Rank-one LMI Approach to Robust Stability of Polynomial Matrices[1]

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Abstract

Necessary and sufficient conditions are formulated for checking robust stability of an uncertain polynomial matrix. Various stability regions and uncertainty models are handled in a unified way. The conditions, stemming from a general optimization methodology similar to the one used in μ-analysis, are expressed as a rank-one LMI, a non-convex problem frequently arising in robust control. Convex relaxations of the problem yield tractable sufficient LMI conditions for robust stability of uncertain polynomial matrices.

Keywords

Polynomial Matrix, Robust Stability, LMI.

1 Introduction

Polynomial matrices play a central role in modern systems theory. Algebraic methods such as the polynomial approach [20] or the behavioral approach [28] heavily rely upon polynomial matrices. Dynamics of many systems (e.g. lightly damped structures such as oil derricks or regional power system models, see [19] and references therein) are most

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naturally represented by polynomial matrices and polynomial matrix fraction descriptions. Unsurprisingly, fundamental system features are captured by properties of polynomial matrices. For example, the zeros[1] of the denominator polynomial matrix in a matrix fraction description characterize system dynamics and performance. Satisfactory transient time response can be ensured as soon as the zeros are located in some specific region of the complex plane.

An important issue in control is to assess to what extent stability and performance of a system can be guaranteed in face of uncertainty or variation of the system parameters. A lot of research efforts has been recently devoted to the investigation of this problem, which has been coined out as the robust stability analysis problem, see e.g. the textbooks [2, 3, 29] and references therein. Quite naturally, the problem of checking robust stability of uncertain linear systems amounts to checking robust stability of uncertain polynomial matrices.

A very few works have been devoted so far to the study of robust stability of polynomial matrices, probably because the problem is particularly difficult to solve in its most general form. For example, it was proved that checking robust stability of a polytope of polynomial matrices (the so-called polytopic uncertainty model, see e.g. [2]) is an NP-hard problem even in the simple case that all vertex matrices are of degree zero (i.e. when they do not depend in the indeterminate) [4]. NP-hardness roughly means that it is very unlikely to find an algorithm that solves the problem in a time which is a polynomial function of the problem dimensions. These negative results have naturally led to the development of conservative, yet tractable, polynomial-time conditions for checking robust stability. One possibility is to strengthen the standard notions of stability and allowable uncertainty model [19]. Another possibility is to pursue the so-called quadratic stability approach in the context of Lyapunov theory [11, 2]. It paved the way for the development of efficient polynomial-time stability tests based on optimization over linear matrix inequalities (LMIs, see [5]). Such stability tests are based on sufficient stability conditions. That is to say, they may guarantee robust stability at the price of a certain amount of conservatism which is difficult to evaluate. Preliminary results on the application of LMI techniques to the study of robust stability of polytopes of polynomial matrices are reported in [18].

In view of this unsatisfactory state of the art, the paper is precisely an attempt to overcome the lack of sufficiently general methods for assessing robust stability of uncertain polynomial matrices. As an extension of the results presented in [18], we provide sufficient but also necessary robust stability conditions. Our approach is sufficiently general to handle in a unified way a fairly large number of uncertainty models and stability regions. The basic idea behind our approach can be found in [9] and can be traced back to μ-analysis [10, 21]. It has been used recently to assess $\mathcal{D}$-stability of polynomial matrices [16] and stability of 2-D polynomial matrices [17]. The stability problem is first expressed as a quadratic optimization problem. Then, several techniques are used to come up with a standard form of the problem. The necessary and sufficient robust stability conditions are expressed as a rank-one LMI problem, a non-convex optimization problem frequently arising in robust control problems, see [15] and references therein. Convex relaxations

[1]The zeros of a polynomial matrix are the roots of its determinant.

of this non-convex rank-one LMI problem yield possibly conservative but tractable LMI conditions for robust stability.

The outline of the paper is as follows. In Section 2 we state the problem to be solved. Then we introduce the stability regions (Section 3) and uncertainty models (Section 4) considered in the paper. In Section 5 we derive a rank-one LMI formulation of the problem. An illustrative example is proposed in Section 6.

Notations: $\mathbb{R}$ and $\mathbb{C}$ are the sets of real and complex numbers, respectively. $A^\star$ means transpose conjugate of complex matrix A. The matrix inequalities $A \succ B$ and $A \succeq B$ mean that matrix $A - B$ is positive definite and positive semidefinite, respectively.

