linearized in order to be minimized. Because of the non-convexity of the objective function, the algorithm may unfortunately get stuck in a local minimum. As the size of the problem increases, the number of local minima is likely to increase and the complexity of the problem increases as well. Technical details are not reproduced here for conciseness. The interested reader may consult [9].

4.2 Frank and Wolfe Algorithm

A second heuristic may be pursued in order to solve problem (5). It is based on the Frank and Wolfe feasible direction method. The algorithm was originally proposed for linear programming [21] and was later on extended to the cone of positive definite matrices by El Ghaoui and co-workers [11]. It was generalized to concave programming and general BMI problems by Apkarian and Tuan [2]. The algorithm proves to be very efficient in practice. For instance, it is one of the best resolution methods to date for static output feedback design, see [24] for a recent overview. The main lines of the approach are sketched below.

Consider the minimization problem

$$\begin{array}{ll} \min & f(z) \\ \text{s.t.} & z \in \mathcal{Z} \end{array} \tag{7}$$

where the concave function f has continuous first-order partial derivatives and is bounded below on the convex set $\mathcal{Z}$. The algorithm of Frank and Wolfe for solving problem (7) is as follows.

Step 1. Let $l = 0$. Solve for a solution z^0 a convex relaxation of problem (7).

Step 2. Find a steepest descent direction by solving the convex problem

$$z^{l+1} = \arg \begin{array}{ll} \min & \nabla f(z^l) \cdot z \\ \text{s.t.} & z \in \mathcal{Z} \end{array}$$

where $\nabla f(z^l)$ stands for the gradient of f evaluated at z^l and the dot stands for the standard inner product.

Step 3. Stop if a local optimum is reached. Otherwise, let $l = l + 1$ and go to Step 2.

Since f is concave, the above algorithm generates strictly decreasing sequences that can only terminate to a point satisfying the minimum principle necessary optimality condition.

As pointed out in [2], the Frank and Wolfe algorithm can be applied to solve our rank-one LMI problem once it is realized that an equivalent concave minimization formulation for problem (5) is to find (if any) the zero optimal solution of the concave program

$$\begin{array}{ll} \min & \mathrm{Tr}(Z - zz') \\ \text{s.t.} & \sum_{j=0}^{n} \sum_{k=0}^{n} z_{jk} H_{ijk} > 0, \quad i = 1, \ldots, N \\ & \begin{bmatrix} Z & z \\ z' & 1 \end{bmatrix} \geq 0. \end{array} \tag{8}$$

Concavity of the objective function can easily be checked by looking at its hypograph, see [2].

In Step 1 of the Frank and Wolfe algorithm, a convex LMI relaxation of concave problem (8) is readily obtained by setting $z = 0$.

In Step 2, the linearized problem becomes

$$z^{l+1} = \arg \begin{array}{ll} \min & \mathrm{Tr} Z - 2z'z^l \\ \text{s.t.} & \sum_{j=0}^{n} \sum_{k=0}^{n} z_{jk} H_{ijk} > 0, \quad i = 1, \ldots, N \\ & \begin{bmatrix} Z & z \\ z' & 1 \end{bmatrix} \geq 0 \end{array}$$

which is a convex LMI problem.

5 Illustrative Examples

In the preceding sections, we have seen that our approach to robust controller design suffer from two potential sources of conservatism, namely

- BMIs (3) are necessary and sufficient stabilizability conditions for zero- and first-order controllers, but only sufficient stabilizability conditions for second- and third-order controllers. In other words, if BMIs (3) have no solution, it does not mean that there is no robustly stabilizing controller of second- or third-order.
- The algorithms proposed in Section 4 to solve rank-one LMIs (5) are heuristics without guarantee of convergence. That is to say, if the algorithms do not yield a feasible point for rank-one LMIs (5) it does not mean that there is no robustly stabilizing controller.

In spite of this seemingly severe restrictions, we now illustrate by numerical examples that our approach turns out to be a satisfactory and computationally cheap robust design method in many instances. In several examples picked up from the literature, our algorithms yielded significant improvements over existing results.

5.1 Implementation

Some comments regarding implementation are in order. Both algorithms proposed in the previous section rely on successive resolution of LMI optimization problems. The potential reduction algorithm was implemented in MATLAB as described in [9], and hinges upon the interior point method of Nesterov and Nemirovski [23]. The Frank and Wolfe algorithm was also implemented in MATLAB. Successive LMI problems were solved with the LMI Toolbox [12], also based on the method of Nesterov and Nemirovski.

