Measures and LMI for impulsive optimal control with applications to orbital rendezvous problems

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Abstract

This paper shows how to use semi-definite programming to find lower bounds on (and sometimes solve globally) a large class of nonlinear optimal control problems. This is done by relaxing an optimal control problem into a linear programming problem on measures, also known as a generalized moment problem. The handling of measures by their moments reduces the problem to a convergent series of standard linear matrix inequality relaxations. After providing a completely worked-through example, we apply the method to the linearized impulsive rendezvous problem between two orbiting spacecraft. As the method provides lower bounds on the global infimum, global optimality of the solutions can be guaranteed numerically by *a posteriori* simulations or by comparison with suboptimal local solutions obtained by other methods. On some problems, we can even recover simultaneously the actual impulse times and amplitudes by simple linear algebra. Finally, our approach can be readily implemented with standard software, as illustrated by numerical examples.

1 Introduction

Optimal control is still an active area of current research despite the availability of powerful theoretical tools such as Pontryagin's maximum principle or the Hamilton-Jacobi-Bellman approach. However, numerical methods based on such optimality conditions rely on a certain number of assumptions that are often not met in practice, and state constraints are particularly hard to handle in the maximum principle framework.

On the other side, many numerical methods have been developed that deliver suboptimal solutions by restricting the search space and parametrizing it. However, the users of these

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methods are often left to wonder if a better solution exists. For example, in the particular case of impulsive controls, one could assume purely impulsive solutions of at most n impulses and obtain a static optimization problem with impulse times and amplitudes as unknowns, but it is often not known if more regular solutions could provide a better cost.

For a recent survey on impulsive control see e.g. [17] and the references therein. See also [12] for a recent application and for more references[1]. For historical works see e.g. [23, 25, 28] and also [13, 5].

In this paper, we describe and test a numerical method following ideas of [18, 15], but which addresses optimal control problems whose optimal solution may now include impulsive controls. Our numerical scheme consists in solving a hierarchy of semi-definite relaxations in the form of Linear Matrix Inequalities (LMIs), whose associated sequence of optimal values provides a monotone nondecreasing sequence of lower bounds on the global minimum of affine-in-the control optimal control problems. In particular, the method may assert the global optimality of local solutions found by other methods, and as importantly, can also provide numerical certificates of infeasibility for ill-posed problems. Finally, in some cases, it is also possible to generate the globally optimal control law.

At the end of the paper, this so-called LMI method is successfully applied to three different problems of orbital rendezvous. The first example shows that the proposed algorithm is able to retrieve the impulsive optimal solution conjectured by running a direct approach based on the solution of a Linear Programming (LP) problem. The other two tackle the cases of hard state constraints and a more realistic fuel model. Without ever assuming the nature of the propulsion (continuous or impulsive), all obtained solutions can be certified to be globally optimal, or nearly so.

1.1 Contributions

The paper improves the model presented in [18, 15] in the following ways:

- The range of applications is much larger as impulsive controls can now be taken into account.
- Because controls are represented by measures and not by variables, the size of semi-definite programming (SDP) blocks composing the LMIs is significantly reduced. This allows to handle larger problems in terms of number of state variables as well as to reach higher LMI relaxations.
- Total variation constraints can be handled very easily.

Altogether, these three improvements permit to handle problems such as consumption minimization for orbital rendezvous, another significant contribution of this paper.

Let us emphasize the fact that the paper does not aim at a comprehensive mathematical treatment of impulsive control problems, in particular we do not investigate here the dual Hamilton-Jacobi-Bellman partial differential equation satisfied by the value function and

[1] We are grateful to Térence Bayen for pointing out this reference to us.

its regularity properties. These developments will be reported elsewhere. We believe that a key contribution of our work is to provide a numerical method based on standard interfaces and solvers, relying on sophisticated, albeit by now relatively standard LMI formulations for measure/moment linear programming problems. As far as we know, this is the first time that a systematic, constructive and reproducible numerical approach is proposed for such constrained control problems. In non-trivial examples it has permitted to validate some results obtained by other local optimization methods and certify that the resulting solution was globally optimal.

1.2 Notations

Integration of a function $f : \mathbb{R}^n \to \mathbb{R}$ with respect to a measure μ on a set $X \subset \mathbb{R}^n$ is written $\int_X f(x)\, d\mu(x)$ or sometimes $\int_X f(x)\, \mu(dx)$ when more convenient. The Lebesgue or uniform measure on X is denoted by λ_X whereas the Dirac measure concentrated at point x^* is denoted by δ_{x^*}. A measure μ is a probability measure on the set X whenever $\int_X d\mu = 1$. The support of measure μ on X is the largest closed set B such that $\mu(X \setminus B) = 0$, and is denoted by $\operatorname{supp} \mu$. The indicator function of set X (equal to one in X and zero outside) is denoted by I_X.

$F(X)$ is the space of Borel measurable functions on X, whereas $BV(X)$ is the space of functions of bounded variation on X. $\mathcal{M}^+(X)$ is the space of finite positive measures on X while $\mathcal{M}(X)$ is the space of finite signed measures. $\mathbb{R}[x]$ is the ring of polynomials in the variable x. $\mathcal{B}(X)$ denotes the Borel σ-algebra associated with X.

If $k \in \mathbb{N}^n$ denotes a vector of indices then x^k with $x \in \mathbb{R}^n$ is the multi-index notation for $\prod x_i^{k_i}$. The degree of the index k is $\deg k = \sum k_i$. Finally, $\mathbb{N}_d^n$ is the set of all indices for which $\deg k \leq d,\ k \in \mathbb{N}^n$.

2 The optimal control problem

This paper deals with the following nonlinear optimal control problem

$$V(x_0) = \inf_{u(t) \in F([0,T])^m} J(x_0, u) = \inf_{u(t)} \int_0^T h(t, x(t))\, dt + \int_0^T H(t)\, u(t)\, dt + \ h_T(x(T)) \qquad (1)$$

such that

$$\begin{aligned} &\dot{x}(t) = f(t, x(t)) + G(t)\, u(t), \quad \text{for a.e. } t \in [0, T] \qquad (2)\\ &x(0) = x_0, \quad x(T) \in X_T, \quad x(t) \in X \subset \mathbb{R}^n, \forall t \in [0, T], \end{aligned}$$

where the dot denotes differentiation with respect to time. Criterion $J(x_0, u)$ is *affine* in the controls u, and V is called the value function. It is assumed straight away that all problem data are polynomials, meaning that all functions are in $\mathbb{R}[t, x]$, and that all sets are compact basic semialgebraic. Recall that such sets are those which may be written as $\{x : a_i(x) \geq 0,\ i = 1, \ldots, q\}$ for some family $\{a_i\}_{j=1}^q$, $a_i \subset \mathbb{R}[x]$. A mild technical condition (implying compactness of X) must be satisfied [19, Assumption 2.1]: There exists a u in the quadratic module $Q(a)$ generated by the family $\{a_i\}$ such that the level set $\{x \in \mathbb{R}^n : u(x) \geq 0\}$ is compact. In practice, this condition is often met, and adding a standard ball constraint $\sum x_i^2 \leq r^2$ to the state constraints will enforce the condition no matter what. The reason for making these assumptions will be apparent in later sections.

Note that the hypotheses of polynomial dynamics, hence locally Lipchitz continuous on compact sets, implies uniqueness of the trajectory for a given pair (x_0, u), given this trajectory exists. This justifies the notation for $J(x_0, u)$.