2 Problem Statement

Suppose we are given a non-singular complex polynomial matrix

$$A(s, \Delta) = A_0(\Delta) + A_1(\Delta)s + \cdots + A_d(\Delta)s^d$$

of size n and degree d, whose matrix coefficients $A_i(\Delta)$ are affected by some uncertainty $\Delta \in \mathbf{\Delta}$ where $\mathbf{\Delta}$ is a given set of allowable uncertainties. Suppose moreover that we are given a region $\mathcal{D} \subset \mathbb{C}$ of the complex plane.

We say that uncertain polynomial matrix $A(s, \Delta)$ is robustly $\mathcal{D}$-stable if and only if the zeros of $A(s, \Delta)$ remain in region $\mathcal{D}$ for all admissible values Δ of the uncertainty. We aim at finding necessary and sufficient conditions for robust $\mathcal{D}$-stability of polynomial matrix $A(s, \Delta)$.

3 Stability Regions

First we describe the class of regions $\mathcal{D}$ we will consider throughout the paper.

Define $\mathcal{D}^C = \{s \in \mathbb{C} : s \notin \mathcal{D}\}$ as the complement of region $\mathcal{D}$ in $\mathbb{C}$. In this paper, we restrict our attention to two-dimensional regions $\mathcal{D}$ whose complement reads

$$\mathcal{D}^C = \{s \in \mathbb{C} : D^{00} + D^{01}s + D^{10}s^\star + D^{11}ss^\star \succeq 0\} \tag{1}$$

where Hermitian matrix

$$D = D^\star = \begin{bmatrix} D^{00} & D^{01} \\ D^{10} & D^{11} \end{bmatrix}$$

is non-singular and has at least one negative eigenvalue. In [16] we show that the above description is fairly general and can cover half-planes, disks, ellipsoids, parabolas and their complements, or possibly non-connected unions. For the sake of simplicity, in this paper we restrict our attention to the following simple regions:

- Half-plane $\mathcal{D} = \{x + jy \in \mathbb{C} : ax + by + c \geq 0\}$ with $a, b, c \in \mathbb{R}$ and

$$D = \begin{bmatrix} 2c & a + jb \\ a - jb & 0 \end{bmatrix}$$

- Disk $\mathcal{D} = \{s \in \mathbb{C} : |s - s_0| \leq r\}$ with $s_0 \in \mathbb{C}$, $r > 0 \in \mathbb{R}$ and

$$D = \begin{bmatrix} r^2 - s_0 s_0^\star & s_0 \\ s_0^\star & -1 \end{bmatrix}$$

- Disk complement $\mathcal{D} = \{s \in \mathbb{C} : |s - s_0| \geq r\}$ with $s_0 \in \mathbb{C}$, $r > 0 \in \mathbb{R}$ and

$$D = \begin{bmatrix} -r^2 + s_0 s_0^\star & -s_0 \\ -s_0^\star & 1 \end{bmatrix}$$

where 2×2 matrix D has exactly one negative eigenvalue and one positive eigenvalue. Note that some care must be taken with zeros at infinity when dealing with open stability regions such as the disk complement or the left half-plane, see [16, §3] for details.

4 Uncertainty Models

In this section, we describe the class of allowable uncertainties studied in the paper.

We consider that uncertainty Δ has a block-diagonal structure, a standard assumption made in the robust control literature, i.e.

$$\Delta = \begin{bmatrix} \Delta_1 & & & 0 \\ & \Delta_2 & & \\ & & \ddots & \\ 0 & & & \Delta_K \end{bmatrix}.$$

Each sub-block Δ_i belongs to $\mathbb{C}^{N_i \times N_i}$. This allows us to write uncertain polynomial matrix $A(s, \Delta)$ using a Linear Fractional Representation (LFR)

$$A(s, \Delta) = A_0 + L\Delta_s(I_N - D\Delta_s)^{-1}R \tag{2}$$

where the indeterminate s has been incorporated into

$$\Delta_s = \begin{bmatrix} sI_{dn} & 0 \\ 0 & \Delta \end{bmatrix}$$

so that $N = dn + N_1 + \cdots + N_K$ [29]. For a given index i, a rather large set of uncertainties can be captured by the quadratic description

$$\mathbf{\Delta_i} = \{\Delta_i \in \mathbb{C}^{N_i \times N_i} : D_i^{00} + D_i^{01}\Delta_i + \Delta_i^\star D_i^{10} + \Delta_i^\star D_i^{11}\Delta_i \succeq 0\}$$

where Hermitian matrix

$$D_i = D_i^\star = \begin{bmatrix} D_i^{00} & D_i^{01} \\ D_i^{10} & D_i^{11} \end{bmatrix}$$

is non-singular and has at least one negative eigenvalue. Note that this class of uncertainties is similar to the stability regions considered in Section 3. In [25], this class is referred to as $\{X, Y, Z\}$-dissipative uncertainties (with an additional sign assumption on D_i^{11}).