The potential reduction algorithm stops when the number of eigenvalues of matrix Z greater than a given threshold is one, or when the Newton descent parameter [23] is less than 0.25, in which case the algorithm gets stuck in a local optimum and no solution of rank one is found. The Frank and Wolfe algorithm stops when the criterion is decreased by less than 10% with respect to the previous iteration.

Computational tests were carried out on a SUN SPARC 30 workstation with MATLAB version 5.2.

5.2 Static and first order controllers

5.2.1 First example

We consider Example 1 in [4], where

$$P(\mathsf{s}) = \frac{[1-\varepsilon, 1+\varepsilon] + \mathsf{s}}{[20-\varepsilon, 20+\varepsilon] + [22-\varepsilon, 22+\varepsilon]\mathsf{s} + [8-\varepsilon, 8+\varepsilon]\mathsf{s}^2 + \mathsf{s}^3}$$

and $\varepsilon \geq 0$ parametrizes the size of the uncertainty intervals. Note that the approach in [4] is restricted to the case $\varepsilon < 6.321$ for which the number of unstable poles of the plant remain unchanged with respect to the nominal case. This restriction can be alleviated with our approach, and it turns out that a first order controller can be found with the Frank and Wolfe algorithm for a value of $\varepsilon = 7.5$, see Tables 3 and 4.

ε	Potential reduction algorithm	Frank and Wolfe algorithm
6	Converged after 67 iterations and 2.5 seconds to the stabilizing controller $$C(\mathsf{s}) = 0.390$$	Converged after 1 iteration and 1 second to the stabilizing controller $$C(\mathsf{s}) = 0$$
6.3	Converged after 70 iterations and 5 seconds to the stabilizing controller $$C(\mathsf{s}) = 0.0380$$	Converged after 1 iteration and 1 second to the stabilizing controller $$C(\mathsf{s}) = 0$$
6.5	LMI problem found infeasible after 50 iterations and 3 seconds.	LMI problem found infeasible after 1 iteration and 0.5 seconds.

Table 3. First example. Static controller design.

ε	Potential reduction algorithm	Frank and Wolfe algorithm
6.5	Converged after 105 iterations and 12 seconds to the stabilizing controller $$C(\mathsf{s}) = \frac{-4.640 + 3.171\mathsf{s}}{2.691 + \mathsf{s}}$$	Converged after 1 iteration and 10 seconds to the stabilizing controller $$C(\mathsf{s}) = \frac{0.669 + 2.470\mathsf{s}}{11.72 + \mathsf{s}}$$
6.6	Rank 2 solution reached after 62 iterations and 11 seconds. No stabilizing controller was found.	Converged after 1 iteration and 10 seconds to the stabilizing controller $$C(\mathsf{s}) = \frac{0.809 + 3.830\mathsf{s}}{11.27 + \mathsf{s}}$$
7.5		Converged after 1 iteration and 10 seconds to the stabilizing controller $$C(\mathsf{s}) = \frac{66.16 + 52.01\mathsf{s}}{38.18 + \mathsf{s}}$$
7.6		Local optimum reached after 3 iterations and 25 seconds. No stabilizing controller was found.

Table 4. First example. First order controller design.

5.2.2 Second example

We consider Example 2 in [4], also proposed as Example 1 in [8], where

$$P(\mathsf{s}) = \frac{[4-\varepsilon, 4+\varepsilon] + 5\mathsf{s}}{[-15-\varepsilon, -15+\varepsilon] + [2-\varepsilon, 2+\varepsilon]\mathsf{s} + \mathsf{s}^2}$$

and $\varepsilon \geq 0$ parametrizes the size of the uncertainty intervals. Note that parameter ε is restricted to $\varepsilon \leq 4$ in [4]. A similar bound was found in [8]. Our approach improves these results and a first order stabilizing controller is found for $\varepsilon = 6.8$ using the potential reduction algorithm, see Tables 5 and 6.

ε	Potential reduction algorithm	Frank and Wolfe algorithm
3.6	Converged after 68 iterations and 4 seconds to the stabilizing controller $$C(\mathsf{s}) = 47.99$$	Converged after 2 iterations and 1.5 second to the stabilizing controller $$C(\mathsf{s}) = 48.83$$
3.9	Converged after 58 iterations and 4 seconds to the stabilizing controller $$C(\mathsf{s}) = 226.5$$	Converged after 2 iterations and 1.5 seconds to the stabilizing controller $$C(\mathsf{s}) = 199.0$$
4	LMI problem found infeasible after 146 iterations and 8 seconds.	LMI problem found infeasible after 1 iteration and 0.5 seconds.

