Robust $\mathcal{D}$ stabilization of a polytope of matrices

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Abstract

The problem of stabilization of a polytope of matrices in a subregion $\mathcal{D}_R$ of the complex plane is revisited. A new sufficient condition of robust $\mathcal{D}_R$ stabilization is given. It implies the solution of an $\mathcal{LMI}$ involving matrix variables constrained by a nonlinear algebraic relation. Some $\mathcal{LMI}$ relaxations are first proposed. Then, it is shown that a cone complementarity formulation of this condition allows to associate an efficient iterative numerical procedure which leads to a low computational burden. This algorithm is tested on different numerical examples for which existing approaches in control literature fail.

Keywords

$\mathcal{D}_R$ stability, parameter-dependent Lyapunov functions, cone complementarity.

1 Introduction

In the synthesis of feedback control systems, it is necessary to guarantee that the stability and some performance properties of the closed-loop system are robust with respect to plant perturbations. If a state-space approach is considered for modeling, plant uncertain parameters may be viewed as perturbations affecting the coefficients of system matrices and defining therefore families of system. This paper focuses on families of linear systems in state-space form where the domain of admissible system matrices is a real convex polytope. The problem of finding stability conditions for a polytope of matrices has received a considerable attention in the literature, [5],[6], [9], [18], [19], [25], [26] and some attempts have been made to solve the synthesis problem [17], [14], [3], [23], [24]. In both cases the problem is known to be $\mathcal{NP}$-hard, [11] and therefore a critical tradeoff has to be faced : find testable precise conditions while keeping a weak computational complexity. Robust stability problems have been attacked via methods which rely heavily on the convexity assumption, (the results based on quadratic stability concept) or on more complex approaches for which branching operations may be required. In the first case, it is well known that we get too much conservative results while for the second case, computational complexity is a major difficulty.

The situation is more awkward in synthesis problems since a constructive method is needed to get a robust controller. The quadratic stability framework, [14], [3], has proven to be a successfull design methodology but still suffers from its conservatism when dealing with structured uncertainty. Recently, a new robust stability analysis condition simultaneously appeared in [22] and [24]. This $\mathcal{LMI}$-based condition involves parameter-dependent Lyapunov functions and extra matrix variables leading to a drastic reduction of the conservatism, (see [24] for comparison results). This analysis result has been used in [21] and [23] to tackle the problems of robust state-feedback synthesis and multiobjective synthesis for discrete-time systems. Unfortunately, such an extension is surprisingly impossible for continuous-time systems. In fact, a linearizing change of variables is proposed in [24] to get sufficient $\mathcal{LMI}$-based conditions of stabilization of a polytope of matrices in particular subregions $\mathcal{D}_R$ of the complex plane. This change of variables is no more valid for $\mathcal{D}_R$ regions such as half-planes and sectors which do not verify some basic technical assumption. For such regions, a Bilinear Matrix Inequality, $\mathcal{BMI}$, formulation may be deduced. There is a considerable amount of literature reformulating robust stabilization problem as BMIs, [16]. Here, a sufficient condition of robust $\mathcal{D}_R$ stabilization of a polytope of matrices is characterized by an $\mathcal{LMI}$ involving matrix variables subject to an additional non linear algebraic constraint. Exploiting the particular structure of this reformulation allows to use a conic complementarity formulation, [13], of the problem and its related numerical procedure. A fundamental point is that our approach always ensures to get better results than the quadratic stability framework except for marginal numerical ill-conditioned problems.

The contribution of this paper is threefold. First, it extends in a very natural way the robust analysis conditions of [24] leading to an effective synthesis procedure for any $\mathcal{LMI}$ regions. Second, it generalizes and improves the results of [21] to continuous-time systems and for any $\mathcal{LMI}$ regions. Although the approach is not purely $\mathcal{LMI}$-based, the computational complexity remains reasonable since each iteration consists in an $\mathcal{LMI}$ optimization and that, in general, the algorithm requires few iterations, (less than 10). Finally, the proposed method allows to deal with intersection of $\mathcal{LMI}$ regions using different Lyapunov functions for each elementary region unlike the method proposed in [8]. This obviously reduces the conservatism of the existing results.

Notation is standard. The transpose of a matrix A is denoted A'. For symmetric matrices, $>$ $(\geq)$ denotes the Löewner partial order, i.e. $A > (\geq) B$ iff $A - B$ is positive (semi) definite. $\mathbf{1}$ stands for the identity matrix and $\mathbf{0}$ for the zero matrix with the appropriate dimensions. $\mathcal{S}_n$ denotes the set of symmetric matrices of $\mathbb{R}^{n\times n}$ and $\mathcal{S}_n^+$, $(\mathcal{S}_n^{+*})$, the cone of positive semi-definite, (definite) matrices in $\mathcal{S}_n$. $\mathbb{C}$ is the set of complex numbers. $\mathbb{R}^+$, $(\mathbb{R}^{+*})$ is the set of positive, (strictly positive), real numbers. $\otimes$ is the Kronecker product of two matrices. We remind that $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$. The symetric part of a square matrix A is denoted $sym[A]$, i.e. $sym[A] = A + A'$. δ is the derivation operator for continuous-time systems, $(\delta[x(t)] = \dot{x}(t))$ and the delay operator for discrete-time ones, $(\delta[x(t)] = x(t+T))$.