In this paper, we restrict ourselves to the following uncertainty models:

- Norm-bounded uncertainties, with

$$\mathbf{\Delta_i} = \{\Delta_i \in \mathbb{C}^{N_i \times N_i} : \|\Delta_i\|_2 \leq \gamma_i\}$$

  and

$$D_i = \begin{bmatrix} \gamma_i^2 I_{N_i} & 0 \\ 0 & -I_{N_i} \end{bmatrix}$$

- Interval uncertainties, with

$$\mathbf{\Delta_i} = \{\Delta_i = (x_i + jy_i) I_{N_i} \in \mathbb{C}^{N_i \times N_i} : x_i \in [a_i, b_i]\}$$

  and

$$D_i = \begin{bmatrix} -2ab & a+b \\ a+b & -2 \end{bmatrix}.$$

Using the same formalism, we can also cope with positive-real uncertainties, sector-bounded uncertainties or β-bounded uncertainties [13, 26, 12]. We can also enforce the uncertain parameters to be purely real (using a technique similar to that exposed in [10]) but this is not exposed here for simplicity.

5 Rank-one LMI Formulation of the Problem

Following these preliminaries, we can now derive a rank-one LMI formulation of a necessary and sufficient condition for robust stability of a polynomial matrix.

Our approach is twofold. First in Paragraph 5.1 we show that checking robust stability amounts to solving a quadratic optimization problem. Second, the LFR of $A(s, \Delta)$ is used in Paragraph 5.2 to take advantage of the special quadratic structure of the optimization problem and to derive an LMI formulation where all the non-convexity is concentrated into the constraint that a matrix has rank one. Convex primal and dual LMI relaxations are then proposed in Paragraph 5.3 and 5.4 respectively.

5.1 Quadratic Optimization Problem

Recall from the problem statement in Section 2 that robust stability of uncertain polynomial matrix $A(s, \Delta)$ holds if and only if its zeros remain in $\mathcal{D}$ for all uncertainty $\Delta \in \mathbf{\Delta}$, or equivalently, if and only if $A(s, \Delta)v \neq 0$ for all non-zero $v \in \mathbb{C}^n$, $s \in \mathcal{D}^C$ and $\Delta \in \mathbf{\Delta}$.

Robust stability is then ensured if and only if the optimal value μ of the quadratic optimization problem

$$\begin{array}{rcl} \mu & = & \min \quad v^\star A^\star(s,\Delta)A(s,\Delta)v \\ & & \text{s.t.} \quad s \in \mathcal{D}^C \\ & & \qquad \Delta \in \mathbf{\Delta} \\ & & \qquad v^\star v = 1 \end{array} \tag{3}$$

is strictly positive.

5.2 Rank-one LMI Problem

Now we show that solving quadratic optimization problem (3) amounts to solving a rank-one LMI optimization problem.

In relation with the LFR (2) of matrix $A(s,\Delta)$, we can define vectors

$$p = \begin{bmatrix} p_0 \\ p_1 \\ \vdots \\ p_K \end{bmatrix} \quad q = \begin{bmatrix} q_0 \\ q_1 \\ \vdots \\ q_K \end{bmatrix}$$

such that

$$\begin{array}{rcl} A(s,\Delta)v & = & A_0 v + Lp \\ q & = & Rv + Dp \\ p & = & \Delta_s q. \end{array} \tag{4}$$

