A hierarchy of LMI inner approximations of the set of stable polynomials

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Abstract

Exploiting spectral properties of symmetric banded Toeplitz matrices, we describe simple sufficient conditions for positivity of a trigonometric polynomial formulated as linear matrix inequalities (LMI) in the coefficients. As an application of these results, we derive a hierarchy of convex LMI inner approximations (affine sections of the cone of positive definite matrices of size m) of the nonconvex set of Schur stable polynomials of given degree $n < m$. It is shown that when m tends to infinity the hierarchy converges to a lifted LMI approximation (projection of an LMI set defined in a lifted space of dimension quadratic in n) already studied in the technical literature. An application to robust controller design is described.

Keywords: stability; positive polynomials; LMI; Toeplitz matrices

1 Introduction

Linear system stability can be formulated algebraically in the space of coefficients of the characteristic polynomial. The region of stability is generally *nonconvex* in this space, and this is a major obstacle when solving fixed-order or robust controller design problems. In the case of discrete-time linear systems, the region of stability is a bounded open set whose boundary consists of (flat) hyperplanes and nonconvex (negatively curved) algebraic varieties. Recent results on real algebraic geometry and generalized problems of moments can be used to build up a hierarchy of convex linear matrix inequality (LMI) outer approximations of the region of stability, with asymptotic convergence to its convex hull, see e.g. [7] for a software implementation and examples. It is generally more difficult to construct convex LMI *inner approximations*, see [6] for a survey. Strict positive realness

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of rational transfer functions and its connection with polynomial positivity conditions are used in [6] to generate inner approximations which are *lifted* LMI sets. For polynomials of degree n, they are projections onto coefficient space $\mathbb{R}^n$ of an LMI set living in a lifted space $\mathbb{R}^{\frac{n^2+3n}{2}}$. The LMI set is built around a particular point, the central polynomial, whose relevance in robust control design is explained in [6]. These lifted LMI regions are also used in signal processing, see e.g. [3, Section 7.3]. They can be derived in a state-space setting with the Kalman-Yakubovich-Popov lemma [4].

Whereas lifted LMIs are a powerful modeling paradigm (it is currently conjectured that every convex semialgebraic set is a lifted LMI set), the introduction of a large number of lifting variables can be seen as a drawback. It is therefore relevant to build convex LMI inner approximations of the nonconvex stability region *without liftings*, namely as affine sections of the cone of positive semidefinite matrices. This is the objective of this paper. We use results of functional analysis on sequences of eigenvalues of Toeplitz matrices to derive sufficient LMI conditions for positivity of trigonometric polynomials, and we apply these results to construct a hierarchy of m-by-m LMI inner approximations of the nonconvex stability domain. Moreover we prove that when m tends to infinity, the hierarchy converges asymptotically to the lifted LMI approximation of [6]. Finally, we describe an application of these results to robust controller design.

2 Trigonometric polynomials and Toeplitz matrices

Let p_k, $k = 0, 1, 2, \ldots, n$ denote real numbers, and define the trigonometric polynomial

$$\begin{aligned} z = e^{i\theta} \mapsto p(\theta) &= p_0 + p_1(z + z^{-1}) + p_2(z^2 + z^{-2}) + \cdots + p_n(z^n + z^{-n}) \\ &= p_0 + 2p_1 \cos\theta + 2p_2 \cos 2\theta + \cdots + 2p_n \cos n\theta \end{aligned}$$

of degree n mapping the unit circle of the complex plane onto the real axis.

For a given integer $m > n$, define the column vector $v_m(z) = [1 \; z \; z^2 \cdots z^{m-1}]^T$ and represent polynomial p as a quadratic form

$$p(\theta) = \frac{1}{m} v_m^T(e^{-i\theta}) P_m v_m(e^{i\theta}) \tag{1}$$

where

$$P_m = \begin{bmatrix} p_0 & \frac{m}{m-1}p_1 & \frac{m}{m-2}p_2 & & \\ \frac{m}{m-1}p_1 & p_0 & \frac{m}{m-1}p_1 & & \\ \frac{m}{m-2}p_2 & \frac{m}{m-1}p_1 & p_0 & & \\ & & & \ddots & \\ & & & & p_0 \end{bmatrix} \tag{2}$$

is an m-by-m symmetric banded Toeplitz matrix.

Define

$$R_m = \begin{bmatrix} p_0 & p_1 & p_2 & & \\ p_1 & p_0 & p_1 & & \\ p_2 & p_1 & p_0 & & \\ & & & \ddots & \\ & & & & p_0 \end{bmatrix}$$

as the m-by-m moment matrix of p, so named for

$$p_k = \frac{1}{2\pi}\int_0^{2\pi} p(\theta)e^{-ik\theta}d\theta$$

is the k-th moment, or Fourier coefficient, of polynomial p. Note that R_m has the same banded symmetric Toeplitz structure as P_m.