2 Preliminaries

2.1 Background

Let us consider the linear uncertain dynamical system,

$$\delta[x(t)] = Ax(t) + Bu(t) = M \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \qquad M = \begin{bmatrix} A & B \end{bmatrix} \in \mathcal{M} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector. The dynamical matrix A and the input matrix B are in the convex polytope $\mathcal{M}$ defined by,

$$\mathcal{M} = co\left\{M^{[1]}, \cdots, M^{[N]}\right\} = \left\{M = \sum_{i=1}^{N} \xi_i M^{[i]} \; : \; \xi_i \geq 0 \; , \; \sum_{i=1}^{N} \xi_i = 1\right\} \tag{2}$$

This representation is quite general and encompasses the well-known case of interval systems defined by,

$$M = [m_{ij}] \quad \underline{m}_{ij} \leq m_{ij} \leq \overline{m}_{ij} \tag{3}$$

Let

$$\mathcal{D}_R = \{z \in \mathbb{C} \;\; : \;\; f_{\mathcal{D}_R}(z) = R_{11} + R_{12}z + R'_{12}z^* + R_{22}zz^* < 0\} \tag{4}$$

be a given region of the complex plane, where $R \in \mathbb{R}^{2d \times 2d}$ is a symmetric matrix partitionned as:

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R'_{12} & R_{22} \end{bmatrix} \;\; : \;\; \begin{array}{l} R_{11} = R'_{11} \in \mathbb{R}^{d \times d} \\ R_{22} = R'_{22} \in \mathbb{R}^{d \times d} \end{array} \tag{5}$$

Following the terminology of [7], the matrix-valued function $f_{\mathcal{D}_R}(z)$ is called the characteristic function of $\mathcal{D}_R$. Without any assumption on the matrix R_{22}, $\mathcal{D}_R$ regions are not convex. With the assumption of positive definiteness, $R_{22} \geq \mathbf{0}$, (4) appears to be a slight modification of the characterisation of $\mathcal{LMI}$ regions, [24]. Usual choices for $\mathcal{D}_R$ are the left half-plane, ($d = 1$, $R_{11} = R_{22} = 0$, $R_{12} = 1$), and the unit disk, ($d = 1$, $R_{11} = -1$, $R_{22} = 1$, $R_{12} = 0$).

• **Remarks 1** As for $\mathcal{LMI}$ regions the $\mathcal{D}_R$ regions are symmetric with respect to the real axis and an intersection of $\mathcal{D}_R$ regions is a $\mathcal{D}_R$ region.

Note that the class of $\mathcal{LMI}$ regions belongs to the class of $\mathcal{D}_R$ regions but our investigations will be restricted to the former, i.e. we assume that $R_{22} \geq \mathbf{0}$. For a more complete description of possible $\mathcal{LMI}$-regions, the interested reader may have a look to the references [7].
A straightforward extension of the usual stability property of dynamical systems may be easily extrapolated.

Definition 1
A matrix $A \in \mathbb{R}^{n \times n}$ is said to be $\mathcal{D}_R$-stable if and only if all its eigenvalues lie in the $\mathcal{D}_R$ region defined by (4).

Usually, two different robust stability concepts may been defined to study the robustness of pole clustering of a matrix A belonging in a convex matrix polytope $\mathcal{A}$ defined as (2), in $\mathcal{D}_R$ regions.

Definition 2

- *$\mathcal{A}$ is robustly $\mathcal{D}_R$-stable, if and only if, for all $A \in \mathcal{A}$, A is $\mathcal{D}_R$-stable.*
- *$\mathcal{A}$ is quadraticaly $\mathcal{D}_R$-stable, if and only if, there exists a matrix $P \in \mathcal{S}_n^{+*}$ such that for all $A \in \mathcal{A}$:*

$$R_{11} \otimes P + R_{12} \otimes (PA) + R'_{12} \otimes (A'P) + R_{22} \otimes (A'PA) < \mathbf{0} \tag{6}$$

These two notions are not in general equivalent except in special cases, e.g., complex or real unstructured uncertainty for the open left half-plane. Quadratic $\mathcal{D}_R$ stability is well-known to be more conservative though it proves useful for analysis and synthesis purpose.