For any sub-vectors p_i, q_i in p, q, it follows from the block-diagonal structure of Δ_s that

$$p_i = \Delta_i q_i. \tag{5}$$

Now define the rank-one positive semidefinite matrix

$$X = xx^\star = \begin{bmatrix} v \\ p \end{bmatrix} \begin{bmatrix} v \\ p \end{bmatrix}^\star \succeq 0 \tag{6}$$

and matrices $\mathcal{A}$, $\mathcal{P}_i$ and $\mathcal{Q}_i$ such that

$$\begin{array}{rcl} A(\Delta_s)v & = & \mathcal{A}x \\ q_i & = & \mathcal{Q}_i x \\ p_i & = & \mathcal{P}_i x \end{array}$$

for $i = 0, 1, \ldots, K$. Define finally the partitioning

$$\begin{bmatrix} X_i^{00} & X_i^{01} \\ X_i^{10} & X_i^{11} \end{bmatrix} = \begin{bmatrix} \mathcal{Q}_i \\ \mathcal{P}_i \end{bmatrix} X \begin{bmatrix} \mathcal{Q}_i \\ \mathcal{P}_i \end{bmatrix}^\star$$

and the linear maps F_i from $\mathbb{C}^{N\times N}$ to $\mathbb{C}^{M_i \times M_i}$ associated with sets $\mathbf{\Delta_i}$ as follows [9]:

- Stability region $\mathbf{\Delta_i} = \{sI_{N_i} \in \mathbb{C}^{N_i \times N_i} : D^{00} + D^{01}s + D^{10}s^\star + D^{11}s^\star s \geq 0\}$

$$F_i(X) = D^{00}X_i^{00} + D^{01}X_i^{10} + D^{10}X_i^{01} + D^{11}X_i^{11}$$

- Norm-bounded uncertainties $\mathbf{\Delta_i} = \{\Delta_i \in \mathbb{C}^{N_i \times N_i} : \|\Delta_i\|_2 \leq \gamma_i\}$

$$F_i(X) = \mathrm{Trace}(\gamma_i^2 X_i^{00} - X_i^{11})$$

- Interval uncertainties $\mathbf{\Delta_i} = \{(x_i + jy_i)I_{N_i} \in \mathbb{C}^{N_i \times N_i} : x_i \in [a_i, b_i]\}$

$$F_i(X) = -2a_i b_i X_i^{00} + (a_i + b_i)(X_i^{01} + X_i^{10}) - 2X_i^{11}.$$

In relation to the above linear maps, equation (5) and rank-one matrix (6), we can state the following central result.

Lemma 1 *Assume vector q_i is non-zero. It holds $p_i = \Delta_i q_i$ for some $\Delta_i \in \mathbf{\Delta_i}$ if and only if*

$$F_i(X) \succeq 0. \tag{7}$$

Proof of Lemma 1 This is a standard result in μ-analysis, see [10, 21]. The proof is not reproduced here for conciseness. □

Using equations (4), (6), Lemma 1 and gathering all linear maps $F_i(X)$ into one block-diagonal linear map

$$F(X) = \begin{bmatrix} F_0(X) & & & \\ & F_1(X) & & \\ & & \ddots & \\ & & & F_K(X) \end{bmatrix},$$

an alternative formulation of problem (3) can now be derived. It reads

$$\begin{array}{rcll} \mu & = & \min & \mathrm{Trace}\mathcal{A}^\star \mathcal{A} X \\ & & \mathrm{s.t.} & F(X) \succeq 0 \\ & & & X = X^\star \succeq 0 \\ & & & \mathrm{Rank} X = 1 \\ & & & \mathrm{Trace} \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix} X = 1. \end{array} \tag{8}$$

Problem (8) is an LMI optimization problem with a non-convex rank constraint. It must be pointed out that rank-constrained LMIs frequently arise in control problems but also in mathematical programming and combinatorial optimization, see [15] for a recent overview. We have shown the main result of this paper.

Theorem 1 *Polynomial matrix $A(s, \Delta)$ is robustly $\mathcal{D}$-stable if and only if $\mu > 0$ in rank-one LMI optimization problem (8).*

5.3 Primal LMI Relaxation

A convex primal LMI relaxation can readily be derived for non-convex rank-constrained LMI problem (8). As a result, we obtain a sufficient condition of robust $\mathcal{D}$-stability of polynomial matrix $A(s,\Delta)$.

Consider the following convex relaxation of rank-one LMI problem (8)

$$\begin{array}{rcll} \nu & = & \min & \mathrm{Trace}\mathcal{A}^\star\mathcal{A}X \\ & & \mathrm{s.t.} & F(X) \succeq 0 \\ & & & X = X^\star \succeq 0 \\ & & & \mathrm{Trace}\left[\begin{array}{cc} I_n & 0 \\ 0 & 0 \end{array}\right] X = 1 \end{array} \tag{9}$$

where the non-convex rank constraint has been dropped. Since the feasible set of problem (8) is included in the feasible set of problem (9), $\nu > 0$ in problem (9) obviously implies $\mu > 0$ in problem (8). This is captured in the following corollary to Theorem 1.