Connections between the spectrum of matrix R_m and the values taken by polynomial p on the unit circle have been studied extensively. In the sequel, $\lambda_{\min}$ denotes the minimum eigenvalue of a symmetric matrix.

Theorem 2.1

$$\lim_{m\to+\infty} \lambda_{\min}(R_m) = \min_\theta p(\theta).$$

Proof: It is a corollary of Gábor Szegő's fundamental eigenvalue distribution theorem, see e.g. [5, Corollary 4.2]. □

In this section we aim at establishing a similar spectral property linking matrix P_m and polynomial p. First let us state a few instrumental results.

Lemma 2.1 *For all θ it holds $\lambda_{\min}(P_m) \leq p(\theta)$ and as a consequence*

$$\limsup_{m\to+\infty} \lambda_{\min}(P_m) \leq \min_\theta p(\theta). \tag{3}$$

Proof: From relation (1) and the identity $v_m^T(e^{-i\theta})v_m(e^{i\theta}) = m$, it follows that

$$\frac{v_m^T(e^{-i\theta})P_m v_m(e^{i\theta})}{v_m^T(e^{-i\theta})v_m(e^{i\theta})} = p(\theta) \tag{4}$$

and hence $\lambda_{\min}(P_m) \leq p(\theta)$. When $m \to \infty$ we obtain the desired result. □

Lemma 2.2

$$\|P_m - R_m\| = O(m^{-\frac{1}{2}}).$$

Proof: Consider

$$P_m - R_m = \begin{bmatrix} 0 & \frac{1}{m-1}p_1 & \frac{1}{m-2}p_2 & & \\ \frac{1}{m-1}p_1 & 0 & \frac{1}{m-1}p_1 & & \\ \frac{1}{m-2}p_2 & \frac{1}{m-1}p_1 & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$$

and hence for the Frobenius norm

$$\|P_m - R_m\|^2 = \sum_{k=1}^{n} \frac{m-k}{(m-k)^2} p_k^2 = \sum_{k=1}^{n} \frac{1}{m-k} p_k^2.$$

□

We are now ready to state our main result.

Theorem 2.2

$$\lim_{m\to+\infty}\lambda_{\min}(P_m)=\min_\theta p(\theta).$$

Proof: Let v be an eigenvector of P_m such that $v^Tv=1$ and $P_mv=\lambda_{\min}(P_m)v$. From the equality

$$v^TP_mv=v^T(P_m-R_m)v+v^TR_mv,$$

we obtain with the help of Lemma 2.2 the following inequality

$$\lambda_{\min}(P_m)\geq O(m^{-\frac{1}{2}})+\lambda_{\min}(R_m).$$

Taking the limit, we obtain

$$\liminf_{m\to+\infty}\lambda_{\min}(P_m)\geq\lim_{m\to+\infty}\lambda_{\min}(R_m).$$

Using Lemma 2.1 and Theorem 2.1, we can see that

$$\liminf_{m\to+\infty}\lambda_{\min}(P_m)\geq\lim_{m\to+\infty}\lambda_{\min}(R_m)=\min_\theta p(\theta)$$

and hence

$$\liminf_{m\to+\infty}\lambda_{\min}(P_m)\geq\min_\theta p(\theta)\geq\limsup_{m\to+\infty}\lambda_{\min}(P_m).$$

If x_m is a real sequence then it is well-known that if $\liminf_{m\to+\infty}x_m=\limsup_{m\to+\infty}x_m$, then the sequence x_m converges to $\lim_{m\to+\infty}x_m=\liminf_{m\to+\infty}x_m=\limsup_{m\to+\infty}x_m$, and this completes the proof. □

Corollary 2.1 *Assume that polynomial p is positive. Then, there exists a sufficiently large integer m_0 such that for $m\geq m_0$, the Toeplitz matrix P_m is positive definite.*

Proof: Use Theorem 2.2. □

Remark 2.1 *Note that when p is positive, matrices P_m are not necessarily positive definite if m is not large enough. As a simple example consider the positive polynomial $p(\theta)=2+2\cos\theta+\frac{8}{5}\cos 2\theta$. We have*

$$P_3=\begin{bmatrix}2&\frac{3}{2}&\frac{12}{5}\\ \frac{3}{2}&2&\frac{3}{2}\\ \frac{12}{5}&\frac{3}{2}&2\end{bmatrix}$$

which is not positive definite, since $\lambda_{\min}(P_3)=-\frac{2}{5}$. Also, the next Toeplitz matrix

$$P_4=\begin{bmatrix}2&\frac{4}{3}&\frac{8}{5}&0\\ \frac{4}{3}&2&\frac{4}{3}&\frac{8}{5}\\ \frac{8}{5}&\frac{4}{3}&2&\frac{4}{3}\\ 0&\frac{8}{5}&\frac{4}{3}&2\end{bmatrix},$$

is not positive definite either, since $\lambda_{\min}(P_4)=\frac{8}{13}-\frac{2\sqrt{509}}{15}\approx-0.3415$. However, one can check that when $m\geq m_0=30$, matrices P_m are indeed positive definite.

