Convex inner approximations of nonconvex semialgebraic sets applied to fixed-order controller design

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Abstract

We describe an elementary algorithm to build convex inner approximations of nonconvex sets. Both input and output sets are basic semialgebraic sets given as lists of defining multivariate polynomials. Even though no optimality guarantees can be given (e.g. in terms of volume maximization for bounded sets), the algorithm is designed to preserve convex boundaries as much as possible, while removing regions with concave boundaries. In particular, the algorithm leaves invariant a given convex set. The algorithm is based on Gloptipoly 3, a public-domain Matlab package solving nonconvex polynomial optimization problems with the help of convex semidefinite programming (optimization over linear matrix inequalities, or LMIs). We illustrate how the algorithm can be used to design fixed-order controllers for linear systems, following a polynomial approach.

Keywords: polynomials; nonconvex optimization; LMI; fixed-order controller design

Introduction

The set of controllers stabilizing a linear system is generally nonconvex in the parameter space, and this is an essential difficulty faced by numerical algorithms of computer-aided control system design, see e.g. [2] and references therein. It follows from the derivation of the Routh-Hurwitz stability criterion (or its discrete-time counterpart) that the set of stabilizing controllers is real basic semialgebraic, i.e. it is the intersection of sublevel sets of given multivariate polynomials. A convex inner approximation of this nonconvex semialgebraic stability region was obtained in [2] in the form of linear matrix inequalities (LMI) obtained from univariate polynomial positivity conditions. Convex inner approximations make it possible to design stabilizing controllers with the help of convex optimization techniques, at the price of loosing optimality w.r.t. closed-loop performance criteria (H_2 norm, H_∞ norm or alike).

Generally speaking, the technical literature abounds of convex *outer* approximations of nonconvex semialgebraic sets. In particular, such approximations form the basis of many branch-and-bound global optimization algorithms [5]. By construction, Lasserre's hierarchy of LMI relaxations for polynomial programming is a sequence of embedded convex inner approximations which are semidefinite representable, i.e. which are obtained by projecting affine sections

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of the convex cone of positive semidefinite matrices, at the price of introducing lifting variables [3]. In [1, Section 6] it is conjectured that all convex semialgebraic sets (and, in particular, convex hulls of nonconvex semialgebraic sets) are semidefinite representable.

After some literature search, we could not locate any systematic constructive procedure to generate convex *inner* approximations of nonconvex semialgebraic sets, contrasting sharply with the many convex outer approximations mentioned above. In the context of fixed-order controller design, inner approximations correspond to a guarantee of stability, at the price of loosing optimality. No such stability guarantee can be ensured with outer approximations.

Our main contribution is therefore an elementary algorithm, readily implementable in Matlab, that generates convex inner approximations of nonconvex sets. Both input and output sets are basic semialgebraic sets given as lists of defining multivariate polynomials. Even though no optimality guarantees can be given in terms of volume maximization for bounded sets, the algorithm is designed to preserve convex boundaries as much as possible, while removing regions with concave boundaries. In particular, the algorithm leaves invariant a given convex set. The algorithm is based on Gloptipoly 3, a public-domain Matlab package solving nonconvex polynomial optimization problems with the help of convex LMIs [4]. We illustrate how the algorithm can be used to design fixed-order controllers for linear systems, following a polynomial approach.

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