Corollary 1 *Polynomial matrix $A(s,\Delta)$ is robustly $\mathcal{D}$-stable if $\nu > 0$ in LMI optimization problem (9).*

5.4 Dual LMI Relaxation

Now we propose a convex dual LMI relaxation for non-convex rank-constrained LMI problem (8).

For $i = 0, 1, \ldots, K$ define the linear maps $F_i^D(P_i)$ dual to the maps $F_i(X)$ introduced above, i.e. such that for any couple of matrices $X \in \mathbb{C}^{N\times N}$ and $P_i \in \mathbb{C}^{M_i\times M_i}$, it holds

$$\mathrm{Trace}F_i^D(P_i)X = \mathrm{Trace}F_i(X)P_i.$$

Then define

$$P = \left[\begin{array}{cccc} P_0 & & & \\ & P_1 & & \\ & & \ddots & \\ & & & P_K \end{array}\right]$$

and the associated linear map

$$F^D(P) = F_0^D(P_0)X + F_1^D(P_1)X + \cdots + F_K^D(P_K)X.$$

Consider the LMI feasibility problem

$$\begin{array}{l} \mathcal{A}^\star\mathcal{A} \succ F^D(P) \\ P = P^\star \succ 0 \end{array} \tag{10}$$

and note that as soon as the above problem is feasible, it holds

$$\mathrm{Trace}\mathcal{A}^\star\mathcal{A}X > \mathrm{Trace}F^D(P)X = \mathrm{Trace}F(X)P$$

for any matrix $X = X^\star \succeq 0$. Since the above inequality is also valid for any rank-one matrix X, it follows that $\mu > 0$ in problem (8).

Using standard semi-definite programming duality arguments [27] it can be shown that LMI feasibility problem (10) is actually dual to relaxed LMI problem (9). Now if $\mathcal{N}$ denotes a matrix whose columns span the right null-space of $\mathcal{A}$, it follows from the Elimination Lemma [5] that feasibility problem (10) can equivalently be written as

$$\begin{array}{l} \mathcal{N}^\star F^D(P)\mathcal{N} \prec 0 \\ P = P^\star \succ 0. \end{array} \tag{11}$$

The following corollary to Theorem 1 follows from the above discussion and provides us with an equivalent sufficient condition of robust $\mathcal{D}$-stability.

Corollary 2 *Polynomial matrix $A(s,\Delta)$ is robustly $\mathcal{D}$-stable if there is a matrix P solution to LMI feasibility problem (11).*

6 Illustration

Let

$$A(s,\Delta) = \underbrace{\begin{bmatrix} 1 & 2 \\ 4 & 0 \end{bmatrix}}_{A_0} + (\underbrace{\begin{bmatrix} 0 & 0 \\ -1 & 0 \end{bmatrix}}_{A_1} + x_1 \underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{B_1})s + (\underbrace{\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix}}_{A_2} + \Delta_2)s^2$$

be a polynomial matrix affected by interval uncertainty

$$x_1 \in [-0.6000, 0.6000] = [a_1, b_1]$$

and norm-bounded uncertainty

$$\|\Delta_2\|_2 \leq \gamma_2 = 0.3000.$$

We are interested in knowing whether the zeros of polynomial matrix $A(s,\Delta)$ stay outside of the closed unit disk

$$\mathcal{D}^C = \{s \in \mathbb{C} : s^\star s \leq 1\}.$$

for all admissible uncertainty.

