Detecting infinite zeros in polynomial matrices

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***Abstract*— A simple necessary and sufficient algebraic condition is given to detect zeros at infinity in a polynomial matrix.**

***Index Terms*— Polynomial matrices, poles and zeros, linear algebra**

I. Introduction

Polynomial matrices are used as modeling tools in linear system theory. Several control problems can be solved by manipulating polynomial matrices, as shown in [5]. A well-known example is polynomial matrix spectral factorization, which has applications in H_2 and H_∞ control, see [2]. In [1] and then later on in [3] several numerical methods are described to perform polynomial matrix spectral factorization. Computing the zeros of polynomial matrices is a key ingredient of these numerical recipes. Most of these algorithms assume that the processed polynomial matrix has no zeros at infinity. In the presence of infinite zeros, a suitable column or row-reduction pre-processing scheme must be applied [1], [3]. Column- or row-reducing a polynomial matrix ensures that it has no zeros at infinity, but the converse is not necessarily true, as shown by a counter-example in [1, Comment 2.1].

In his paper [1], after reviewing eigenstructure properties of polynomial matrices, Callier points out that "The least one can say is that it is difficult to count the McMillan degree of the zero at infinity [of a polynomial matrix]" and concludes by mentioning that "a sufficient test for the absence of zeros at infinity based on direct data of [the polynomial matrix] is most welcome".

In this note we show that early results by Van Dooren and colleagues on the Smith-MacMillan form of a rational matrix [6] can be used to derive a simple necessary and sufficient algebraic test for detecting infinite zeros in a polynomial matrix. These results were already used in a companion paper [4] to evaluate the degree of the determinant of a polynomial matrix.

II. Main result

Let

$$A(s) = A_0 + A_1 s + \cdots + A_\alpha s^\alpha \tag{1}$$

be a non-singular n-by-n polynomial matrix of degre α. We can define the zeros at infinity of $A(s)$ in several equivalent ways. The standard definition is based on the Smith-MacMillan form at infinity [6] which is defined as

$$B_1(s)A(s)B_2(s) = \begin{bmatrix} s^{-\gamma_1} & & & \\ & s^{-\gamma_2} & & \\ & & \ddots & \\ & & & s^{-\gamma_n} \end{bmatrix} \tag{2}$$

with possibly negative integers $\gamma_i \leq \gamma_{i+1}$, $i = 1, 2, \ldots, n$, and where $B_1(s)$ and $B_2(s)$ are biproper rational matrices, namely non-singular when $s \to \infty$. If γ_i is negative, we say that $A(s)$ has a pole at infinity of order γ_i. If γ_i is positive, we say that $A(s)$ has a zero at infinity of order γ_i. The sum denoted by

$$z_\infty = \sum_{i=k}^{n} \gamma_i, \tag{3}$$

with index k such that $\gamma_i > 0$ for $i \geq k$ is called the MacMillan degree at infinity, or number of zeros at infinity of $A(s)$.

Let

$$T_i = \begin{bmatrix} A_\alpha & \cdots & A_{-i+1} & A_{-i} \\ & \ddots & & \vdots \\ & & A_\alpha & A_{\alpha-1} \\ 0 & & & A_\alpha \end{bmatrix} \tag{4}$$

denote a block Toeplitz matrix built from matrix coefficients of $A(s)$, where it is assumed that $A_i = 0$ when $i < 0$ and $T_i = 0$ when $i < -\alpha$. Finally, let

$$r_i = \operatorname{rank} T_i - \operatorname{rank} T_{i-1}. \tag{5}$$

Theorem 1: Polynomial matrix $A(s)$ has no zeros at infinity if and only if $r_0 = n$.

Proof: First note that matrix T_i is built up by appending a block column to the right of matrix T_{i-1}. It follows that $\operatorname{rank} T_i = \operatorname{rank} T_{i-1} + r_i$ where r_i is the number of linearly independent columns newly introduced when passing from T_{i-1} to T_i. Note that r_i cannot exceed n, the total number of new columns. Moreover, because of the block Toeplitz structure of T_i, the sequence r_i is monotonically increasing, i.e.

$$r_{i-1} \leq r_i \leq n. \tag{6}$$

Second, using Corollary 3.6 in [6], it can be shown that there exists a finite index β for which $r_i = n$ for all $i \geq \beta$.

Third, using Corollaries 3.6 and 3.7 and Remark 4 in Section IV in [6], it can be shown that

$$z_\infty = \sum_{i=0}^{\beta} (n - r_i). \tag{7}$$

Finally, it follows that $z_\infty = 0$, i.e. there are no zeros at infinity, if and only if $r_0 = n$. ∎

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III. Numerical examples

A. First example

Let

$$A(s) = \left[\begin{array}{cc} 1 & s \\ 0 & 1 \end{array}\right]. \tag{8}$$

Note that $\det A(s) = 1$ so that $A(s)$ is a unimodular matrix with no finite zeros.

Then

$$T_{-1} = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] \tag{9}$$

and

$$T_0 = \left[\begin{array}{cc|cc} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline & & 0 & 1 \\ & & 0 & 0 \end{array}\right] \tag{10}$$

so

$$r_0 = \operatorname{rank} T_0 - \operatorname{rank} T_{-1} = 2 - 1 = 1 \neq 2 \tag{11}$$

from which it follows that $A(s)$ has zeros at infinity. Indeed, the Smith-MacMillan form at infinity of $A(s)$ has diagonal terms s and s^{-1}.

B. Second example

Let

$$A(s) = \left[\begin{array}{cc} s+1 & -(s+1)^2 \\ 0 & s+1 \end{array}\right] \tag{12}$$

be the polynomial matrix used in [1] to show that absence of infinite zeros does not imply column- or row-reducedness.

Then

$$T_{-1} = \left[\begin{array}{rr|rr} 0 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 \\ \hline & & 0 & -1 \\ & & 0 & 0 \end{array}\right] \tag{13}$$

and

$$T_0 = \left[\begin{array}{rr|rr|rr} 0 & -1 & 1 & -2 & 1 & -1 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ \hline & & 0 & -1 & 1 & -2 \\ & & 0 & 0 & 0 & 1 \\ \hline & & & & 0 & -1 \\ & & & & 0 & 0 \end{array}\right] \tag{14}$$

so that

$$r_0 = \operatorname{rank} T_0 - \operatorname{rank} T_{-1} = 4 - 2 = 2 \tag{15}$$

implying that polynomial matrix $A(s)$ has no zeros at infinity. Indeed, the Smith-MacMillan form at infinity of $A(s)$ has diagonal terms s^2 and 1.

IV. Conclusion

In this short note we described a simple algebraic condition to detect infinite zeros in a polynomial matrix. When applied to the denominator of a polynomial matrix fraction describing linear differential equations, this condition allows to dectect impulsive modes [7]. Absence of zeros at infinity in a polynomial matrix also renders superfluous the pre-processing column- or row-reduction schemes of the spectral factorization algorithm of [1], [3].

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