2.2 The robust pole clustering analysis problem

The problem addressed in this paper is to find a robust state-feedback matrix $K \in \mathbb{R}^{m \times n}$ such that every closed-loop matrix $A_{cl} = A+BK$ belonging to the convex polytope $\mathcal{A}$ defined as,

$$\mathcal{A} = \left\{ A_{cl} = \sum_{i=1}^{N} \alpha_i (A^{[i]} + B^{[i]} K) \; : \; \alpha_i \geq \mathbf{0}, \; \sum_{i=1}^{N} \alpha_i = 1 \right\} \tag{7}$$

has all its eigenvalues in the $\mathcal{D}_R$ region. $\mathcal{A}$ is robustly $\mathcal{D}_R$-stable if and only if, for each $A_{cl} \in \mathcal{A}$, there exists a symmetric positive definite matrix P such that (6) holds. It is well-known that deciding whether or not every member of the polytope maintains eigenvalue locations in the specified $\mathcal{D}_R$ region is equivalent to solve an $\mathcal{NP}$-hard problem, [11]. The related problem of robust stabilization via state-feedback of the convex polytope $\mathcal{M}$ is therefore equivalent to an $\mathcal{NP}$-hard problem. Most of the approaches dealing with the synthesis problem are based on the quadratic stability notion which leads to inherently conservative stability tests. In [8], analysis results with respect to unstructured uncertainty are given in the quadratic $\mathcal{D}_R$ stability framework. Parameter-dependent Lyapunov functions are then used to derive sufficient conditions of robust pole clustering. Finally, the output-feedback synthesis problem is tackled in a somewhat framework. Note that our approach is based on a different parametrization of the parameter-dependent Lyapunov functions and relies on very different conditions where no scaling matrices are involved.

Theorem 1 *[24]*
If, there exists two matrices $H_1 \in \mathbb{R}^{dn \times dn}$, $H_2 \in \mathbb{R}^{dn \times n}$ and N matrices $P_i \in \mathcal{S}_n^{+*}$ such that, $\forall \; i = 1, \cdots, N$:

$$R \otimes P^{[i]} + \begin{bmatrix} sym\left[(\mathbf{1}_d \otimes A_{cl}^{[i]}) H_1\right] & (\mathbf{1}_d \otimes A_{cl}^{[i]}) H_2 - H_1' \\ \star & -H_2 - H_2' \end{bmatrix} < \mathbf{0} \tag{8}$$

then $\mathcal{A}$ is robustly $\mathcal{D}_R$-stable.

Proof
Theorem 2 is a dual, (transposed) version of theorem 4 in [24] and the proof is easily transposable from this reference. ■

- **Remarks 2** *:*

If the $\mathcal{D}_R$ region of pole clustering consists in the intersection of L elementary $\mathcal{D}_{R_i}$ regions, independent parameter-dependent Lyapunov functions involving L sets of N positive definite matrices P_i may be used. The interest of this new condition is that this feature may be not only used for analysis purpose but also when synthesizing a controller as will be seen in the next section. This is a major difference with the approach proposed in [8].

The closed-loop matrix $A_{cl} = A+BK$ is affine in the controller parameter K. For the synthesis problem where we are looking for the gain matrix K, the inequality (8) is therefore a bilinear matrix inequality with respect to the unknown matrices due to the products between H_1, H_2 and the gain matrix K. The next section proposes a sufficient condition of robust $\mathcal{D}_R$ stabilization implying the solution of an $\mathcal{LMI}$ involving matrix variables constrained by a nonlinear algebraic condition.

3 State feedback $\mathcal{D}_R$ stabilization

3.1 Pseudo $\mathcal{LMI}$-based formulation

In [21], for stability of discrete-time systems, it is proposed to choose $H_1 = \mathbf{0}$ and to generalize a well-known linearizing change of variables, [14], by letting $S = KH_2$. An $\mathcal{LMI}$ formulation of the robust state-feedback stabilization problem is then possible. This change of variables is generalized in [24] where it is shown that it is possible to perform such a linearization only for $\mathcal{D}_R$ regions satisfying the additional assumption $R_{22} > 0$. This assumption is not satisfied by $\mathcal{D}_R$ regions of major importance such as half-planes, conic sectors, or their intersection. In particular, no equivalent condition exists for the robust stabilization of polytopes of continuous-time systems. The next result recasts the original bilinear problem as an $\mathcal{LMI}$ feasibility problem subject to a nonlinear algebraic constraint.