Defining

$$\Delta_s = \begin{bmatrix} sI_4 & & \\ & x_1 I_2 & \\ & & \Delta_2 \end{bmatrix}$$

and using the construction principle described in [29], one possible LFR (2) for the above polynomial matrix is given by

$$\left[\begin{array}{c|c} A_0 & L \\ \hline R & D \end{array}\right] = \left[\begin{array}{c|cccc} A_0 & A_1 & A_2 & B_1 & I_2 \\ \hline I_2 & 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 \end{array}\right].$$

Projection matrices in LMI problem (9) are as follows

$$\mathcal{Q}_0 = \left[\begin{array}{ccccc} I_2 & 0 & 0 & 0 & 0 \\ 0 & I_2 & 0 & 0 & 0 \end{array}\right] \quad \mathcal{P}_0 = \left[\begin{array}{ccccc} 0 & I_2 & 0 & 0 & 0 \\ 0 & 0 & I_2 & 0 & 0 \end{array}\right]$$

$$\mathcal{Q}_1 = \left[\begin{array}{ccccc} 0 & I_2 & 0 & 0 & 0 \end{array}\right] \quad \mathcal{P}_1 = \left[\begin{array}{ccccc} 0 & 0 & 0 & I_2 & 0 \end{array}\right]$$

$$\mathcal{Q}_2 = \left[\begin{array}{ccccc} 0 & 0 & I_2 & 0 & 0 \end{array}\right] \quad \mathcal{P}_2 = \left[\begin{array}{ccccc} 0 & 0 & 0 & 0 & I_2 \end{array}\right].$$

With the notation

$$\mathcal{A} = \left[\begin{array}{ccccc} A_0 & A_1 & A_2 & B_1 & I_2 \end{array}\right],$$

LMI optimization problem (9) reads

$$\begin{array}{rcll} \nu & = & \min & \mathrm{Trace}\mathcal{A}^\star\mathcal{A}X \\ & & \text{s.t.} & F_0(X) = \mathcal{Q}_0 X \mathcal{Q}_0^\star - \mathcal{P}_0 X \mathcal{P}_0^\star \succeq 0 \\ & & & F_1(X) = -2a_1b_1\mathcal{Q}_1 X \mathcal{Q}_1^\star + (a_1+b_1)(\mathcal{Q}_1 X \mathcal{P}_1^\star + \mathcal{P}_1 X \mathcal{Q}_1^\star) - 2\mathcal{P}_1 X \mathcal{Q}_1^\star \succeq 0 \\ & & & F_2(X) = \mathrm{Trace}(-\mathcal{P}_2 X \mathcal{P}_2^\star + \gamma_2^2 \mathcal{Q}_2 X \mathcal{Q}_2^\star) \succeq 0 \\ & & & \mathrm{Trace}\left[\begin{array}{cc} I_2 & 0 \\ 0 & 0 \end{array}\right] X = 1, \quad X = X^\star \succeq 0. \end{array}$$

With the help of the interior-point algorithm of the SDPHA package 3.0 [6] called from the user-friendly interface LMITOOL 2.0 for MATLAB [8] we obtained

$$\nu = 0.004635$$

as the optimal value of the above problem, with

$$X = \left[\begin{array}{cccccccccc}
0.1562 & -0.3551 & 0.1556 & -0.3602 & 0.1539 & -0.3628 & -0.0934 & 0.2082 & 0.1116 & -0.0106 \\
-0.3551 & 0.8438 & -0.3473 & 0.8407 & -0.3371 & 0.8316 & 0.2162 & -0.5043 & -0.2720 & 0.0205 \\
0.1556 & -0.3473 & 0.1562 & -0.3551 & 0.1556 & -0.3602 & -0.0923 & 0.2021 & 0.1079 & -0.0112 \\
-0.3602 & 0.8407 & -0.3551 & 0.8438 & -0.3473 & 0.8407 & 0.2177 & -0.4987 & -0.2683 & 0.0223 \\
0.1539 & -0.3371 & 0.1556 & -0.3473 & 0.1562 & -0.3551 & -0.0906 & 0.1944 & 0.1035 & -0.0117 \\
-0.3628 & 0.8316 & -0.3602 & 0.8407 & -0.3551 & 0.8438 & 0.2176 & -0.4895 & -0.2626 & 0.0240 \\
-0.0934 & 0.2162 & -0.0923 & 0.2177 & -0.0906 & 0.2176 & 0.0562 & -0.1278 & -0.0687 & 0.0060 \\
0.2082 & -0.5043 & 0.2021 & -0.4987 & 0.1944 & -0.4895 & -0.1278 & 0.3038 & 0.1643 & -0.0111 \\
0.1116 & -0.2720 & 0.1079 & -0.2683 & 0.1035 & -0.2626 & -0.0687 & 0.1643 & 0.0889 & -0.0058 \\
-0.0106 & 0.0205 & -0.0112 & 0.0223 & -0.0117 & 0.0240 & 0.0060 & -0.0111 & -0.0058 & 0.0011
\end{array}\right].$$

In virtue of Corollary 1, $\nu > 0$ implies that $A(s, \Delta)$ has no zero within the closed unit disk for any admissible uncertain parameters x_1 and Δ_2. Hence polynomial matrix $A(s, \Delta)$ is robustly $\mathcal{D}$-stable. This can be checked graphically in Figure 1, where zeros of $A(s, \Delta)$ are represented for 1000 randomly chosen admissible uncertain parameters.