Table 5. Second example. Static controller design.

ε	Potential reduction algorithm	Frank and Wolfe algorithm
4	Converged after 113 iterations and 11 seconds to the stabilizing controller $$C(\mathsf{s}) = \frac{179.4 + 119.1\mathsf{s}}{-45.01 + \mathsf{s}}$$	Local optimum reached after 9 iterations and 54 seconds. No stabilizing controller was found.
6.8	Converged after 89 iterations and 11 seconds to the stabilizing controller $$C(\mathsf{s}) = \frac{397.4 + 214.4\mathsf{s}}{-135.8 + \mathsf{s}}$$	
6.9	Rank 2 solution reached after 51 iterations and 11 seconds. No stabilizing controller was found.	

Table 6. Second example. First order controller design.

5.3 PI, PID and second order controllers

5.3.1 Third example

We consider the model of an experimental oblique wing aircraft reproduced as Example 11.5.1 in [3]. The aircraft transfer function is

$$P(\mathsf{s}) = \frac{[90, 166] + [54, 74]\mathsf{s}}{[-0.1, 0.1] + [30.1, 33.9]\mathsf{s} + [50.4, 80.8]\mathsf{s}^2 + [2.8, 4.6]\mathsf{s}^3 + \mathsf{s}^4}.$$

As shown in Tables 7–10, robustly stabilizing controllers are easily found for this plant. For example, in Table 8, two robustly stabilizing PI controllers are obtained at low cost, without pursuing the intricate graphical approach described in [3, §11.5].

Potential reduction algorithm	Frank and Wolfe algorithm
Converged after 99 iterations and 6 seconds to the stabilizing controller $$C(\mathsf{s}) = 0.381$$	Converged after 1 iteration and 2 seconds to the stabilizing controller $$C(\mathsf{s}) = 0.00111$$

Table 7. Third example. Static controller design.

Potential reduction algorithm	Frank and Wolfe algorithm
Converged after 112 iterations and 9 seconds to the stabilizing controller $$C(\mathsf{s}) = \frac{0.907}{\mathsf{s}} + 1.142$$	Converged after 2 iterations and 11 seconds to the stabilizing controller $$C(\mathsf{s}) = \frac{1}{\mathsf{s}} + 1$$

Table 8. Third example. PI controller design.

Potential reduction algorithm	Frank and Wolfe algorithm
Converged after 117 iterations and 12 seconds to the stabilizing controller $$C(\mathsf{s}) = \frac{0.975 + 0.559\mathsf{s}}{0.950 + \mathsf{s}}$$	Converged after 2 iterations and 27 seconds to the stabilizing controller $$C(\mathsf{s}) = \frac{0.937 + 0.156\mathsf{s}}{36.39 + \mathsf{s}}$$

Table 9. Third example. First order controller design.

Potential reduction algorithm	Frank and Wolfe algorithm
Rank 2 solution reached after 147 iterations and 86 seconds. No stabilizing controller was found.	Local optimum reached after 2 iterations and 113 seconds. Resulting stabilizing controller is $$C(\mathsf{s}) = \frac{0.594}{\mathsf{s}} + 0.748 + 0.690\mathsf{s}$$

Table 10. Third example. PID controller design.

5.3.2 Fourth example

We consider the model of a two-phase servo-motor, given as Example 13.3 in [5]. The transfer function from the control voltage to the displacement of the motor shaft is given by

$$P(\mathsf{s}) = \frac{[43.5 - \varepsilon, 43.5 + \varepsilon]}{0.777 + [16.9 - 0.7\varepsilon, 16.9 + 0.7\varepsilon]\mathsf{s}}$$

where $\varepsilon \geq 0$ parametrizes the size of uncertainty intervals. In [5], a controller a given for a perturbation $\varepsilon \leq 9.4$. As shown in Tables 11–15, stabilizing controllers can be found for considerably higher values of ε. We stopped our experimentations to values of ε for which no degree dropping occurs in the numerator, i.e. $\varepsilon < 43.5$.