3 LMI inner approximations of stability domain

Consider a monic polynomial

$$d(z) = d_0 + d_1 z + \cdots + d_{n-1} z^{n-1} + z^n$$

of degree n, with coefficient vector $d \in \mathbb{R}^n$ and let us define the set

$$\mathcal{S} = \{d \in \mathbb{R}^n \ : \ d(z) \text{ stable}\}$$

where stability is meant in the discrete-time, or Schur sense, i.e. all the roots of $d(z)$ belong to the open unit disk. Many control problems (e.g. fixed-order or robust controller design) can be formulated as linear programming problems in $\mathcal{S}$. Unfortunately $\mathcal{S}$ is nonconvex when $n > 2$, which renders controller design difficult in general. It can therefore be relevant to describe convex inner approximations of $\mathcal{S}$, in particular by exploiting the modeling flexibility of linear matrix inequalities (LMIs), see [6] and references therein.

An approach consists in choosing a monic polynomial

$$c(z) = c_0 + c_1 z + \cdots + c_{n-1} z^{n-1} + z^n$$

which is stable. Once c is given, we define the trigonometric polynomial

$$\begin{aligned} z = e^{i\theta} \mapsto p^{c,d}(\theta) &= c(z^{-1})d(z) + c(z)d(z^{-1}) \\ &= 2\sum_{l=0}^{n} \sum_{\substack{j,k=0 \\ |j-k|=l}}^{n} c_j d_k \cos l\theta \end{aligned}$$

and the set

$$\mathcal{P}^c = \{d \in \mathbb{R}^n \ : \ p^{c,d}(\theta) > 0 \quad \forall\, \theta \in \mathbb{R}\}.$$

Lemma 3.1 *Let $c(z)$ be a given stable polynomial. Then $\mathcal{P}^c \subset \mathcal{S}$.*

Proof: A geometric proof is as follows. Since polynomial $c(z)$ is Schur stable, when $z = e^{i\theta}$ varies along the unit circle, complex number $c(e^{i\theta})$ has a net increase of argument of $2n\pi$, or equivalently the plot of $c(e^{i\theta})$ encircles the origin n times, see e.g. [2, Section 1.3.3] or use Cauchy's argument principle. Notice that the real number $p^{c,d}(\theta) = c(e^{-i\theta})d(e^{i\theta}) + c(e^{i\theta})d(e^{-i\theta})$ is equal to $2|c(e^{i\theta})d(e^{i\theta})|\cos(c(e^{i\theta}), d(e^{i\theta}))$ where the last term is the cosine of the oriented angle between vectors $c(e^{i\theta})$ and $d(e^{i\theta})$ in the complex plane. Therefore $p^{c,d}(\theta)$ positive implies that the cosine is positive and hence that the angle between $c(e^{i\theta})$ and $d(e^{i\theta})$ is less than $\frac{\pi}{2}$ in absolute value for any given value of θ. This means that complex number $d(e^{i\theta})$ also encircles the origin n times when θ range from 0 to 2π, and hence that polynomial $d(z)$ is Schur stable. $\square$

Let $P_m^{c,d}$ be the symmetric banded Toeplitz matrix corresponding to polynomial $p^{c,d}$, built as in (2), and define the set

$$\mathcal{P}_m^c = \{d \in \mathbb{R}^n \ : \ P_m^{c,d} \succ 0\}$$

where $\succ 0$ means positive definite. Note that symmetric matrix $P_m^{c,d}$ depends affinely on d, so that $\mathcal{P}_m^c$ is a convex LMI set.

Theorem 3.1 *Let $c(z)$ be a given stable polynomial of degree n, and let $m > n$. Then $\mathcal{P}^c_m \subset \mathcal{S}$.*

Proof: Since $mp^{c,d}(\theta) = v^T_m(e^{-i\theta})P^{c,d}_m v_m(e^{i\theta})$, positive definiteness of matrix $P^{c,d}_m$ implies positivity of polynomial $p^{c,d}(\theta)$. Then use Lemma 3.1. □

In Section 2 we remarked that matrices P_m and R_m share many similar properties. From the proof of Theorem 3.1 we see however that only matrix P_m allows to build up a valid convex inner approximation of the nonconvex stability region. Basically, P_m is the Gram matrix in basis $v_m(e^{i\theta})$ of polynomial $p^{c,d}(\theta)$ (up to scaling by m), whereas R_m does not allow such a representation. Note also that, for a given m, the geometry of the convex set $\mathcal{P}^c_m$ depends only on the choice of a stable polynomial $c(z)$.