Theorem 2 *:*
If there exist N matrices $P^{[i]} \in \mathcal{S}_n^{+*}$ and four matrices $H_1 \in \mathbb{R}^{dn\times dn}$, $G_1 \in \mathbb{R}^{dm\times dn}$, $h_2 \in \mathbb{R}^{n\times n}$, $g_2 \in \mathbb{R}^{m\times n}$ solutions of the following linear inequalities, $\forall\ i = 1, \cdots, N$

$$R\otimes P^{[i]} + \begin{bmatrix} sym\left[(\mathbf{1}_d\otimes A^{[i]})H_1\right] + sym\left[(\mathbf{1}_d\otimes B^{[i]})G_1\right] & -H_1' + \mathbf{1}_d\otimes\left(A^{[i]}h_2 + B^{[i]}g_2\right) \\ \star & -\mathbf{1}_d\otimes(h_2 + h_2') \end{bmatrix} < \mathbf{0} \tag{9}$$

under the nonlinear equation,

$$G_1 - (\mathbf{1}_d\otimes(g_2h_2^{-1}))H_1 = \mathbf{0} \tag{10}$$

then the polytope $\mathcal{M}$ is $\mathcal{D}_R$ stabilizable and a robust state-feedback matrix is given by :

$$K_{\mathcal{D}_R} = g_2h_2^{-1} \tag{11}$$

Let us note inequality (9),

$$\mathcal{L}^{[i]}(P^{[i]}, G_1, H_1, g_2, h_2) < \mathbf{0} \tag{12}$$

<u>**Proof**</u>

Suppose there exist N matrices $P^{[i]} \in \mathcal{S}_n^{+*}$ and matrices $H_1 \in \mathbb{R}^{dn\times dn}$, $G_1 \in \mathbb{R}^{dm\times dn}$, $h_2 \in \mathbb{R}^{n\times n}$, $g_2 \in \mathbb{R}^{m\times n}$ solutions of (9), $\forall\ i = 1, \cdots, N$ under the nonlinear algebraic constraint (10) then replacing G_1 in (9) leads to the following inequality after some elementary algebraic manipulations, $\forall\ i = 1, \cdots, N$:

$$R\otimes P^{[i]} + \begin{bmatrix} sym\left[(\mathbf{1}_d\otimes(A^{[i]} + B^{[i]}g_2h_2^{-1}))H_1\right] & -H_1' + \left(\mathbf{1}_d\otimes(A^{[i]} + B^{[i]}g_2h_2^{-1})\right)H_2 \\ \star & -\mathbf{1}_d\otimes(h_2 + h_2') \end{bmatrix} < \mathbf{0} \tag{13}$$

It therefore exist N matrices $P^{[i]} \in \mathcal{S}_n^{+*}$ and matrices $H_1 \in \mathbb{R}^{dn\times dn}$, $G_1 \in \mathbb{R}^{dm\times dn}$, $H_2 = \mathbf{1}_d\otimes h_2 \in \mathbb{R}^{dn\times dn}$, $G_2 = \mathbf{1}_d\otimes g_2 \in \mathbb{R}^{dm\times dn}$ solutions of (8) where the robust state-feedback gain is given by:

$$K_{\mathcal{D}_R} = g_2h_2^{-1} \tag{14}$$

This proves that the gain (14) robustly stabilizes the polytope of matrices $\mathcal{M}$. ■

The proof of theorem 2 shows that the unknown matrix H_2 must have a bloc structure given by the constraint $H_2 = \mathbf{1}_d\otimes h_2$ which induces extra conservatism in the condition for regions which have an order greater than one, i.e. $d > 1$. It is important to note that in the conditions of theorem 2, the non convex feature of the original problem is entirely enclosed in the nonlinear algebraic equality (10). One way to overcome this difficulty is to find convex $\mathcal{LMI}$ relaxations of the condition.

3.2 $\mathcal{LMI}$ Relaxations

Similarly to the previous change of variables, a conservative change of variables is considered in [21] and [24], $H_1 = R_{12} \otimes H$, $H_2 = R_{22} \otimes H$ and $S = KH$. This leads to the following sufficient condition of robust $\mathcal{D}_R$ stabilization for regions such that $R_{22} > \mathbf{0}$.

Theorem 3 *[24]*
If there exist two matrices $H \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{m\times n}$ and N matrices $P^{[i]} \in \mathcal{S}_n^{+*}$ such that $\forall i \in \{i, \cdots, N\}$:

$$R \otimes P^{[i]} + \begin{bmatrix} sym\left[R_{12} \otimes (A^{[i]}H + B^{[i]}S)\right] & -R'_{12} \otimes H' + R_{22} \otimes (A^{[i]}H + B^{[i]}S) \\ \star & -R_{22} \otimes (H + H') \end{bmatrix} < \mathbf{0} \qquad (15)$$

then the polytope $\mathcal{M}$ is robustly $\mathcal{D}_R$-stabilisable by state feedback and an admissible gain is:

$$K = SH^{-1} \qquad (16)$$

Another way to relax the problem is to give a structure to the matrices H_1 and G_1 in order to enforce the nonlinear condition (10). Obviously, a simple condition to verify (10) is to choose $H_1 = x \otimes h_2$ and $G_1 = x \otimes g_2$ for a given matrix $x \in \mathbb{R}_{d\times d}$.