Now if we set

$$\gamma_2 = 0.4000$$

the optimal value of LMI problem (9) is

$$\nu = 5.2982 \cdot 10^{-10} \approx 0$$

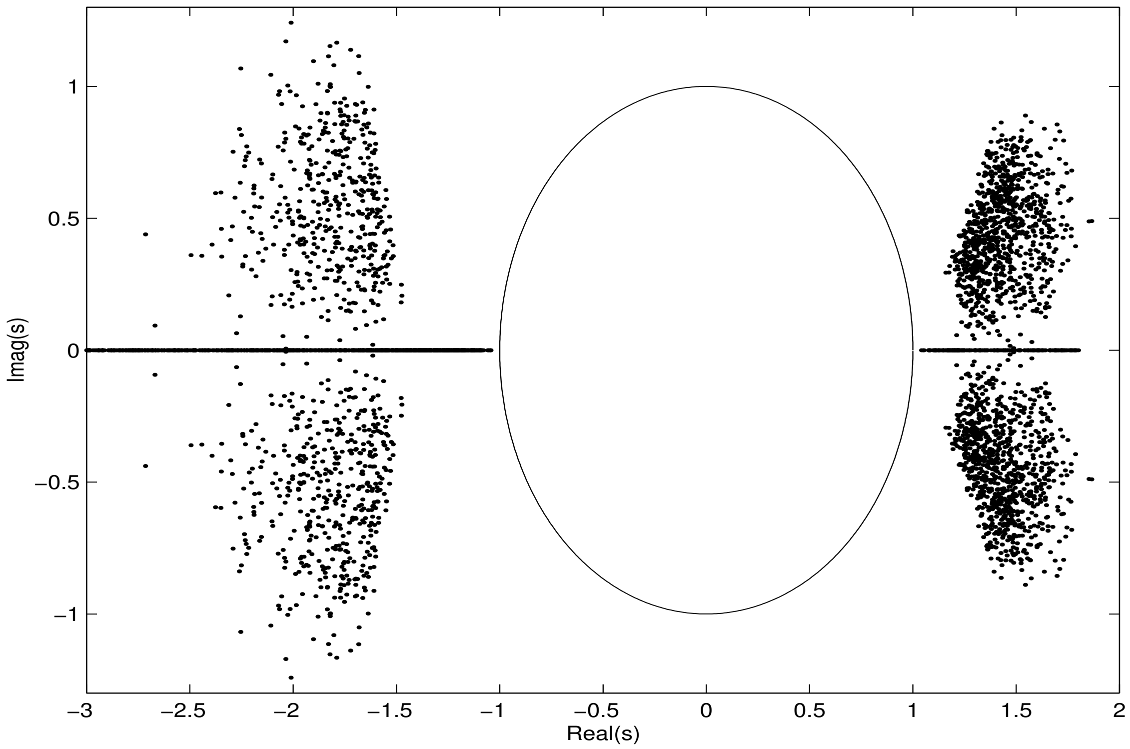

Figure 1: Case $\gamma_2 = 0.3000$. Zeros of $A(s,\Delta)$ for 1000 randomly chosen admissible uncertain parameters x_1 and Δ_2.