Theorem 3.2 *Let $c(z)$ be a given stable polynomial. Then $\mathcal{P}^c = \lim_{m\to+\infty} \mathcal{P}^c_m$.*

Proof: Use Theorem 2.2. □

Finally we make the connection with the results in [6]. Recall that a discrete-time rational function is strictly positive real (SPR) whenever its real part is strictly positive when evaluated along the unit circle.

Theorem 3.3

$$\mathcal{P}^c = \{d \in \mathbb{R}^n \ : \ \frac{d(z)}{c(z)} \ \text{SPR}\}.$$

Proof: Since $c(z)$ is stable, the SPR inequality

$$\text{Re}\,\frac{d(e^{i\theta})}{c(e^{i\theta})} = \frac{1}{2}\left(\frac{d(e^{i\theta})}{c(e^{i\theta})} + \frac{d(e^{-i\theta})}{c(e^{-i\theta})}\right) = \frac{c(e^{-i\theta})d(e^{i\theta}) + c(e^{i\theta})d(e^{-i\theta})}{2|c(e^{i\theta})|^2} > 0$$

is equivalent to positivity of trigonometric polynomial $p^{c,d}(\theta)$. □

Polynomial $c(z)$ is referred to as a central polynomial in [6] since set $\mathcal{P}^c$ is built around $c(z)$ in the coefficient space. Note however that there is no guarantee that $c(z)$ belongs to $\mathcal{P}^c_m$ if m is not large enough, see Remark 2.1.

4 Examples

4.1 Second-order polynomials

We consider second-order polynomials for which the exact stability region is a triangle with vertices $(z+1)^2$, $(z-1)(z+1)$ and $(z-1)^2$ [1, Example 11.13].

Choosing $c(z) = z^2$, we have $p^{c,d}(\theta) = 2 + 2d_1\cos\theta + 2d_0\cos 2\theta$. The first LMI inner approximation is

$$\mathcal{P}^{c,d}_3 = \{(d_0, d_1) \ : \ P^{c,d}_3 = \begin{bmatrix} 2 & \frac{3}{2}d_1 & 3d_0 \\ \frac{3}{2}d_1 & 2 & \frac{3}{2}d_1 \\ 3d_0 & \frac{3}{2}d_1 & 2 \end{bmatrix} \succ 0\}$$