Without additional assumptions, the infimum in problem (1)-(2) is not attained in general because of possible concentration (impulsive) effects [23]. This is why we embed the controls $u(t)$ in the space of the weak derivatives of functions of bounded variation $w(t)$:

$$V_R(x_0) = \inf_{w(t) \in BV([0,T])^m} J_R(x_0, w) = \inf_{w(t)} \int_0^T h(t, x(t))\, dt + \int_0^T H(t)\, dw(t) + h_T(x(T)) \quad (3)$$

such that

$$\begin{aligned} &dx(t) = f(t, x(t))\, dt + G(t)\, dw(t), \quad t \in [0, T] \\ &x(0) = x_0, \quad x(T) \in X_T, \quad x(t) \in X \subset \mathbb{R}^n, \end{aligned} \quad (4)$$

where V_R stands for the relaxed value function. In particular, controls in problem (3)-(4) may now be impulsive: The controls can be seen as a distribution of the first order and it is therefore the (vector) distributional derivative $dw(t)$ of some (vector) function of bounded variation $w(t) \in BV([0, T])^m$, see e.g. [28] and [24, §4] or also [7, Prop. 8.3].

Note that in the state-constrained case as treated in this paper, there might be a strict gap between V and V_R as the next example, inspired by one in [27], demonstrates:

Example 1 (Relaxation gap)**.** Consider the following dynamics:

$$\begin{aligned} \dot{x} =& (t - \frac{1}{2})^2\, u \\ \dot{y} =& u \end{aligned} \quad (5)$$

with constraints $x(t) = 0$, $y(0) = 0$, $y(1) = 1$ and $u(t) \geq 0$. Its relaxed version reads:

$$\begin{aligned} dx =& (t - \frac{1}{2})^2\, dw(t) \\ dy =& dw(t) \end{aligned} \quad (6)$$

with $w(t)$ a non-decreasing function of bounded variation. It is pretty clear that all the control action needed to steer y from 0 to 1 needs to be applied at $t = \frac{1}{2}$ if $x(t)$ is to remain identically zero. Therefore, the only admissible relaxed control is the weak derivative of the unit step function jumping at $t = \frac{1}{2}$. However, any function approximating this relaxed control, no matter how close (one could take very large values on a small time interval around $t = \frac{1}{2}$), will slightly violate the constraint $x(t) = 0$ and is therefore not admissible. For any optimal control problem where the cost associated to relaxed dynamics (6) is finite, an infinite relaxation gap exists (where a cost of $V = \infty$ is assigned to infeasible problems).

Existence of such a relaxation gap is not considered in this paper as we directly concentrate on the relaxed optimal control problem (3)-(4). This is however of minor inconvenience, because the method described in this paper relaxes problem (3)-(4) to yield a tractable one, and because it takes very degenerate dynamics and constraints for this gap to occur.

3 The problem on measures

In this section, we formulate problem (3)-(4) as an equivalent infinite-dimensional linear programming problem on measures, a particular instance of the so-called *generalized moment problem* (see [19] for an introduction on the subject). This is a necessary intermediate step towards obtaining a tractable SDP problem for our method, but measure formulations arise quite naturally in optimal control. In fact, optimal control problems involving measures have been introduced to accept solutions that are ruled out or ill-defined in classical optimal control, see e.g. [31]. Multiple solutions, impulsive or chattering controls can be handled naturally by the associated problem on measures. This section outlines the main ideas behind this transformation.

First of all, a few remarks are worth pointing out before further developments:

- It is crucial that $G(t)$ be a matrix of smooth functions, an hypothesis automatically fulfilled by polynomials. As a matter of fact, multiplying distributions with such functions is a well-defined operation (unlike e.g. the product of two distributions). Therefore, except for some very particular cases [21], G cannot be a function of states x_j that could potentially present jump discontinuities. To simplify notations, we have simply assumed that G depends on t only[2].

- In the absence of impulses, the distributional derivative is the traditional one, and the dynamics are classical differential equations with controls $dw(t) = u(t)\,dt$ which generate absolutely continuous trajectories with respect to time.

- State trajectories $x(t)$ are themselves functions of bounded variations, and this is their broadest class in the sense of distributions [28], as trajectories could not be the integral of distributions of higher order than the Dirac distribution.

- Because distributional derivatives of functions of bounded variation on compact supports may be identified with measures [24, §50], the dynamics (4) could be interpreted as a measure differential equation. In fact, it is quite natural to write

$$dx = f(t, x)\,dt + G(t)\,w(dt) \tag{7}$$

  for dynamics (4), see [30] for example, with measure $w(dt)$ defined as $w(A) = \int_A dw(t)$, $\forall A \in \mathcal{B}([0, T])$. This justifies the slight abuse of notation that defines the measure $w(dt)$ from the function of bounded variation $w(t)$.

[2]Rigorously, it could be possible to include state jumps in G, but this requires a careful definition of what is meant by integration, as done e.g. for studying stochastic differential equations. This goes well beyond the scope of this paper.

- Those measures-as-control should also not be confused with generalized controls, where controls are relaxed as *measured variables* of a (Young) measure (see for instance [13]). With those measures, fast oscillating behavior (chattering) can be captured. In the case of original dynamics (2), these Young relaxations are unnecessary because the set of available vector fields is convex due to the affine dependence on control and the convexity of the control set.

Dynamics (7) already encompass measures on time as controls, but trajectory $x(t)$ for instance still appears as a function of bounded variation. We will therefore introduce new measures, so called *occupation measures*, to frame all elements of dynamical constraint (7) into constraints on measures. Those measures will measure subsets of time and space, and as $X \subset \mathbb{R}^n$ is compact by hypothesis, these measures can be put in duality correspondence with all continuous functions $v(t, x(t))$ supported on $[0, T] \times X$ by one of the Riesz representation theorems [16, §36.6]. We will use these test functions precisely to define those linear relations between the measures. Note that because continuous functions on compact sets can be uniformly approached by polynomials by virtue of the Stone-Weierstrass theorem, it is enough to consider polynomial test functions $v(t, x(t)) \in \mathbb{R}[[0, T] \times X]$.

By Lebesgue's decomposition theorem [16, §33.3], the control measures $w(dt)$ may be split into two parts: Their absolutely continuous (with respect to the Lebesgue measure on $[0, T]$) parts $w^{AC}(dt) = u(t)\,dt$ with density $u : [0, T] \to \mathbb{R}^m$, and their purely singular parts $w^S(dt) = \sum_{j\in J} u_{t_j}\, \delta_{t_j}(dt)$ with *jump amplitude* vectors $u_{t_j} \in \mathbb{R}^m$ supported at impulsive *jump times* t_j, $j \in J$, with J a subset of Lebesgue measure zero of $[0, T]$, not necessarily countable. We conjecture that for the control problems studied in this paper, subset J can be assumed countable without loss of generality. Indeed, under reasonable assumptions, it was shown in [1] that Cantor sets cannot be sets of switching points for optimal control laws[3]. However, a careful treatment of this question is beyond the scope of this paper.

We write

$$w(dt) = u(t)\,dt + \sum_{j\in J} u_{t_j}\, \delta_{t_j}(dt)$$

where jump end-points are linked by the relationship

$$x^+(t_j) = x^-(t_j) + G(t_j) u_{t_j}, \quad \forall\, j \in J.$$

Obviously, $x^+(t) = x^-(t)$ on continuous arcs.