Theorem 4
If there exist two matrices $h \in \mathbb{R}^{n\times n}$, $g \in \mathbb{R}^{m\times n}$ and N matrices $P^{[i]} \in \mathcal{S}_n^{+*}$ such that $\forall i \in \{i, \cdots, N\}$ and for a given matrix $x \in \mathbb{R}^{d\times d}$:

$$R \otimes P^{[i]} + \begin{bmatrix} sym\left[x \otimes (A^{[i]}h + B^{[i]}g)\right] & -x' \otimes h' + \mathbf{1}_d \otimes (A^{[i]}h + B^{[i]}g) \\ \star & -\mathbf{1}_d \otimes (h + h') \end{bmatrix} < \mathbf{0} \qquad (17)$$

then the polytope $\mathcal{M}$ is robustly $\mathcal{D}_R$-stabilisable by state feedback and an admissible gain is:

$$K = gh^{-1} \qquad (18)$$

Proof

This result is readily proved by remarking that a particular solution of the nonlinear equation (10) is given by $H_1 = x \otimes h_2$ and $G_1 = x \otimes g_2$ for a given matrix x. ■
The previous result gives an $\mathcal{LMI}$ condition when the matrix x is a priori given. It is clear that the relevance of the condition is strongly related to the choice of x. For regions of order 1, this choice is equivalent to the choice of a scalar $x \in \mathbb{R}$.
Finally, a necessary and sufficient condition of quadratic $\mathcal{D}_R$ stabilizability is easily recovered from (15) by choosing $H = P$ and $P^{[i]} = P$, $\forall\ i = 1, \cdots, n$. In this regard, the relationships between these different conditions are investigated in the following corollary.

Corollary 1 *:*

1- *If the order of the $\mathcal{D}_R$ region is equal to 1 then the quadratic $\mathcal{D}_R$ stability condition is a sufficient condition for the condition of theorem 2.*

2- *If $R_{22} > \mathbf{0}$ then the quadratic $\mathcal{D}_R$ stability condition is a sufficient condition for the condition of theorem 3.*

3- *If $R_{22} > \mathbf{0}$ and the order of the $\mathcal{D}_R$ region is equal to 1 then the quadratic $\mathcal{D}_R$ stability condition is a sufficient condition for the condition of theorem 3 which implies the condition of theorem 2.*

4- *The condition given in theorem 4 is a sufficient condition for the condition of theorem 2.*

- **Remarks 3** *:*

  - *Note that in the previous corollary little is said about regions of order greater than 1. In fact, the main regions of concern when imposing explicit constraints on the closed-loop dynamics are the disk, the half-plane, the conic sector which is considered as the intersection of two half-planes and their intersection which are or may be converted into $\mathcal{D}_R$ regions of order 1. In the sequel, the case of intersection of elementary $\mathcal{D}_R$ regions is carefully studied. $\mathcal{D}_R$ regions of order 2 are, for instance, the ellipse and the hyperbolic sector for which the condition of theorem 3 cannot be applied, ($R_{22} \geq \mathbf{0}$).*

  - *The relationships between the quadratic stability condition and the condition of theorem 4 are not so clear to establish. In particular, it is easy to find instances for which quadratic stability is better than the condition of theorem 4 for some x. In general, the condition is better than the quadratic one but it may be a non trivial work to find a good guess for x for some instances.*

3.3 Intersection of $\mathcal{D}_R$ regions

One of the main features of the new proposed condition is that it is possible to deal with the intersection of $\mathcal{D}_R$ regions by considering a parameter-dependent Lyapunov function for each elementary $\mathcal{D}_R$ regions. This is in stark contrast with the approach proposed in [8]. In the case of an intersection of regions of identical orders, this may be done by choosing single extra variables G_1, g_2, H_1, h_2. When the regions defining the intersection are of different orders, it is necessary to give a structure to the extra variables defined for each j region H_{1j}, G_{1j} as is show in the following theorem.

Theorem 5 *:*
Let us define the region $\mathcal{D}_R$ of the complex plane as the intersection of L elementary regions $\mathcal{D}_{R_j}$ of respective order d_j and characterized by the symmetric matrix $R_j \in \mathbb{R}^{d_j \times d_j}$.

$$\mathcal{D}_R = \bigcap_{j=1}^{L} \mathcal{D}_{Rj} \tag{19}$$

1- *Suppose that $d_j = d$, $\forall\ j = 1, \cdots, L$. If there exist $L \times N$ matrices $P_j^{[i]} \in \mathcal{S}_n^{+*}$ and matrices $H_1 \in \mathbb{R}^{dn \times dn}$, $G_1 \in \mathbb{R}^{dm \times dn}$, $h_2 \in \mathbb{R}^{n \times n}$, $g_2 \in \mathbb{R}^{m \times n}$ solutions of the following linear inequalities, $\forall\ i = 1, \cdots, N$ and $\forall\ j = 1, \cdots, L$,*