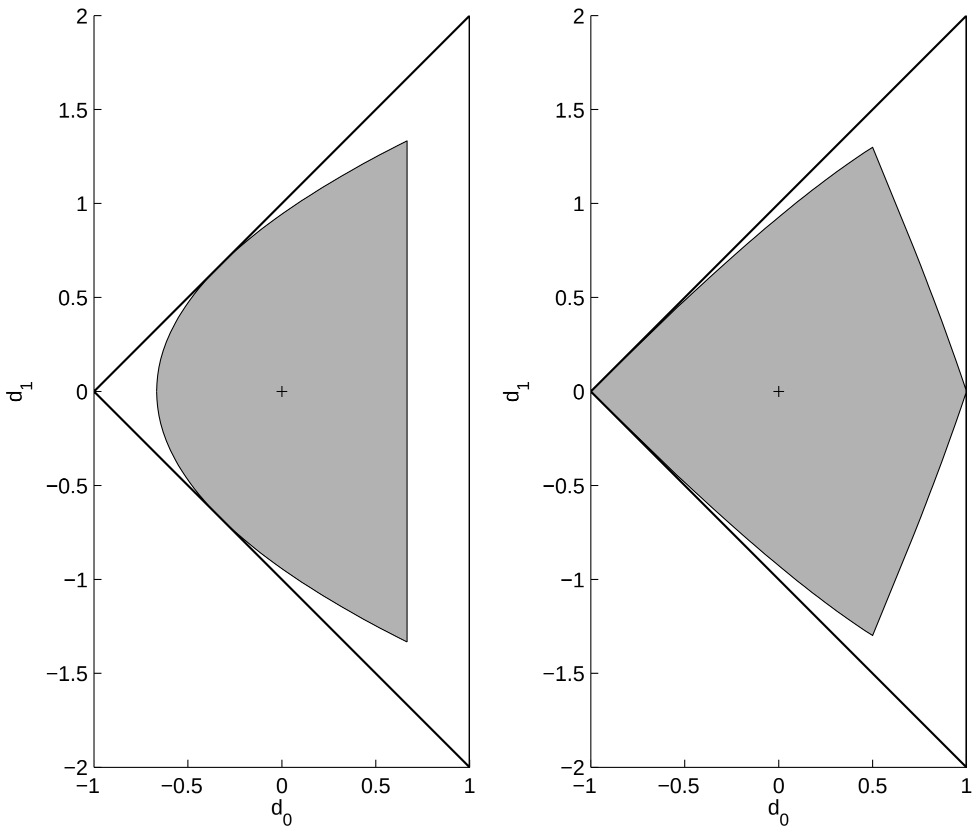

Figure 1: 3-by-3 LMI set (shaded gray, left) and 4-by-4 LMI set (shaded gray, right) within second-order discrete-time stability region (triangle).

and it is represented on the left of Figure 1 within the stability triangle, as claimed by Theorem 3.1.

The second LMI inner approximation is

$$\mathcal{P}_4^{c,d} = \{(d_0, d_1) \; : \; P_4^{c,d} = \begin{bmatrix} 2 & \frac{4}{3}d_1 & 2d_0 & 0 \\ \frac{4}{3}d_1 & 2 & \frac{4}{3}d_1 & 2d_0 \\ 2d_0 & \frac{4}{3}d_1 & 2 & \frac{4}{3}d_1 \\ 0 & 2d_0 & \frac{4}{3}d_1 & 2 \end{bmatrix} \succ 0\},$$

see the right of Figure 1.

On the left of Figure 2 we represent the LMI set

$$\mathcal{P}_7^{c,d} = \{(d_0, d_1) \; : \; P_7^{c,d} = \begin{bmatrix} 2 & \frac{7}{6}d_1 & \frac{7}{5}d_0 & 0 & 0 & 0 & 0 \\ \frac{7}{6}d_1 & 2 & \frac{7}{6}d_1 & \frac{7}{5}d_0 & 0 & 0 & 0 \\ \frac{7}{5}d_0 & \frac{7}{6}d_1 & 2 & \frac{7}{6}d_1 & \frac{7}{5}d_0 & 0 & 0 \\ 0 & \frac{7}{5}d_0 & \frac{7}{6}d_1 & 2 & \frac{7}{6}d_1 & \frac{7}{5}d_0 & 0 \\ 0 & 0 & \frac{7}{5}d_0 & \frac{7}{6}d_1 & 2 & \frac{7}{6}d_1 & \frac{7}{5}d_0 \\ 0 & 0 & 0 & \frac{7}{5}d_0 & \frac{7}{6}d_1 & 2 & \frac{7}{6}d_1 \\ 0 & 0 & 0 & 0 & \frac{7}{5}d_0 & \frac{7}{6}d_1 & 2 \end{bmatrix} \succ 0\}.$$