ε	Potential reduction algorithm	Frank and Wolfe algorithm
23	Converged after 76 iterations and 3.5 seconds to the stabilizing controller $$C(\mathsf{s}) = 8.628 \cdot 10^{-4}$$	Converged after 1 iteration and 1 second to the stabilizing controller $$C(\mathsf{s}) = 0.413 \cdot 10^{-4}$$
24	Converged after 84 iterations and 3.5 seconds to the stabilizing controller $$C(\mathsf{s}) = 1.773 \cdot 10^{-4}$$	Converged after 1 iteration and 1 second to the stabilizing controller $$C(\mathsf{s}) = 0.480 \cdot 10^{-4}$$
24.15	LMI problem found infeasible after 97 iterations and 5 seconds.	LMI problem found infeasible after 1 iteration and 0.5 seconds.

Table 11. Fourth example. Static controller design.

ε	Potential reduction algorithm	Frank and Wolfe algorithm
5	Converged after 117 iterations and 5 seconds to the stabilizing controller $$C(\mathsf{s}) = (\frac{21.34}{\mathsf{s}} + 1.933)10^{-4}$$	Converged after 1 iteration and 2 seconds to the stabilizing controller $$C(\mathsf{s}) = (\frac{0.574}{\mathsf{s}} + 0.671)10^{-4}$$
6	Rank 2 solution reached after 351 iterations and 14 seconds. No stabilizing controller was found.	Converged after 1 iteration and 2 seconds to the stabilizing controller $$C(\mathsf{s}) = (\frac{0.576}{\mathsf{s}} + 0.660)10^{-4}$$
23		Converged after 1 iteration and 2 seconds to the stabilizing controller $$C(\mathsf{s}) = (\frac{0.137}{\mathsf{s}} + 0.455)10^{-4}$$
25		LMI problem found infeasible after 1 iteration and 1 second.

Table 12. Fourth example. PI controller design.

ε	Potential reduction algorithm	Frank and Wolfe algorithm
27	Converged after 109 iterations and 13 seconds to the stabilizing controller $$C(\mathrm{s}) = \frac{1.300 + 26.88\mathrm{s}}{5.439 + \mathrm{s}}$$	Converged after 1 iteration and 5 seconds to the stabilizing controller $$C(\mathrm{s}) = \frac{0.836 + 152.7\mathrm{s}}{1260 + \mathrm{s}}$$
28	Converged after 107 iterations and 13.5 seconds to the stabilizing controller $$C(\mathrm{s}) = \frac{1.390 + 37.61\mathrm{s}}{6.041 + \mathrm{s}}$$	LMI problem found infeasible after 1 iteration and 4.5 seconds
43.4	Converged after 96 iterations and 13 seconds to the stabilizing controller $$C(\mathrm{s}) = \frac{0.0387 + 6077\mathrm{s}}{34.00 + \mathrm{s}}$$	

Table 13. Fourth example. First order controller design.

ε	Potential reduction algorithm	Frank and Wolfe algorithm
1	Converged after 289 iterations and 103 seconds to the stabilizing controller $$C(\mathrm{s}) = \frac{1.327 \cdot 10^{-4}}{\mathrm{s}} + 0.183 + 5.143\mathrm{s}$$	Converged after 1 iteration and 14 seconds to the stabilizing controller $$C(\mathrm{s}) = (\frac{14.11}{\mathrm{s}} + 20.77 + 0.107\mathrm{s})10^{-4}$$
1.5	Rank 2 solution reached after 257 iterations and 96 seconds. No stabilizing controller was found.	Converged after 1 iteration and 13 seconds to the stabilizing controller $$C(\mathrm{s}) = (\frac{28.89}{\mathrm{s}} + 42.32 + 0.423\mathrm{s})10^{-4}$$
23		Converged after 1 iteration and 13 seconds to the stabilizing controller $$C(\mathrm{s}) = (\frac{32.03}{\mathrm{s}} + 94.70 + 28.95\mathrm{s})10^{-4}$$
43.4		Converged after 2 iterations and 25 seconds to the stabilizing controller $$C(\mathrm{s}) = \frac{60.65}{\mathrm{s}} + 97.12 + 207.6\mathrm{s}$$

Table 14. Fourth example. PID controller design.