Now, *given an initial state* x_0 and *given an admissible relaxed control* $w(t) \in BV([0, T])^m$, denote by $x(t) \in BV([0, T])^n$ the corresponding feasible trajectory. Then, for smooth test functions $v : [0, T] \times \mathbb{R}^n \to \mathbb{R}$, it holds that

$$\begin{aligned} \int_0^T dv(t, x(t)) &= v(T, x^+(T)) - v(0, x^-(0)) \\ &= \int_0^T \left(\frac{\partial v}{\partial t} + \left(\frac{\partial v}{\partial x}\right)' (f + G\,u) \right) dt + \sum_{j\in J} v(t_j, x^+(t_j)) - v(t_j, x^-(t_j)) \end{aligned} \tag{8}$$

[3]We are grateful to Emmanuel Trélat for pointing out this reference to us.

where the prime denotes matrix transposition. We are going to express the above *temporal* integration (8) along the trajectory in terms of *spatial* integration with respect to appropriate *occupation measures*.

For this purpose, the *time-state occupation measure*, measuring the occupation of $A \times B$ by the pair $(t, x(t))$ all along the trajectory, is defined as:

$$\mu[x(0), w(t)](A \times B) = \int_A \xi(B \,|\, t)\, dt$$

$\forall A \in \mathcal{B}([0,T])$, $\forall B \in \mathcal{B}(X)$. In particular, $\mu([0,T], B)$ is the time spent by the trajectory $x(t)$ in the set B (whence the name "occupation measure"). Note that we write $\mu[x(0), w(t)]$ to emphasize the dependence of μ on initial state $x(0)$ and relaxed control $w(t)$. However, for notational simplicity, we may use the notation μ. The stochastic kernel ξ, also known as conditional measure, is defined as:

$$\xi(B|t) = \begin{cases} I_B(x(t)) \quad \left(= \int\limits_B \delta_{x(t)}(dx)\right), & \forall t \in [0,T] \setminus J \\ \frac{\lambda([x^-(t_j), x^+(t_j)] \cap B)}{\lambda([x^-(t_j), x^+(t_j)])}, & \forall t_j \in J. \end{cases} \tag{9}$$

That is, $\xi(\cdot \,|\, t)$ is the Dirac measure concentrated at state $x(t)$ along continuous trajectory arcs, while during jumps, it is uniformly distributed along the segment linking the state before and after the jump, hence the Lebesgue measure defined on a line segment in X (not to be confused with the Lebesgue measure on the n-dimensional space X). The above denominator ensures that $\xi(\cdot \,|\, t)$ has unit mass for all t and therefore remains a probability measure during jumps.

The *control-state occupation measure* can be defined in a similar fashion as:

$$\omega[x_0, w(t)](A \times B) \;=\; \int_A \xi(B \,|\, t)\, dw(t)$$

$\forall A \in \mathcal{B}([0,T])$ and $\forall B \in \mathcal{B}(X)$, and with $\xi(\cdot \,|\, t)$ defined as in Eq. (9).

Finally, the *final state occupation measure* is defined as

$$\mu_T[x_0, w(t)](B) = I_B((x(T)), \qquad \forall B \in \mathcal{B}(X). \tag{10}$$

These definitions lead to the central theorem of this paper:

Theorem 1. If $w(t)$ is an admissible relaxed control for a trajectory starting at x_0 and satisfying relaxed dynamics (4) on $[0,T] \times X$, then its corresponding occupation measure $\mu[x_0, w(t)]$, final state measure $\mu_T[x_0, w(t)]$ and control measure $\omega[x_0, w(t)]$ satisfy the linear equation

$$\int\limits_{X_T} v(T,x)\, d\mu_T(x) - v(0,x_0) = \int\limits_{[0,T]\times X} \left(\frac{\partial v}{\partial t} + \left(\frac{\partial v}{\partial x}\right)' f(t,x) \right) d\mu(t,x) + \int\limits_{[0,T]\times X} \left(\frac{\partial v}{\partial x}\right)' G(t)\, d\omega(t,x) \tag{11}$$

for all continuously differentiable test functions $v(t,x)$ on $[0,T] \times X$.

Proof. We first work out the first line of relation (8) to obtain the left-hand side of relation (11):

$$\int_{X_T} v(T,x)\,d\mu_T(x) - v(0,x_0) = \int_{X_T} v(T,x)\,\delta_{x^+(T)}(dx) - v(0,x_0)$$
$$= v(T,x^+(T)) - v(0,x^-(0)).$$

We proceed similarly with the second line of relation (8) to obtain the right-hand side of relation (11):

$$\int_{[0,T]\times X} \left(\frac{\partial v}{\partial t} + \left(\frac{\partial v}{\partial x}\right)' f(t,x)\right) d\mu(t,x) + \int_{[0,T]\times X} \left(\frac{\partial v}{\partial x}\right)' G(t)\,d\omega(t,x)$$
$$= \int_0^T \int_X \left(\frac{\partial v}{\partial t} + \left(\frac{\partial v}{\partial x}\right)' f(t,x)\right) \xi(dx|t)\,dt + \int_0^T \int_X \left(\frac{\partial v}{\partial x}\right)' G(t)\,\xi(dx|t)\,dw(t)$$
$$= \int_0^T \int_X \left(\frac{\partial v}{\partial t} + \left(\frac{\partial v}{\partial x}\right)' f(t,x)\right) \xi(dx|t)\,dt + \int_0^T \int_X \left(\frac{\partial v}{\partial x}\right)' G(t)\,\xi(dx|t)\,(u(t)\,dt + dw^S(t))$$
$$= \int_0^T \int_X \left(\frac{\partial v}{\partial t} + \left(\frac{\partial v}{\partial x}\right)' (f(t,x) + G(t)\,u(t))\right) \xi(dx|t) + \sum_{j\in J} \int_X \left(\frac{\partial v}{\partial x}\right)' G(t_j)\,\xi(dx|t_j)$$
$$= \int_0^T \left(\frac{\partial v}{\partial t} + \left(\frac{\partial v}{\partial x}\right)' (f(t,x(t)) + G(t)\,u)\right) dt + \sum_{j\in J} v(t_j,x^+(t_j)) - v(t_j,x^-(t_j)).$$

This proves that relation (8) implies (11). □

The following obvious lemma readily provides a characterization of infeasible impulsive optimal control problems:

Lemma 1. If no triplet of arbitrary finite measures $(\mu,\mu_T,\omega) \in (\mathcal{M}^+,\mathcal{M}^+,\mathcal{M})$ satisfies (11), neither original problem (1)-(2) or relaxed problem (3)-(4) are feasible.

Turning now to feasible problems, the standard measure relaxation that is the topic of this section consists in enlarging the search among all triplets of measure satisfying (11), instead of only considering the occupation measures as defined above:

Lemma 2 (Measure relaxation)**.** Consider the minimization problem on arbitrary finite measures:

$$V_M(x_0) = \inf_{\mu,\mu_T\in\mathcal{M}^+,\ \omega\in\mathcal{M}} J_M(x_0,\mu,\mu_T,\omega) = \inf_{\mu,\mu_T,\omega} \int_{[0,T]\times X} h\,d\mu + \int_{[0,T]\times X} H\,d\omega + \int_{X_T} h_T\,d\mu_T \qquad (12)$$

under constraints (11). Then $V(x_0) \geq V_R(x_0) \geq V_M(x_0)$.

Proof. First, by construction, problem (3)-(4) is a relaxation of problem (1)-(2), since we optimize over derivatives of functions of bounded variations, i.e. measures, instead of measurable functions. This proves the first inequality.

Secondly, by Theorem 1, every admissible trajectory for problem (12) generates an occupation and control measure satisfying (11). This proves the second inequality. □

It should be noted that for a well-defined control problem (3)-(4), one expects that in fact $V_M(x_0) = V_R(x_0)$ and that an optimal solution of the relaxed problem will be a triplet of occupation measures corresponding to an optimal trajectory of relaxed problem (3)-(4) with given initial state x_0 and relaxed control $w(t)$. However, this will be proved in subsequent works. Note that for the standard polynomial optimal control problem (1)-(2), without impulsive controls, and under additional convexity assumptions, it has been proved in [18] that indeed $V_M(x_0) = V_R(x_0) = V(x_0)$. See also [11].