$$R_j \otimes P_j^{[i]} + \begin{bmatrix} sym\left[(\mathbf{1}_d \otimes A^{[i]})H_1 + (\mathbf{1}_d \otimes B^{[i]})G_1\right] & -H_1' + \mathbf{1}_d \otimes \left(A^{[i]}h_2 + B^{[i]}g_2\right) \\ \star & -\mathbf{1}_d \otimes (h_2 + h_2') \end{bmatrix} < \mathbf{0} \tag{20}$$

under the nonlinear equation,

$$G_1 - (\mathbf{1}_d \otimes (g_2 h_2^{-1}))H_1 = \mathbf{0} \tag{21}$$

then the polytope $\mathcal{M}$ is $\mathcal{D}_R$ stabilizable and a robust state-feedback matrix is given by:

$$K_{\mathcal{D}_R} = g_2 h_2^{-1} \tag{22}$$

2- *Suppose now that the $\mathcal{D}_{R_j}$ are of different order. If there exist $L\times N$ matrices $P_j^{[i]} \in \mathcal{S}_n^{+*}$ and matrices $h_1 \in \mathbb{R}^{n\times n}$, $g_1 \in \mathbb{R}^{m\times n}$, $h_2 \in \mathbb{R}^{n\times n}$, $g_2 \in \mathbb{R}^{m\times n}$ solutions of the following linear inequalities, $\forall\ i = 1, \cdots, N$ and $\forall\ j = 1, \cdots, L$,*

$$R_j \otimes P_j^{[i]} + \begin{bmatrix} sym\left[R_{12j} \otimes (A^{[i]}h_1 + B^{[i]}g_1)\right] & -R'_{12j} \otimes h'_1 + \mathbf{1}_d \otimes \left(A^{[i]}h_2 + B^{[i]}g_2\right) \\ \star & -\mathbf{1}_d \otimes (h_2 + h'_2) \end{bmatrix} < \mathbf{0} \tag{23}$$

under the nonlinear equation,

$$g_1 - (g_2 h_2^{-1})h_1 = \mathbf{0} \tag{24}$$

then the polytope $\mathcal{M}$ is $\mathcal{D}_R$ stabilizable and a robust state-feedback matrix is given by:

$$K_{\mathcal{D}_R} = g_2 h_2^{-1} \tag{25}$$

Proof

1- It is easily deduced from the proof of the theorem 2 by writing the conditions for each region $\mathcal{D}_{R_j}$.

2- Writing the conditions of theorem 2, we get the $\mathcal{LMI}$ (20) with the L additional nonlinear constraints:

$$G_{1j} - (\mathbf{1}_d \otimes (g_2 h_2^{-1}))H_{1j} = \mathbf{0} \tag{26}$$

by choosing $H_{1j} = R_{12j} \otimes h_1$ with $G_{1j} = R_{12j} \otimes (g_2 h_2^{-1} h_1) = R_{12j} \otimes g_1$, it leads to the result.

■

• **Remarks 4** *:*
For intersection of regions of different order, the condition is deduced using the additional change of variables: $H_{1j} = R_{12j} \otimes h_1$. Another choice is possible, $H_1 = \mathbf{1}_{d_j} \otimes h_1$ which leads to a different sufficient condition. The first one is particularly interesting since it ensures that the quadratic stability as well as the result in [24] are included by this one. Another way to tackle the problem of the intersection of $\mathcal{D}_R$ regions consists in applying results of theorem 2 with the matrix R defined as the concatenation of elementary R_j matrices. Of course, in that case, a single parameter-dependent Lyapunov function is used for the different regions.

4 A conic complementarity problem

The problem of synthetizing a $\mathcal{D}_R$ stabilizing gain via the condition of theorem 2 is not trivial due to the nonlinear condition (10). One way to tackle this problem is to recast it as a *conic complementarity problem*:

Problem 1 *: CCP*

$$min \quad Trace \underbrace{\begin{bmatrix} T_1 & T_2 \\ T_2' & T_3 \end{bmatrix}}_{T} \overbrace{\begin{bmatrix} Z_1 & Z_2 \\ Z_2' & Z_3 \end{bmatrix}}^{Z}$$

under

$$\left[\begin{array}{cc|cc} Z_1 & Z_2 & G_1 & \mathbf{1}_d \otimes g_2 \\ Z_2' & Z_3 & H_1 & \mathbf{1}_d \otimes h_2 \\ \hline \star & \star & \mathbf{1} & \mathbf{0} \\ \star & \star & \mathbf{0} & \mathbf{1} \end{array}\right] \geq \mathbf{0}$$

$$\begin{bmatrix} T_1 & T_2 \\ T_2' & T_3 \end{bmatrix} \geq \mathbf{0} \tag{27}$$

$$T_1 \geq \mathbf{1}$$

$$P_i > \mathbf{0} \quad \forall\ i = 1, \cdots, N$$

$$\mathcal{L}^{[i]}(P^{[i]}, G_1, H_1, g_2, h_2) < \mathbf{0} \quad \forall\ i = 1, \cdots, N$$

The relationships between theorem 2 and problem 1 are now stated more formally.