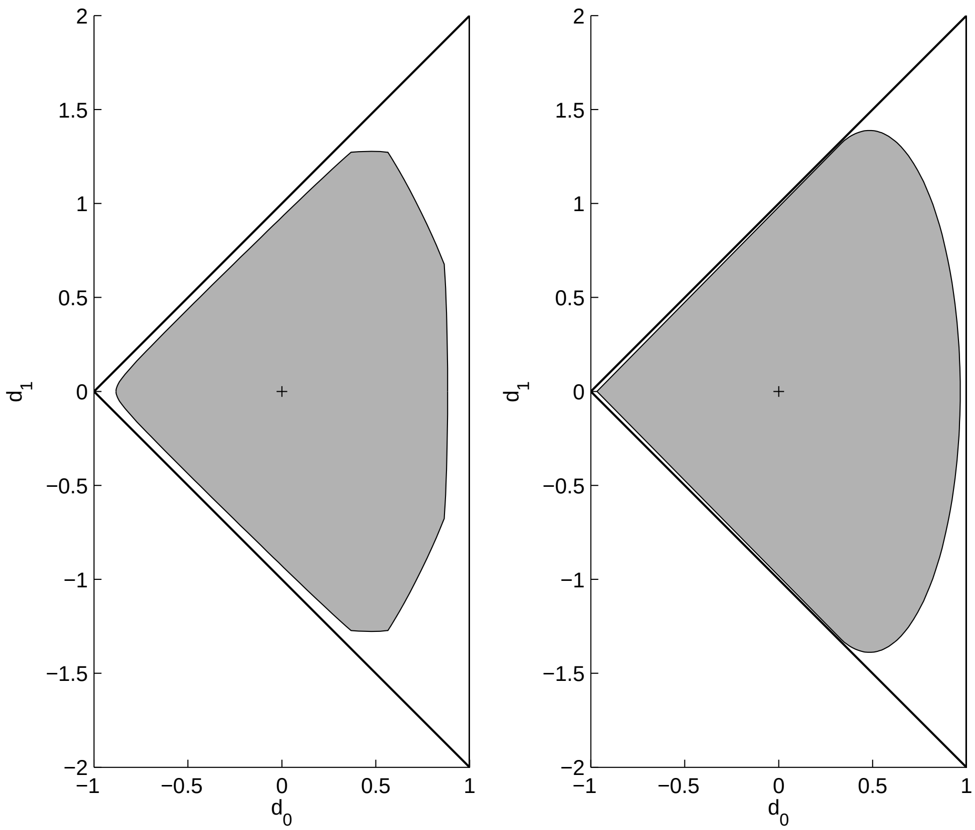

Figure 2: 7-by-7 LMI set (shaded gray, left) and 50-by-50 LMI set (shaded gray, right) within second-order discrete-time stability region (triangle).

The boundary of this set is piecewise polynomial, defined by two algebraic plane curves whose irreducible defining polynomials $-7200 + 5040d_0 + 3528d_0^2 + 4900d_1^2 - 5145d_0d_1^2$ (a cubic) and $6480000 + 4536000d_0 - 9525600d_0^2 - 8820000d_1^2 - 4445280d_0^3 + 7717500d_0d_1^2 + 3111696d_0^4 - 4321800d_0^2d_1^2 + 1500625d_1^4$ (a quartic) factor the determinant of the 7-by-7 pencil $P_7^{c,d}$. See Figure 3 for a representation of this set and the algebraic components of its boundary.

On the right of Figure 2 we represent the LMI set $\mathcal{P}_{50}^{c,d}$ which, according to Theorem 3.2, is almost equal to the lifted LMI set

$$\mathcal{P}^{c,d} = \{(d_0, d_1) \ : \ \exists\, (q_0, q_1, q_2) \ : \ \left[\begin{array}{ccc} q_0 & q_1 & d_0 \\ q_1 & q_2 - q_0 & d_1 - q_1 \\ d_0 & d_1 - q_1 & 2 - q_2 \end{array}\right] \succ 0\},$$

the projection onto $\mathbb{R}^2$ of an LMI living in $\mathbb{R}^5$, and which is the union of an ellipse and a triangle, as studied in [6].

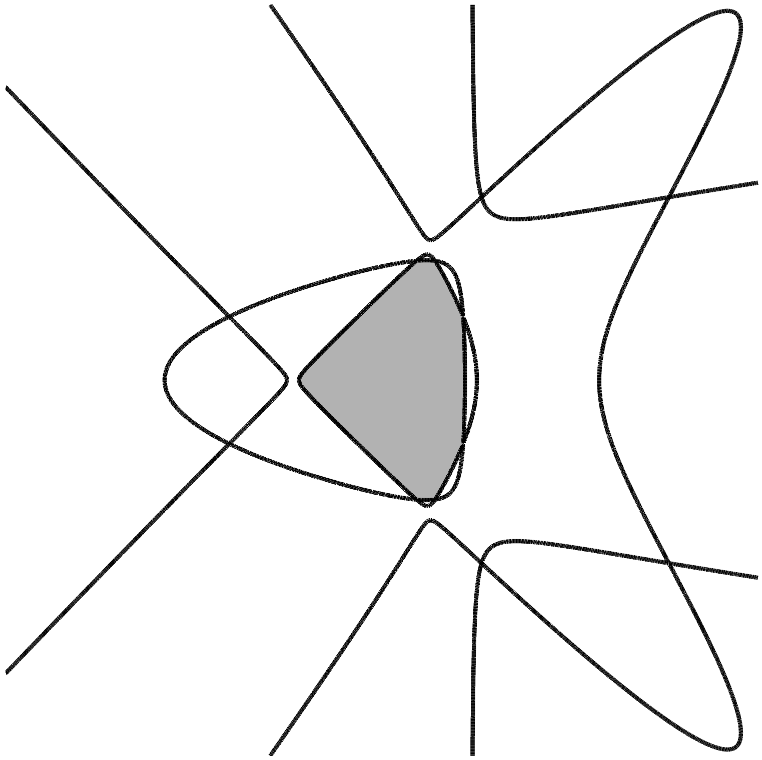

Figure 3: 7-by-7 LMI set (shaded gray) and the algebraic components of its boundary (thick black lines).

4.2 Third order

We consider third-order polynomials for which the exact stability region is delimited by a nonconvex hyperbolic parabolic embedded in a tetrahedron with vertices $(z+1)^3$, $(z+1)^2(z-1)$, $(z+1)(z-1)^2$ and $(z-1)^3$, see [1, Example 11.14].