We provide now a few extensions of our canonical problem that are easily captured by the occupation measure formalism.

3.1 Free initial state

We consider now the case where x_0 is itself a decision variable of our optimization problem instead of being given, taking its values in the compact set X_0. This initial state incurs an additional initial cost of $h_0(x_0)$ to the total cost. We introduce the *initial state* occupation measure μ_0 as

$$\mu_0[x(0), w(t)](B) = I_B\left(x(0)\right), \qquad \forall B \in \mathcal{B}(X)$$

which is obviously the analogue of the final-state occupation measure μ_T. Measure μ_0 can be seen as an *unknown* probability measure on X_0.

Then, the relaxed problem with measures, very similar to Eq. (12), reads :

$$V_M = \inf_{\mu,\mu_0,\mu_T \in \mathcal{M}^+,\ \omega \in \mathcal{M}} \int\limits_{[0,T]\times X} h\, d\mu + \int\limits_{[0,T]\times X} H\, d\omega + \int\limits_{X_0} h_0\, d\mu_0 \int\limits_{X_T} h_T\, d\mu_T \tag{13}$$

such that

$$\int\limits_{X_T} v(T,x)\, d\mu_T - \int\limits_{X_0} v(0,x)\, d\mu_0 = \int\limits_{[0,T]\times X} \left(\frac{\partial v}{\partial t} + \left(\frac{\partial v}{\partial x}\right)' f\right) d\mu + \int\limits_{[0,T]\times X} \left(\frac{\partial v}{\partial x}\right)' G\, d\omega$$

$$\int\limits_{X_0} d\mu_0 = 1.$$

The last constraint is necessary to impose that μ_0 and μ_T are probability measures on X_0 and X_T respectively. In constraints (11) for problem (12), this was implicitly imposed on μ_T by taking test function $v(t,x) = 1$.

3.2 Initial state with a given distribution

We can even go a step further than a free initial state and impose a given probability distribution on the initial state. Recall that the occupation measures defined in the previous section all depend on x_0. Observe that if μ_0 is now a given probability measure on $X_0 \subset \mathbb{R}^n$ and if one defines:

$$\begin{array}{rcl} \mu(A \times B) & = & \int_{X_0} \mu[x_0](A \times B)\, d\mu_0(x_0), \\ \omega(A \times B) & = & \int_{X_0} \omega[x_0](A \times B)\, d\mu_0(x_0), \\ \mu_T(B) & = & \int_{X_0} \mu_T[x_0](B)\, d\mu_0(x_0) \end{array}$$

for all $A \in \mathcal{B}([0,T])$ and $B \in \mathcal{B}(X)$, then

$$J(\mu[\mu_0], \nu[\mu_0], \mu_T[\mu_0]) = \int_{X_0} J(\mu[x_0], \nu[x_0], \mu_T[x_0]) d\mu_0(x_0)$$

becomes the expected average cost associated with the trajectories and with respect to the probability measure μ_0 on X_0.

Therefore, the relaxed problem with measures is nearly identical to (13) except that μ_0 is given instead of being an optimization variable:

$$V_M(\mu_0) = \inf_{\mu, \mu_T \in \mathcal{M}^+,\, \omega \in \mathcal{M}} \int\limits_{[0,T]\times X} h\, d\mu + \int\limits_{[0,T]\times X} H\, d\omega + \int\limits_{X_0} h_0\, d\mu_0 \int\limits_{X_T} h_T\, d\mu_T \qquad (14)$$

under the usual dynamic constraints.

Note that in this case, the stochastic kernel $\xi(dx|t)$ along continuous arcs of the trajectory is generally not a Dirac measure as in (9), unless μ_0 is a Dirac measure supported at x_0 and the optimal control w is unique.

3.3 Decomposition of control measures and handling of total variation

All measures in problem (12) are positive measures, except for the signed measures ω which deserve special treatment. In fact, in the next section and in the rest of the paper, only positive measures will be considered. A standard way to proceed then is to split those measures into a positive part ω^+ and negative part ω^-, that is $\omega = \omega^+ - \omega^-$, both components being positive measures. This way of proceeding is equivalent to substituting $u(t) = u^+(t) - u^-(t)$ in original problem (1)-(2), with $u^+(t)$ and $u^-(t)$ both positive functions, then relaxing with positive measures only. Note that those measures have identical support, and this decomposition is thus not unique; Adding an arbitrary finite positive measure ν to both ω^+ and ω^- yields the same ω. Despite this uniqueness issue, this standard lifting technique does not add any extra element of relaxation for original problem (1)-(2) which is fully linear in $u(t)$.

One obvious advantage of this decomposition is that problems of minimization of the $\mathcal{L}_1$ norm of the original controls $u(t)$, such as those taken from the orbital rendezvous literature and presented in section 7, can now be handled. Indeed, consider the new problem

$$V = \inf_{u(t)} \sum_i \int_0^T |u_i(t)|\, dt.$$

under dynamical constraints (2). Its relaxed companion involves the total variation of the relaxed controls $w(t)$

$$V_R = \inf_{w(t)} \sum_i \int_0^T d|w_i(t)|$$

whose control occupation measure version reads

$$V_M = \inf_{\omega} \sum_i \int_{[0,T]\times X} d|\omega_i| = \inf_{\omega} \sum_i \int_{[0,T]\times X} d\left(\omega_i^+ + \omega_i^-\right).$$

In this case, there will be no problem arising for the non-unique measure decomposition because we are minimizing the sum of masses of positive measures. Hence, the solutions ω^+ and ω^- will naturally tend to the standard Hahn-Jordan decomposition of measure ω, which is unique except on sets of null Lebesgue measure (see [4]).

It could then be appealing to perform the substitution $u(t) = u^+(t) - u^-(t)$ and $|u(t)| = u^+(t) + u^-(t)$ for any linear control problem in both u and $|u|$, in order to obtain a formulation of the problem compatible with the LMI method. However, for some other cost functional or different dynamics, this decomposition may lead to a gap between the original problem and its relaxation, as shown in the following example.

Example 2. Consider the problem

$$\inf_{u(t)} \int_0^1 -|u(t)|\, dt$$

such that

$$\begin{aligned} &\dot{x}(t) = u(t) \\ &x(t) = 0, \quad u(t) \in \mathbb{R} \quad \forall t \in [0,1] \end{aligned}$$

whose cost is 0 since $u(t) = 0$ is the only control admissible with respect to the state constraint. Performing now the decomposition of $u(t)$, one obtains the new problem

$$\inf_{u^+(t),u^-(t)} \int_0^1 -u^+(t) - u^-(t)\, dt$$

such that

$$\begin{aligned} &\dot{x}(t) = u^+(t) - u^-(t) \\ &x(t) = 0, \quad u^+(t), u^-(t) \in \mathbb{R}^+ \quad \forall t \in [0,1]. \end{aligned}$$

The cost of this problem can be made arbitrarily small; With M an arbitrary positive constant, admissible controls $u^+(t) = u^-(t) = M$ generate a cost of $-2M$. An infinite relaxation gap thus exists between the two problems, caused by the non convexity of the cost in u and the "thin" state constraint, which prevents any admissible velocity vector to point strictly "inside" the allowed state-space region. In fact, cast in the formalism of differential inclusions, the substitution $|u(t)| = u^+(t) + u^-(t)$ is equivalent to convexifying dynamics and cost, and the impossibility of approaching convexified differential inclusions by non-convexified ones on closed domains is a necessary condition for the existence of a relaxation gap. See [10] for instance for additional conditions insuring the equivalence between convexified and non-convexified differential inclusions.