Lemma 1 *:*
The sufficient condition of theorem 3 is verified if and only if the global minimum of problem 1 is 0.

Proof

The proof is easily deduced from [12] and [13] by noting that the nonlinear constraint (10) is equivalently written as a rank constraint on the matrix:

$$\Psi = \begin{bmatrix} G_1 & \mathbf{1}_d \otimes g_2 \\ H_1 & \mathbf{1}_d \otimes h_2 \end{bmatrix} \tag{28}$$

■

Note that this formulation is a generalization of the ones presented in [2] and in [12] and originates from [13]. The conic complementarity formulation is a generalizaion of the linear complementarity problems for which exist many different numerical approaches. Some CCP can be solved using primal-dual interior-point algorithms. Here, a linearization algorithm such as the constrained gradient, (known as the Franck and Wolfe algorithm) may then be used to find local solutions of this problem.

Algorithm 1 *: CCA*

> **Step 1:** *Let $k = 0$. Find a feasible point T_0, Z_0, Ψ_0, $P_0^{[i]}$ for $\mathcal{LMI}$ problem (27). If there is no solution, stop. Problem (12)-(10) is not feasible.*

Step 2: *Solve the $\mathcal{LMI}$ problem:*

$$\begin{array}{ll} \min & trace(TZ_k + T_kZ) \\ s.t. & (27) \end{array}$$

for T_{k+1}, Z_{k+1}.

Step 3: *Let* $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n+m}$ *denote the singular values of matrix* M. *If* $\sigma_{n+1} \leq \epsilon\sigma_n$ *then a solution to problem (12)-(10) is found and the* $\mathcal{D}$*-stabilizing state-feedback matrix is* $K = YG^{-1}$, *(*ϵ *defines some accuracy level).*

Step 4: *Let* $k = k + 1$. *If* $k > k_{max}$, *then matrix* K *was not found. Otherwise go to step 2.*

The initialization step amounts to solve an $\mathcal{LMI}$ problem. The solvability of this first step is therefore a necessary condition for the feasibility of the original $\mathcal{BMI}$ problem. In [12], it is shown that the sequence $t_k = trace(T_kZ + Z_kT)$ is a decreasing sequence bounded below by 0 and therefore always converges.
The iterative process is stopped as soon as the ratio of two successive singular values, (testing the rank), of the matrix M is less than some accuracy level or as soon as a prescribed maximum number of iterations is exceeded or as soon as the relative variation of the criterion is less than 0.01%. With this stopping criteria, the behaviour of this algorithm has been carefully studied and extensively compared with existing methods: quadratic-based conditions and conditions of theorem 3. Thousands of random polytopes of matrices have been tested for the continuous-time and discrete-time stabilization problem. Due to the vertexization of the different conditions, the generated systems are limited to 4 states, 5 vertices and 2 inputs. For continuous-time polytopes, the new algorithm is compared to the quadratic approach while for discrete-time, it is compared to the $\mathcal{LMI}$-based method from [21]. For both cases the behavior is similar and the new proposed approach stabilizes between 15 % and 25 % of polytopes for which all the other methods fails. It is important to note that in each case, the algorithm is "plateauing" for less than 5 % of polytopes which are stabilized by the quadratic approach or by the method from [21]. This behavior has been noticed in [1] where an efficient hybrid algorithm based on a combination of conditional gradient and second-order Newton methods is proposed. The interest of the new proposed approach is now illustrated by three numerical examples corresponding to three characteristic cases.

5 Illustrative examples

These numerical examples are intended to illustrate three main features of the proposed approach. First, a continuous-time polytope of matrices example is considered and a comparison between quadratic state-feedback stabilization, condition of theorem 2 and $\mathcal{LMI}$ relaxations is done.
The second example shows that the cone complementarity approach is a valuable extension of the purely $\mathcal{LMI}$-based one proposed in [21]. A discrete-time polytope of matrices is defined for which this last method fails while the one proposed in this article succeeds in few iterations.
Second, a particular region of regional pole placement is considered. It consists in the intersection of three subregions, a disk, a half-plane and a sector. Such a region cannot be considered by the method of [21] and has therefore to be approximated by a disk. In each case, the number of outer iterations, (number of step 2) is given to show the weak complexity of the algorithm.