Choosing $c(z) = z^3$, we have $p^{c,d}(\theta) = 2+2d_2\cos\theta+2d_1\cos 2\theta+2d_0\cos 3\theta$. The first LMI inner approximation is

$$\mathcal{P}_4^{c,d} = \{(d_0, d_1, d_2) \ : \ P_4^{c,d} = \begin{bmatrix} 2 & \frac{4}{3}d_2 & 2d_1 & 4d_0 \\ \frac{4}{3}d_2 & 2 & \frac{4}{3}d_2 & 2d_1 \\ 2d_1 & \frac{4}{3}d_2 & 2 & \frac{4}{3}d_2 \\ 4d_0 & 2d_1 & \frac{4}{3}d_2 & 2 \end{bmatrix} \succ 0\}$$

and it is represented on the left of Figure 4 within the nonconvex stability region, as claimed by Theorem 3.1. On the right of Figure 4 is represented the LMI set $\mathcal{P}_{50}^{c,d}$ which, according to Theorem 3.2, is almost equal to the lifted LMI set $\mathcal{P}^{c,d}$.

4.3 Robust control design

As in [6] we consider the problem of designing a robust controller for the approximate ARMAX model of a transfer function in a robotic disk grinding process. The open loop

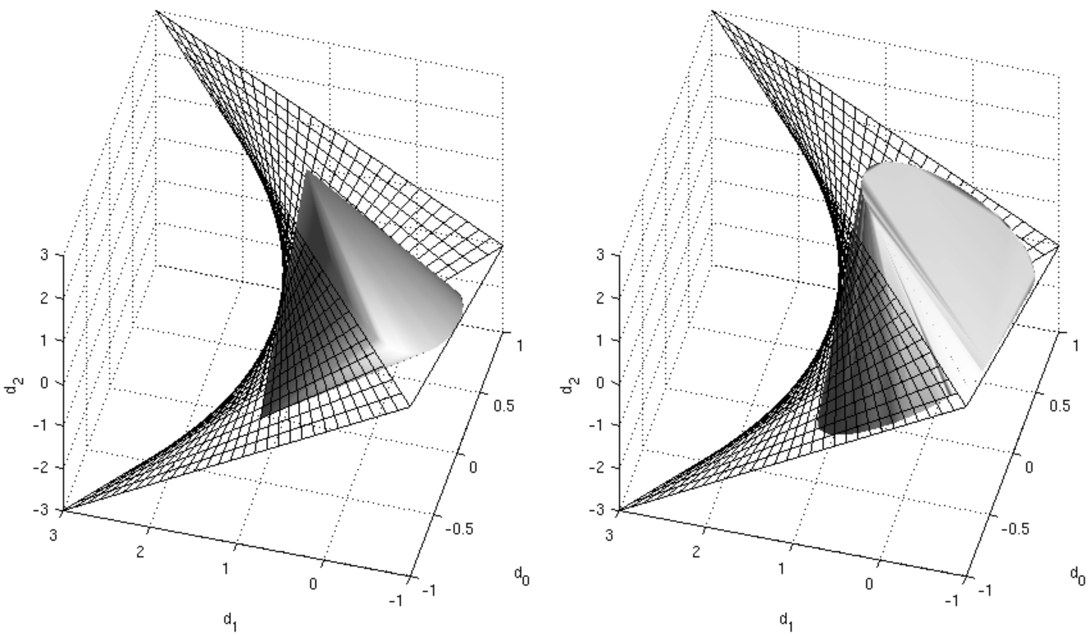

Figure 4: 4-by-4 LMI set (shaded gray, left) and 50-by-50 LMI set (shaded gray, right) within third-order discrete-time stability region (delimited by a meshed hyperbolic parabolic embedded in a tetrahedron).

plant transfer function is

$$\frac{b(z^{-1})}{a(z^{-1})} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}}{1 - 1.914z^{-1} + 1.779z^{-2} - 1.027z^{-3} + 0.2508z^{-4}}$$

and the numerator coefficients are subject to relative uncertainty

$$|b_i - \bar{b}_i| \leq \varepsilon \bar{b}_i$$

around nominal values $\bar{b}_i$, $i = 0, \ldots, 3$, with given $\varepsilon \geq 0$. These coefficients are assumed to be linearly independent, hence generating a polytopic uncertainty model with $2^4 = 16$ vertices. We want to find a p-order controller

$$\frac{y(z^{-1})}{x(z^{-1})} = \frac{y_0 + y_1 z^{-1} + \ldots + y_p z^{-p}}{1 + x_1 z^{-1} + \ldots + x_p z^{-p}}$$

with monic denominator such that the polytopic characteristic polynomial

$$d(z) = z^{12}((1 - z^{-1})a(z^{-1})x(z^{-1}) + z^{-5}b(z^{-1})y(z^{-1}))$$

remains stable (in the sense that all its roots have modulus less than one) for all possible values of the uncertainty. To design a robust controller, we can use the LMI inner approximations of the stability domain of Theorem 3.1 since d depends linearly on unknown polynomials x and y. We choose

$$c(z) = z^{12+p}$$

as a stable central polynomial. We use Matlab 7.7 on a Linux PC and the LMI problems are solved with SeDuMi 1.3.