3.4 Summary

To summarize, the advantages for introducing the relaxed control problem (12) with measures are the following:

- controls are allowed to be measures with absolutely continuous components and singular components including impulses;
- state constraints are handled abstractly via support constraints;
- the initial state may have a fixed given distribution on some specified domain;
- a free initial state in some specified domain could also be allowed.

4 The associated moment problem

So far, the hypothesis of polynomial data has not been used, but this crucial assumption is necessary for this section, where measures are manipulated through their moments. This leads to a semi-definite programming (SDP) problem with countably many linear constraints.

Define the moments of a measure $\mu(dz)$ on $Z \subset \mathbb{R}^n$ as

$$y_k^\mu = \int_Z z^k \, d\mu(z) := \int_Z z_1^{k_1} \cdots z_n^{k_n} \, d\mu(z). \tag{15}$$

Then, with a sequence $y = (y_k)$, $k \in \mathbb{N}^n$, let $L_y : \mathbb{R}[z] \to \mathbb{R}$ be the (Riesz) linear functional

$$f \left(= \sum_k f_k z^k \right) \quad \mapsto \quad L_y(f) \; = \; \sum_k f_k y_k, \quad f \in \mathbb{R}[z].$$

Define the moment matrix of order $d \in \mathbb{N}$ associated with y as the real symmetric matrix $M_d\,(y)$ whose (i,j)th entry reads

$$M_d(y)[i,j] = L_y\left(z^{i+j}\right) = y_{i+j}, \quad \forall i,j \in \mathbb{N}_d^n.$$

Similarly, define the localizing matrix of order d associated with y and $h \in \mathbb{R}[z]$ as the real symmetric matrix $M_d(h\,y)$ whose (i,j)th entry reads

$$M_d(h\,y)[i,j] = L_y\left(h(z)\,z^{i+j}\right) = \sum_k h_k\,y_{i+j+k}, \quad \forall i,j \in \mathbb{N}^n_d.$$

As a last definition, a sequence $y^\mu = (y^\mu_k)$ is said to have a representing measure if there exists a finite Borel measure μ on X, such that relation (15) holds for every $k \in \mathbb{N}^n$.

Now comes the crucial result of the section which uses the SDP formulation characterizing sequences of moments having a representing measure supported on a given semi-algebraic set. Consider the infinite-dimensional SDP problem

$$\begin{array}{rll} V_M^\infty = & \inf\limits_y & (b^\mu)'y^\mu + (b^\omega)'y^\omega + (b^{\mu_T})'y^{\mu_T} = b'y \\ & \text{s.t.} & A^\mu y^\mu + A^\omega y^\omega + A^{\mu_T} y^{\mu_T} = Ay = c \\ & & M_\infty(y^\mu) \succeq 0, \quad M_\infty(a_i^\mu\, y^\mu) \succeq 0, \\ & & M_\infty(y^\omega) \succeq 0, \quad M_\infty(a_i^\omega\, y^\omega) \succeq 0, \\ & & M_\infty(y^{\mu_T}) \succeq 0, \quad M_\infty(a_i^{\mu_T}\, y^{\mu_T}) \succeq 0, \end{array} \tag{16}$$

where the operator M_d^∞ could be seen as a matrix of infinite size by an abuse of notation. This SDP problem is obtained as follows. First, since all problem data were assumed to be polynomial, criterion (12) is transformed into a linear combination of moments to be minimized, where the infimum is now over the aggregated sequence y of moments of all the measures. Second, because the test functions were also restricted to be polynomials, dynamic constraints (11) can be turned into countably many linear constraints on the moments, denoted by $Ay = c$. Finally, the only nonlinear part are the convex SDP constraints for measure representativeness, to be satisfied $\forall d \in \mathbb{N}$. Indeed, it follows from [19, Theorem 3.8] that a sequence of moments y^μ has a representing measure defined on a semi-algebraic set $X^\mu = \{x \,:\, a_i^\mu(x) \geq 0,\ i = 1, 2, \ldots\}$ if and only if $M_d(y^\mu) \succeq 0,\ \forall\, d \in \mathbb{N}$ and $M_d(a_i^\mu\, y^\mu) \succeq 0,\ \forall\, d \in \mathbb{N}$ and $\forall a_i^\mu$ defining set X^μ. We have proved the following result:

Theorem 2. The relaxed problem on measures (12) and the infinite-dimensional SDP moment problem (16) share the same optimum:

$$V_M = V_M^\infty.$$

For the rest of the paper, we will therefore use V_M to denote the cost of the Measure LP problem (12) or Moment SDP problem (16) indifferently.

5 LMI relaxations

The final step to reach a tractable problem is relatively obvious: We simply truncate the problem to its first few moments. Let $d_0 \in \mathbb{N}$ be the smallest integer such that all criterion, dynamics and constraint monomials belong to $\mathbb{N}^{n+1}_{2d_0}$. This is the degree of the so called *first relaxation*. For each relaxation, we get a standard LMI problem that can be solved numerically by off-the-shelf software by simply truncating in problem (16) to involve only moments in $\mathbb{N}^{n+1}_{2d}$, with $d \geq d_1$ the relaxation order.

Theorem 3. Let us denote by $V_M^{d_0+i}$ the optimum obtained by solving the finite-dimensional truncation to moments of degree up to $2(d_0+i)$ of SDP problem (16), for $i = 0, 1, \ldots$ Then

$$V_M^{d_0} \leq V_M^{d_0+1} \leq \cdots V_M^{\infty} = V_M.$$

Proof. By construction, observe that $j > i \Rightarrow V_M^{d_0+j} \geq V_M^{d_0+i}$, i.e. the sequence $V_M^{d_0+i}$ is monotonically non-decreasing. Asymptotic convergence to V_M follows from [19, Theorem 3.8] as in the proof of Theorem 2. □

Therefore, by solving the truncated problem for ever greater relaxation orders, we will obtain a monotonically non-decreasing sequence of lower bounds to the true optimal cost. In the examples below, we will see that in practice, the optimal cost is usually reached after a few relaxations.

6 Academic example

In this section, a basic example is worked out to illustrate how the method works. The example uses GloptiPoly [14] for building the truncated LMI moment problems and SeDuMi [29] for its numerical solution, as with all other numerical examples in this paper.

Before proceeding, define the marginal $M_d(y, z)$ of a moment matrix with respect to variable z as the moment matrix of the subsequence of moments concerning polynomials of z only.

Example 3 (Basic impulsive problem)**.** Consider

$$V = \inf_{u(t)} \int_0^2 x^2(t)\, dt$$

such that

$$\begin{aligned} &\dot{x}(t) = u(t), \\ &x(0) = 1, \quad x(2) = \frac{1}{2}, \\ &x^2(t) \leq 1. \end{aligned}$$

In this introductory example, it is straightforward to notice that the optimal solution consists in reaching the turnpike $x(t) = 0$ by an impulse at initial time $t = 0$, and likewise, departing from it by an impulse at final time $t = T = 2$.