ε	Potential reduction algorithm	Frank and Wolfe algorithm
2	Converged after 347 iterations and 102 seconds to the stabilizing controller $$C(\mathrm{s}) = \frac{-0.344\mathrm{s}}{1.652 + 0.294\mathrm{s} + \mathrm{s}^2}$$	Converged after 2 iterations and 135 seconds to the stabilizing controller $$C(\mathrm{s}) = \frac{3.859 - 0.00261\mathrm{s}}{1003 + 2.973\mathrm{s} + \mathrm{s}^2}$$
3	Rank 2 solution reached after 283 iterations and 82 seconds. No stabilizing controller was found.	Converged after 1 iteration and 68 seconds to the stabilizing controller $$C(\mathrm{s}) = \frac{8.278 + 0.0234\mathrm{s}}{4280 + 5.251\mathrm{s} + \mathrm{s}^2}$$
20		Converged after 1 iteration and 45 seconds to the stabilizing controller $$C(\mathrm{s}) = \frac{159.8 + 16.31\mathrm{s}}{25140 + 110.8\mathrm{s} + \mathrm{s}^2}$$
25		Converged after 1 iteration and 64 seconds to the stabilizing controller $$C(\mathrm{s}) = \frac{53.42 + 170.0\mathrm{s}}{4972 + 59.08\mathrm{s} + \mathrm{s}^2}$$
26		Local optimum reached after 2 iterations and 130 seconds. Resulting controller is not stabilizing.

Table 15. Fourth example. Second order controller design.

5.4 Discussion

As illustrated by the above numerical examples, both potential reduction and Frank and Wolfe algorithms prove to be on the whole efficient low-order controller design methods. Generally speaking, we faced less numerical problems with the Frank and Wolfe algorithm than with the potential reduction algorithm. Sometimes the Frank and Wolfe algorithm gets stuck to a local optimum, but very frequently converges more quickly that the potential reduction algorithm.

The local nature of the algorithms is well illustrated by the fourth example. Indeed, a static controller was found by the potential reduction algorithm for $\varepsilon = 23$, whereas no PI or PID controllers were found by the same algorithm for the same value of ε, in spite of the fact that a static controller is nothing but a particular case of a PI or PID controller. The reason behind this is that convergence is less easily ensured when the number of decision variables is increased. Practically speaking, we ran into difficulty with optimization problems involving more than three or four decision variables. Actually, none of the methods converged to a third order controller (five decision variables) for any of the examples provided above. In this case, more sophisticated local optimization methods may be of some use.

The dimension, and most importantly, the number of LMIs featured in the relaxation of BMI problem (3) is also a crucial parameter. As underlined in Section 2.1, we do not think that currently available LMI solvers are suited for handling 256 or 1024 medium-sized LMIs, thus precluding the use of our approach for designing controllers of order greater than three. We rather recommend the use of conservative approaches based on $\mathcal{H}_\infty$ control theory [4, 8] when high-order controllers are really required.

Finally, it must be noted that the formulation of the LMI optimization problems can slightly be simplified by using reduced Hermite-Fujiwara matrices, as described in [20, Chapter 3].

6 Conclusion

In this paper, we proposed a general methodology for addressing the difficult problem of robust stabilization of a plant subject to interval uncertainty. We formulated sufficient non-convex BMI conditions of robust stabilization and tractable computationally cheap LMI heuristics were described to tackle instances of this problem involving a reasonable number of design parameters. In spite of the conservative nature of the stabilizability condition and the lack of guarantee of convergence of the LMI heuristic, we demonstrated on several numerical examples that our methods provided significant improvements over results found in the literature.

As illustrated by our previous work [17, 18], the approach is general enough to handle other challenging design problems arising in robust control. As an additional advantage, other requirements on controller parameters or closed-loop system performance can readily be incorporated into our methodology as additional linear or bilinear matrix inequalities. This should result in a simple methodology for PI and PID tuning under performance requirements. However, this possibility was not investigated in this paper. On the other hand, we do not think that our approach proves useful when controllers of high orders are required. The number of BMIs turns out to be prohibitive and the computational burden would not fit into reasonable requirements.

Our work hints of several potential directions for future research. We strongly believe that further work must be devoted to the study of algorithms for solving the rank-one LMI problem. Indeed, this key problem appears to encompass in an elegant way a huge range of practical robust control design issues, but also difficult problems arising in combinatorial optimization. Investigations are underway for showing that standard problems such as static output feedback stabilization and robust pole placement into regions of the complex plane are also amenable to this unifying treatment.

It should also be insightful to assess the applicability to our problem of the various rank minimization heuristics proposed in the literature. For instance, the cutting plane technique for rank-one linear programming proposed in [25] may prove useful. Recent theoretical results on the rank minimization problem [22] also deserve to be considered.

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