for

$$X = \begin{bmatrix} 0.1423 & -0.3261 & 0.1256 & -0.2777 & 0.1357 & -0.3425 & -0.0794 & 0.1819 & 0.1212 & -0.0215 \\ -0.3261 & 0.8577 & -0.2500 & 0.5999 & -0.2842 & 0.8230 & 0.1995 & -0.5045 & -0.3460 & 0.0319 \\ 0.1256 & -0.2500 & 0.1412 & -0.3250 & 0.1249 & -0.2771 & -0.0628 & 0.1345 & 0.0878 & -0.0214 \\ -0.2777 & 0.5999 & -0.3250 & 0.8550 & -0.2497 & 0.5998 & 0.1477 & -0.3427 & -0.2293 & 0.0382 \\ 0.1357 & -0.2842 & 0.1249 & -0.2497 & 0.1395 & -0.3231 & -0.0705 & 0.1509 & 0.0984 & -0.0242 \\ -0.3425 & 0.8230 & -0.2771 & 0.5998 & -0.3231 & 0.8505 & 0.1961 & -0.4639 & -0.3122 & 0.0464 \\ -0.0794 & 0.1995 & -0.0628 & 0.1477 & -0.0705 & 0.1961 & 0.0490 & -0.1160 & -0.0781 & 0.0096 \\ 0.1819 & -0.5045 & 0.1345 & -0.3427 & 0.1509 & -0.4639 & -0.1160 & 0.3049 & 0.2093 & -0.0131 \\ 0.1212 & -0.3460 & 0.0878 & -0.2293 & 0.0984 & -0.3122 & -0.0781 & 0.2093 & 0.1477 & -0.0070 \\ -0.0215 & 0.0319 & -0.0214 & 0.0382 & -0.0242 & 0.0464 & 0.0096 & -0.0131 & -0.0070 & 0.0087 \end{bmatrix}.$$

However, Corollary 1 cannot be used to conclude about robust stability or instability of polynomial matrix $A(s,\Delta)$.

Using a trial and error method, we found that matrix $A(s,\Delta)$ has an unstable zero at $\bar{s} = 0.9903$ for the following choice of admissible uncertain parameters

$$\bar{x}_1 = -0.6000 \quad \bar{\Delta}_2 = \begin{bmatrix} 0.1600 & -0.3600 \\ 0.0800 & 0.0400 \end{bmatrix}.$$

The normalized vector $\bar{v}$ such that $(A_0 + (A_1 + \bar{x}_1)\bar{s} + (A_2 + \bar{\Delta}_2)\bar{s}^2)\bar{v} = 0$ gives rises to a vector

$$x = \begin{bmatrix} \bar{v} \\ \bar{s}\bar{v} \\ \bar{s}^2\bar{v} \\ \bar{x}_1\bar{s}\bar{v} \\ \bar{\Delta}_2\bar{s}^2\bar{v} \end{bmatrix} = \begin{bmatrix} -0.3785 \\ 0.9256 \\ -0.3749 \\ 0.9166 \\ -0.3712 \\ 0.9077 \\ 0.2249 \\ -0.5500 \\ -0.3862 \\ 0.0066 \end{bmatrix}$$

such that rank-one matrix $X = xx'$ satisfies $\mu = 0$ in rank-constrained LMI problem (8). In virtue of Theorem 1, uncertain polynomial matrix $A(s, \Delta)$ is not robustly $\mathcal{D}$-stable.

7 Conclusion

Following an idea found in [9, Chapter 1] and that can be traced back to μ-analysis [10, 21], we have proposed a general methodology for determining whether the zeros of a given uncertain polynomial matrix stay within a given region of the complex plane for all admissible uncertainty. Several stability regions and uncertainty models can be covered in a unified way. Necessary and sufficient conditions are formulated as a rank-constrained LMI optimization problem. Sufficient robust stability conditions are readily derived as convex LMI optimization or feasibility problems.

The main motivation behind formulating the robust stability problem as a rank-constrained LMI problem is in that our results can be extended in various directions:

- Necessary robust stability conditions may also be obtained, using geometric properties of the intersection of ellipsoids [15].
- The gap between necessity and sufficiency of the LMI conditions can be narrowed thanks to recent results on the full-block $\mathcal{S}$-procedure [24] and quadratic separators [14, 22].
- Rank-one LMI problems are special kinds of non-convex optimization problems for which tailored global optimization algorithms have recently been designed [1]. Quite efficient local optimization algorithms based on successive linearizations can also be used [7], but without guarantee of convergence to the global optimum.
- The LFR used to represent the uncertain polynomial matrix is not necessarily minimal, thus problem dimensions in the LMIs can be decreased via any (sub-optimal) LFR reduction procedure (see [16, 17] for an example of such a procedure).
- Numerical aspects of solving the LMI conditions must also be carefully checked. Intensive numerical experiments are currently performed and will result in several new macros to be implemented in the next release 3.0 of the Polynomial Toolbox for MATLAB, see [23].

- Another interesting extension could eventually be to consider robust synthesis problems and Diophantine equations over polynomial matrices [20].

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