The associated problem on measure reads:

$$V_M = \inf_{\mu, \omega^+, \omega^- \in \mathcal{M}^+} \int_{[0,T]\times X} x^2\, d\mu$$

such that

$$
\begin{aligned}
v(T, x_T) - v(0, x_0) &= \int\limits_{[0,T]\times X} \frac{\partial v}{\partial t}\, d\mu + \int\limits_{[0,T]\times X} \frac{\partial v}{\partial x}\, d\left(\omega^+ - \omega^-\right), \qquad \forall\, v \in \mathbb{R}[t, x] \\
x_0 &= 1, \quad x_T = \frac{1}{2}, \quad T = 2, \\
X &= \left\{x \in \mathbb{R} : 1 - x^2 \geq 0\right\}.
\end{aligned}
$$

Using the procedure outlined above, one obtains a series of truncated moment problems that can be solved by semi-definite programming. Letting $y^{\mu}_{ij} = \int t^i x^j \, d\mu$, the first LMI relaxation is

$$
V_M^1 = \inf_y y^{\mu}_{02}
$$

subject to the linear constraints associated to the dynamics:

$$
\begin{aligned}
2 - 0 &= y^{\mu}_{00} & [v(t, x) = t] \\
\frac{1}{2} - 1 &= y^{\omega^+}_{00} - y^{\omega^-}_{00} & [v(t, x) = x] \\
4 - 0 &= 2y^{\mu}_{10} & [v(t, x) = t^2] \\
1 - 0 &= y^{\mu}_{01} + y^{\omega^+}_{10} - y^{\omega^-}_{10} & [v(t, x) = tx] \\
\frac{1}{4} - 1 &= 2y^{\omega^+}_{01} - 2y^{\omega^-}_{01} & [v(t, x) = x^2] \\
8 - 0 &= 3y^{\mu}_{20} & [v(t, x) = t^3] \\
2 - 0 &= 2y^{\mu}_{11} + y^{\omega^+}_{20} - y^{\omega^-}_{20} & [v(t, x) = t^2 x] \\
\frac{1}{2} - 0 &= y^{\mu}_{02} + 2y^{\omega^+}_{11} - 2y^{\omega^-}_{11} & [v(t, x) = tx^2] \\
\frac{1}{8} - 1 &= 3y^{\omega^+}_{02} - 3y^{\omega^-}_{02} & [v(t, x) = x^3]
\end{aligned}
$$

and to the moment SDP constraints that hold for $\tau = \{\mu, \omega^+, \omega^-\}$:

$$
\begin{bmatrix} y^{\tau}_{00} & y^{\tau}_{10} & y^{\tau}_{01} \\ y^{\tau}_{10} & y^{\tau}_{20} & y^{\tau}_{11} \\ y^{\tau}_{01} & y^{\tau}_{11} & y^{\tau}_{02} \end{bmatrix} \succeq 0, \qquad y^{\tau}_{00} - y^{\tau}_{02} \geq 0.
$$

It turns out that the optimal value $V_M = 0$ is estimated correctly (within numerical tolerance) from the first relaxation on and that the optimal trajectory $x(t) = 0$ can easily be recovered. Indeed, the marginal $M_d\left(y^{\mu}, x\right)$ is the length of the time interval multiplying a truncated moment matrix of a Dirac measure concentrated at $x = 0$, while its marginal with respect to t equals a truncated Lebesgue moment matrix on the $[0, 2]$ interval. More importantly, one can recover the optimal controls as the marginal $M_d(y^{\omega}, t)$ is the weighted sum of Dirac measures located at the impulse times, the weights being the impulse amplitudes. In summary, we can recover numerically the optimal measures:

$$
\begin{aligned}
\mu(dt, dx) &= \lambda_{[0,2]}(dt)\, \delta_0(dx) \\
\omega(dt, dx) &= -\delta_0(dt)\, \lambda_{[0,1]}(dx) + \delta_2(dt)\, \lambda_{[0,\frac{1}{2}]}(dx).
\end{aligned}
$$

7 The fuel-optimal linear impulsive guidance rendezvous problem

In this section, the moment approach is applied to the far-range rendezvous in a linearized gravitational field. This problem is defined as a fixed-time minimum-fuel impulsive orbital transfer between two known circular orbits. Under Keplerian assumptions and for a circular rendezvous, the complete rendezvous problem may be decoupled between the out-of-plane rendezvous problem, for which an analytical solution may be found in [8], and the coplanar problem. Therefore, only coplanar circular rendezvous problems based on the Hill-Clohessy-Wiltshire equations and associated transition matrix [9] will be considered for numerical illustration of the moment approach.

7.1 Minimizing the $\mathcal{L}_1 - l_1$ norm

The first important case to consider is that of a satellite whose attitude is uncontrolled, relying on two perpendicular sets of thrusters to change its velocities. In the Local Vertical Local Horizontal (LVLH) reference frame, this leads to the minimization of the $\mathcal{L}_1$-norm of the controls, or the $\mathcal{L}_1 - l_1$ norm following Ross' terminology for space trajectory optimization [26].

The problem becomes:

$$V_M = \inf_{w(\nu)} \int_{\nu_0}^{\nu_f} |dw_1|(\nu) + |dw_2|(\nu)$$

such that:

$$dx = \underbrace{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 2 \\ 0 & 3 & -2 & 0 \end{bmatrix}}_{A} x(\nu)\, d\nu + \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}}_{B} dw(\nu)$$

$$x(\nu_0) = x_0, \quad x(\nu_f) = x_f$$

where ν, the true anomaly, is the independent variable taking values in the $[\nu_0, \nu_f]$ interval. The (x_1, x_2) components of the state vector are relative positions (X, Z) in the orbital plane, and (x_3, x_4) are their respective relative velocities $(\dot{X}, \dot{Z})$. The cost is a direct image of fuel consumption which is to be minimized. The general framework for this problem is recalled in [8] and [2], where an indirect method based on primer vector theory is used.

Before applying the moment approach, it is easy to notice that the problem at hand is fully linear. As such, it is possible to use a direct method based on Linear Programming (LP) as a baseline solution, as in [22]. The method discretizes the control in N time steps and seeks a solution amongst all purely impulsive solutions at the prescribed anomalies.

Table 1: Impulse times and amplitudes for Ex. 4

LMI method			LP method		
ν_i	$(u_1)_{\nu_i}$	$(u_2)_{\nu_i}$	ν_i	$(u_1)_{\nu_i}$	$(u_2)_{\nu_i}$
0	-0.0386	0	0	-0.0392	0
1.791	+0.109	0	1.795	+0.109	0
4.495	-0.109	0	4.488	-0.109	0
6.283	+0.0389	0	6.283	+0.0392	0

Using a classical transcription method [6] [20], the genuine infinite-dimensional problem may then be approximated by a finite-dimensional LP problem:

$$\begin{aligned} V_{LP} = \quad & \min_u \sum_{i=0}^{N} \|u_{\nu_i}\|_1 \\ \text{s.t.} \quad & x(\nu_f) = \Phi(\nu_f, \nu_0)x(\nu_0) + \sum_{i=0}^{N} \Phi(\nu_f, \nu_i)Bu_{\nu_i} \\ & x(\nu_0) = x_0,\ x(\nu_f) = x_f \end{aligned} \tag{17}$$

where Φ is the Hill-Clohessy-Wiltshire transition matrix, B is defined as before and u_{ν_i} is the vector of velocity increments at true anomaly ν_i in the LVLH frame [2].

Note that the hypothesis of a purely impulsive solution as assumed in the baseline LP method is valid for the unconstrained case only. In fact, a proof by Neustadt [23] shows that for minimum $\mathcal{L}_1 - l_p$ norm ($1 \leq p \leq \infty$) problems with linear time-varying dynamics, such as the one above, optimal controls have at most n impulses, n being the number of end-constrained states. However, this falls obviously apart as soon as other state constraints are present. Consider for instance a problem when at some intermediary anomaly, the values of the state vector are prescribed; optimal solutions may then possess up to $2n$ impulses. It is also easy to imagine constraints where the optimal trajectories will have to be continuous.