First let $\varepsilon = 0$ (no uncertainty) and $p = 4$ (fourth-order controller). We have $n = 16$ and SeDuMi provides a (dual) certificate of emptiness of the LMI set $\mathcal{P}_m^c$ for $17 \leq m \leq 40$. For $m = 41$ the LMI set is not empty anymore, and SeDuMi finds a point yielding the controller polynomials $x(z^{-1}) = 1 + 2.344z^{-1} + 2.844z^{-2} + 2.184z^{-3} + 0.8139z^{-4}$ and $y(z^{-1}) = -3.300 + 4.822z^{-1} - 0.4120z^{-2} - 1.869z^{-3} + 0.6898z^{-4}$ that indeed stabilize the system. The maximum modulus of the closed-loop roots is equal to 0.9879.

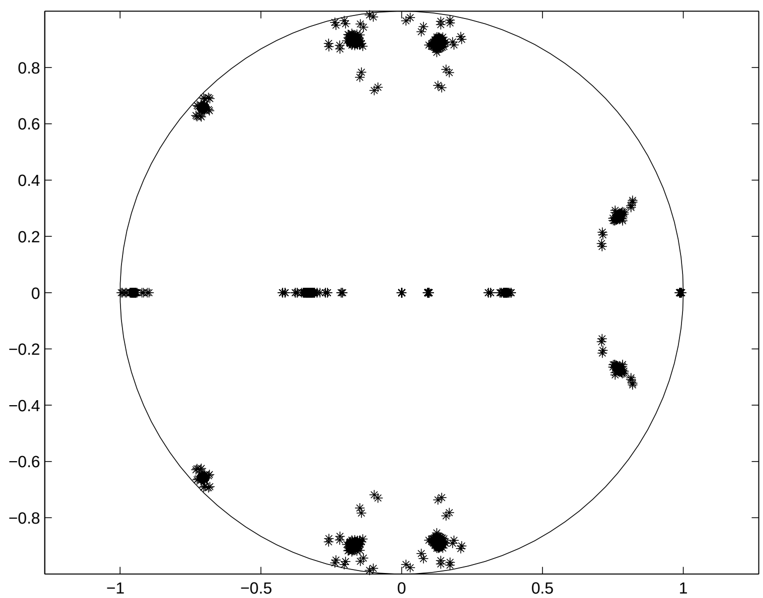

Figure 5: Robust root locus of polytopic uncertain system with a fifth-order controller.

Now let $\varepsilon = \frac{1}{2}$ (50% uncertainty) and $p = 5$ (fifth-order controller). When $m = 168$, SeDuMi solves after a few seconds the 16 LMIs of size 168-by-168 in 11 variables and finds a point in $\mathcal{P}_m^c$ corresponding to the robustly stabilizing controller polynomials $x(z^{-1}) = 1+2.691z^{-1}+4.067z^{-2}+4.067z^{-3}+2.658z^{-4}+0.9446z^{-5}$ and $y(z^{-1}) = -1.755+0.7829z^{-1}+1.033z^{-2} + 0.1742z^{-3} - 0.3218z^{-4} + 0.02688z^{-5}$. On Figure 5 we represent the root locus of 100 random closed-loop characteristic polynomials in the uncertainty polytope, and we can observe that indeed all of them are stable.

5 Concluding Remarks

We have used results on spectra of Toeplitz matrices to construct a hierarchy of convex inner approximations of the nonconvex set of stable polynomials, with potential applications in fixed-order robust controller design. The main difference with respect to previous results is that the inner sets are defined by LMIs (affine sections of the cone of positive definite matrices) without the need to resort to projections and lifting variables. Moreover, our LMI sets belong to a hierarchy converging asymptotically to a lifted LMI inner approximation described previously in [6].

It is likely that our results can be extended to deal with positive trigonometric polynomial matrices and block Toeplitz matrices, with potential applications in multi-input multi-output control systems.

Sufficient conditions ensuring that a real polynomial is a sum-of-squares (and hence that it is positive) have been proposed in [8], so it could be insightful to transpose these conditions to trigonometric polynomials and compare with our approach. Results in [8] are also valid for multivariate polynomials, and this may have applications in fixed-order or robust controller design for multi-dimensional systems.

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