5.1 Example 1

The previous algorithm is now applied to the robust stabilization problem of an uncertain continuous-time system defined by the following system matrices:

$$A = \begin{bmatrix} 0 & \alpha - 1 \\ \beta & 0 \end{bmatrix} \quad B = \begin{bmatrix} \alpha \\ 1-\beta \end{bmatrix}$$

The uncertain parameters are defined as $|\alpha - 0.5| \leq \gamma \quad |\beta - 0.5| \leq \gamma$. This defines a polytope of matrices (A, B) with four vertices.
This example is borrowed from [10] where a quadratic state feedback stabilizing gain is computed. Moreover, it is shown that the set of quadratic state feedback stabilizing gains is not empty for all $\gamma \in [0\ ,\ 0.36]$. For $\gamma = 0.36$ a quadratic state-feedback stabilizing gain is given by $K_{quad.} = \begin{bmatrix} -0.26 & -4.94 \end{bmatrix}$.
Applying the algorithm 1 based on cone complementarity, we are able to find robust state feedback stabilizing gains for all $\gamma \in [0\ ,\ 0.5-\epsilon]$ where ϵ is of the order of the relative accurracy of the $\mathcal{LMI}$ solver. Note that two of the vertices of the polytope are not controllable pairs for $\gamma = 0.5$. For $\epsilon = 0.002$, after 4 iterations, the obtained robust state-feedback gain is $K_{rob.} = \begin{bmatrix} -0.00038 & -0.88784 \end{bmatrix}$. Note that the maximum real part of the closed-loop poles is equal to -0.00071.
For continuous-time instances, it is not possible to apply condition of theorem 3. On the contrary, the $\mathcal{LMI}$ condition of theorem 4 leads to similar results for $x = 1$. For $\epsilon = 0.0002$, we get a stabilizing gain $K_{\mathcal{LMI}} = \begin{bmatrix} -41.899 & -3.5356 \end{bmatrix}$.

5.2 Example 2

The second example is given as a discrete-time polytope of matrices with three vertices defining the three following couples of $(A^{[i]}, B^{[i]})$ matrices:

$$\begin{array}{llll} A_1 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ -1 & 2 & 0 \end{bmatrix} & B_1 = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} & A_2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ -2 & 0 & 0 \end{bmatrix} & B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \\ A_3 = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} & B_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} & & \end{array} \tag{29}$$

This polytope of matrices is not quadratically stabilizable and the $\mathcal{LMI}$ approaches proposed in [21] and in the theorem 4 also fail, (the associated $\mathcal{LMI}$'s are found infeasible). After 3 iterations, our algorithm gives the following robust state-feedback gain $K = \begin{bmatrix} -0.0592 & 0.5758 & -0.1080 \end{bmatrix}$.

5.3 Example 3

Let the continuous-time polytope of matrices be given by,

$$A_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \tag{30}$$

The polytope of matrices defined by the two vertices A_1, A_2 is not stable. The $\mathcal{D}_R$ region of pole placement is the intersection between the disk centered at $\alpha_1 = -0.4$ with radius $r = 1$, the conic sector defined by its angle with the vertical $\theta = \pi/6$ and its apex $\alpha_2 = -0.25$ and the left half-plane defined for $x = \alpha_3 = -0.75$. After 7 iterations, the algorithm gives a $\mathcal{D}_R$ stabilizing gain $K = \begin{bmatrix} -0.0809 & -0.3849 \end{bmatrix}$ and $6 = 3 \times 2$ Lyapunov matrices. The figure 2 shows the closed-loop poles of the polytope in the considered region along the convex combination of the closed-loop matrices $\lambda(A_1 + B_1 K) + (1 - \lambda)(A_2 + B_2 K) \quad 0 \leq \lambda \leq 1$. Note that it is impossible to find a disk included in this region for which the method of [21]

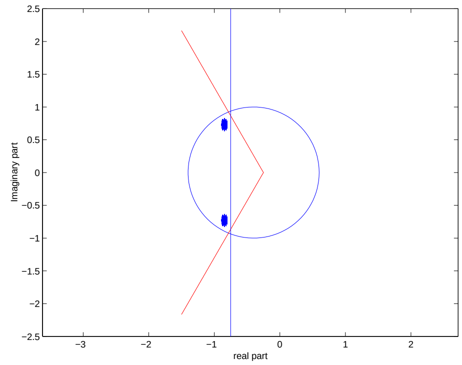

Figure 1: Uncertain system closed-loop poles

or [24] succeeds.

6 Conclusions

In this paper, a tractable state-feedback synthesis method has been proposed for robust regional pole placement in $\mathcal{D}_R$ regions. A recently developped framework based on parameter-dependent Lyapunov functions is used to generalize existing conditions. The main result consists in a sufficient condition involving an $\mathcal{LMI}$ condition and a nonlinear equality relating the different matrix variables. Some pure $\mathcal{LMI}$ relaxations are first proposed. Then, the original nonlinear problem is formulated as a conic complementarity problem for which an efficient linearization approach is used to get at least local solutions. The relevance of the approach is then illustrated by different numerical examples. Note that the proposed algorithm may be numerically improved by considering simple modifications proposed in [20].
The particular class of CCP is likely to encourage us to find generalizations of the numerical methods proposed for LCP and to apply it to our particular control problems. Another current area of research is the synthesis of output-feedback controller assigning the closed-loop poles in a prescribed $\mathcal{D}_R$ region via parameter-dependent Lyapunov functions.

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