Example 4 ($\mathcal{L}_1 - l_1$ minimization)**.** We now compare both approaches using the third case presented in [8]. It consists in a coplanar circle-to-circle rendezvous with zero eccentricity. The rendezvous maneuver must be completed in one orbital period ($\nu_f = 2\pi$) with boundary conditions $x_0 = [\ 1 \quad 0 \quad 0 \quad 0\ ]'$ and $x_f = [\ 0 \quad 0 \quad 0 \quad 0.427\ ]'$.

With a grid of $N = 50$ points, the LP algorithm converges to the four-impulse control policy presented in Tab. 1, giving the trajectory depicted in Fig. 1.

Using the LMI method and adding a sufficiently large ball constraint on the trajectory to have a problem defined on compact sets, we reached the same criterion (within numerical tolerance) after the fourth relaxation, see Tab. 2. As with academic example 3, controls can be inferred from the moment matrix of the ω measures. Indeed, ω_1 converges to a measure whose anomaly marginal is $\sum (u_{\nu_i})_1\, \delta_{\nu_i}(d\nu)$, with impulse amplitudes $(u_{\nu_i})_1$ and anomaly ν_i taken from Tab. 1, while ω_2 converges to the zero measure. Not only does

Figure 1: Optimal trajectory for example 4

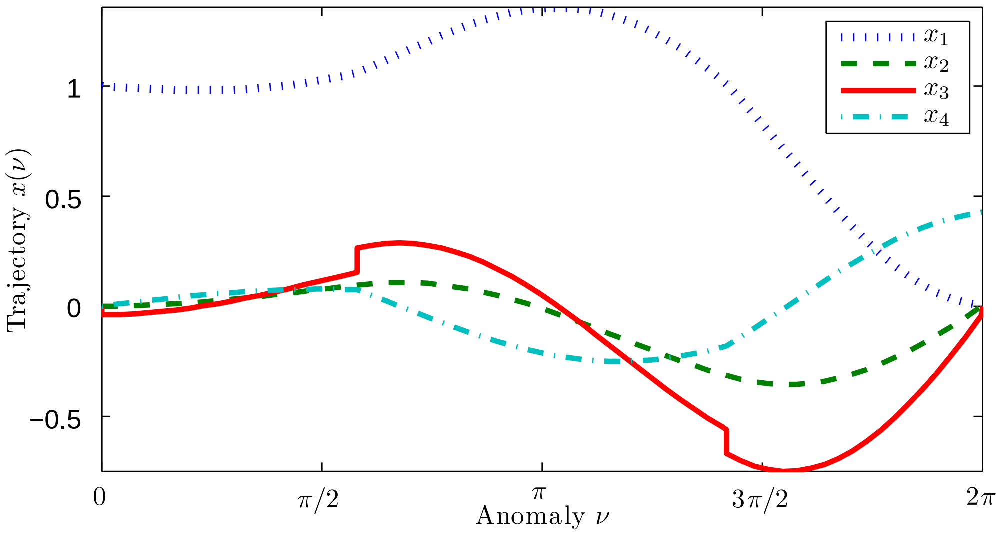

Table 2: Criterion as a function of LMI relaxation order for Ex. 4

d	V_M^d
1	0.0463
2	0.0680
3	0.2188
4	0.2972

this result prove the global optimality of the LP solution within the class of all-impulsive solutions no matter the number of impulses, but it also shows that it is optimal over all measure thrust solutions, including continuous thrust.

7.2 Minimizing the $\mathcal{L}_1 - l_2$ norm

The other case to consider is that of a satellite whose attitude is controlled, and therefore only one thruster is needed to achieve all thrust directions. This leads to the minimization of the square of the $\mathcal{L}_1 - l_2$ norm following Ross' terminology [26]. Using Neustadt [23] again, the optimal solution of the unconstrained problem will be purely impulsive with at most n impulses, but this is of course never assumed in the LMI method.

In this problem, controls appear in a non-linear fashion, which prevents the use of the formulation outlined in this paper. However, by considering a single impulsive non-decreasing relaxed control w for thrust magnitude and a bounded control θ in the $[0, 2\pi]$ interval, the problem becomes:

$$V_M = \inf_{w(\nu)\in BV} \int_{\nu_0}^{\nu_f} |dw|(\nu) \tag{18}$$

Table 3: Criterion as a function of LMI relaxation order for Ex. 5

d	V_M^d
1	0.0059
2	0.1038
3	0.1038
4	0.1059

such that:

$$\begin{aligned} dx &= A\,x(\nu)\,d\nu + B\begin{bmatrix}\cos\theta\\ \sin\theta\end{bmatrix} dw(\nu) \\ x(\nu_0) &= x_0, \quad x(\nu_f) = x_f \end{aligned} \tag{19}$$

where the A and B matrices are defined as in Ex. 4. The requirement that all data be polynomial is met by posing $u_1 := \cos\theta$, $u_2 := \sin\theta$ and adding the compactly defined constraint $u_1^2 + u_2^2 = 1$.

This problem mixes the impulsive formulation as outlined in this paper with the bounded control formulation as in [18], but the extension is easy to infer. In fact, the occupation measure version of (18) reads:

$$J_m = \inf_{\mu,\omega} \int_{[0,T]\times X\times U} d\omega(t,x,u)$$

such that, $\forall\, v \in \mathbb{R}[t,x]$,

$$\begin{aligned} v(T,x_T) - v(0,x_0) &= \int_{[0,T]\times X} \left(\frac{\partial v}{\partial t} + \left(\frac{\partial v}{\partial x}\right)' A\,x\right) d\mu(t,x) \; + \int_{[0,T]\times X\times U} \left(\frac{\partial v}{\partial x}\right)' B\,u\,d\omega(t,x,u) \\ X &= \left\{x \in \mathbb{R}^4 : r^2 - \sum_i x_i^2 \geq 0\right\} \\ U &= \left\{u \in \mathbb{R}^2 : 1 - u_1^2 - u_2^2 = 0\right\} \end{aligned}$$

where r is again a sufficiently large real number.

Example 5 ($\mathcal{L}_1 - l_2$ minimization)**.** Consider the first case presented by Carter in [8]. This rendezvous maneuver must be completed in one orbital period ($\nu_f = 2\pi$) with boundary conditions $x_0 = [\ 1 \quad 0 \quad 0 \quad 0\ ]'$ and $x_f = [\ 0 \quad 0 \quad 0 \quad 0\ ]'$. This problem is usually difficult to handle by other methods because of the high number of symmetries that are present.

Using the LMI method, we obtain the cost evolution with respect to relaxation order of Tab. 3. At the fourth relaxation, we can extract with the usual method the control

Table 4: Impulse times, amplitudes and directions for Ex. 5

ν_i	u_{ν_i}	$\cos\theta_{\nu_i}$	$\sin\theta_{\nu_i}$
$0.0043 \cdot 2\pi$	0.0530	0.9983	-0.0568
$0.9961 \cdot 2\pi$	0.0530	-0.9983	-0.0568

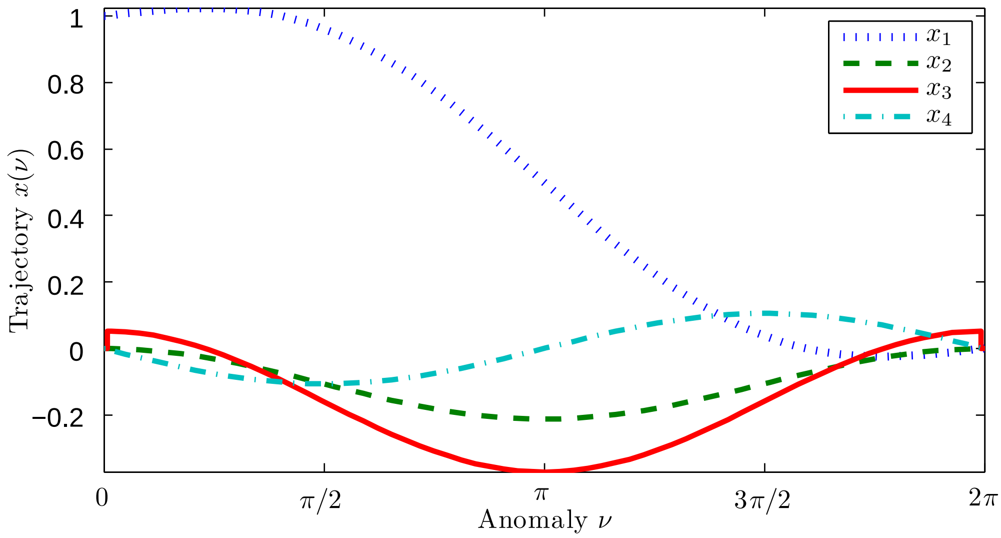

Figure 2: Optimal trajectory for example 5

policy of Tab. 4 whose corresponding trajectory is given on Fig. 2. This trajectory is (numerically) admissible and therefore optimal, and is consistent with the analytical two-impulse solution derived in [3]. Therefore, this result numerically proves that for this particular problem, only 2 impulses (instead of the upper bound of 4) are needed, which is a new result.

8 Conclusion

The focus of this work is on actual computation of optimal impulsive controls for systems described by ordinary differential equations with polynomial dynamics and polynomial (semi-algebraic) constraints on the state. State trajectory and controls are captured by measures which are linearly constrained, resulting in an infinite-dimensional Linear Programming (LP) problem consistent with the formalism of our GloptiPoly software [14]. This LP problem on measures can then be solved numerically via a hierarchy of Linear Matrix Inequality (LMI) relaxations, for which off-the-shelf Semi-Definite Programming (SDP) solvers can be used. If the solution is purely impulsive, the optimal impulse sequence can then be retrieved by simple linear algebra, and global optimality can be verified by *a posteriori* simulation or comparison with suboptimal control sequences computed by alternative techniques.

For orbital rendezvous, our technique can be readily adapted to cope with state (e.g. obstacle avoidance) constraints, as soon as they define a basic semi-algebraic set. Other criteria than the total variation can also be handled. Smoother solutions can be expected, maybe consisting of a mix of absolutely continuous and singular controls, including impulsive controls.

References

[1] Agrachev A. A., On regularity properties of extremal controls, *J. Dynamical Control Systems*, 1(3):319-324, 1995.

[2] Arzelier D., Kara-Zaitri M., Louembet C., Delibaşı A, Using polynomial optimization to solve the fuel-optimal linear impulsive rendezvous problem, *Journal of Guidance, Control, and Dynamics*, 34(5), 2011.

[3] Arzelier D., Louembet C., Rondepierre A., Kara-Zaitri M., A new nixed iterative algorithm to solve the fuel-optimal linear impulsive rendezvous problem, *J. Optimization Theory and Applications*, submitted, preprint available at http://hal.archives-ouvertes.fr/hal-00687499

[4] Ash R. B., *Real analysis and probability*, Academic Press, San Diego, 1972

[5] Bensoussan A., Lions J.-L., *Contrôle impulsionnel et inéquations variationnelles*, Gauthier-Villars, Paris, 1982.

[6] Betts J.T., *Practical methods for optimal control using nonlinear programming*, SIAM, Philadelphia, 2001.

[7] Brezis H., *Functional analysis, Sobolev spaces and partial differential equations*, Springer, Berlin, 2011.

[8] Carter T. E., Optimal impulsive space trajectories based on linear equations, *J. Optim. Theory Appl.*, 70(2):277-297, 1991.

[9] Clohessy W. H., Wiltshire R. S., Terminal guidance system for satellite rendezvous, *J. Aerospace Sci.*, 27(9):653-658, 1960.

[10] Frankowska H., Rampazzo F., Filippov's and Filippov-Ważewski's theorems on closed domains, *Journal of Differential Equations*, 161:449-478, 2000.

[11] Gaitsgory V., Quincampoix M., Linear programming approach to deterministic infinite horizon optimal control problems with discounting, *SIAM J. Control Optim.* 48(4):2480-2512, 2009.

[12] Gajardo P., Ramírez C. H., Rapaport A., Minimal time sequential batch reactors with bounded and impulse controls for one or more species, *SIAM J. Control Optim.* 47(6):2827-2856, 2008.

[13] Gamkrelidze, R. V., *Principles of optimal control theory*, English translation of a 1975 Russian original, Plenum Press, New York, 1978.

[14] Henrion D., Lasserre J.-B., Löfberg J., GloptiPoly 3: Moments, Optimization and Semidefinite Programming, *Optim. Methods and Software*, 24(4-5):761-779, 2009.

[15] Henrion D., Lasserre J.-B., Savorgnan C., Nonlinear optimal control synthesis via occupation measures, *Proc. IEEE Conf. Decision and Control*, Cancún, Mexico, Dec. 2008.

[16] Kolmogorov A. N., Fomin S. V., *Introductory real analysis*, English translation of a 1968 Russian original, Dover Publications, New York, 1970.

[17] Kurzhanski A. B., Daryin A. N., Dynamic programming for impulse controls. *Annual Rev. Control*, 32:213-227, 2008.

[18] Lasserre J.-B., Henrion D., Prieur C., Trélat E., Nonlinear optimal control via occupation measures and LMI relaxations. *SIAM J. Control Opt.*, 47(4):1643-1666, 2008.

[19] Lasserre J.-B., *Moments, positive polynomials and their applications.* Imperial College Press, London, UK, 2009.

[20] Louembet C., Arzelier D., Deaconu G., Blanc-Paques P., Robust rendezvous planning under navigation and manoeuvering errors, *International ESA Conference on Guidance and Navigation Control Systems*, Karlovy Vary, Czech Republic, June 2011.

[21] Miller B., The generalized solutions of ordinary differential equations in the impulse control problems, *J. Math. Systems, Estimation, and Control*, 4(3):1-21, 1994.

[22] Mueller J. B., Larsson R., Collision avoidance manoeuver planning with robust optimization, *International ESA Conference on Guidance, Navigation and Control Systems*, Tralee, County Kerry, Ireland, June 2008.

[23] Neustadt L. W., Optimization, a moment problem and nonlinear programming. *SIAM J. Control*, 2(1):33–3, 1964.

[24] Riesz F., Sz.-Nagy B., *Leçons d'analyse fonctionnelle.* 3ème édition. Gauthier-Villars, Paris, Akadémiai Kiadó, Budapest, 1955.

[25] Rischel R., An extended Pontryagin principle for control systems whose control laws contain measures, *SIAM J. Control*, 3:191-205, 1965.

[26] Ross I. M., Space trajectory optimization and L^1 optimal control problems, *in Modern Astrodynamics*, chapter 6, edited by P. Gurfil, Elsevier, 2006.

[27] Roubíček T., Numerical Techniques in Relaxed Optimization Problems, *Nonconvex Optimization and Its Applications*, 8:157:178, 2006

[28] Schmaedeke W.W., Optimal control theory for nonlinear vector differential equations containing measures, *SIAM J. Control*, 3:231-280, 1965.

[29] Sturm J. F., Using SeDuMi 1.02, a Matlab toolbox for optimization over symmetric cones, *Optim. Methods and Software*, Vol. 11-12:625-653, 1999.

[30] Vinter R. B., A maximum principle for optimal processes with discontinuous trajectories, *SIAM J. Control and Optimization*, 1:205-229, 1988.

[31] Young L. C., *Lectures on the calculus of variations and optimal control theory*, W. B. Saunders Co., Philadelphia, 1969.