Output Feedback Robust Stabilization of Uncertain Linear Systems with Saturating Controls: An LMI Approach

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Abstract : The problem of robust controller design is addressed for an uncertain linear system subject to control saturation. No assumption is made concerning open-loop stability and no a priori information is available regarding the domain of stability. A saturating linear output feedback law and a safe set of initial conditions are determined using a heuristic based on iterative LMI relaxation procedures. A readily implementable algorithm based on standard numerical techniques is described and illustrated on two numerical examples.

1 Introduction

During the last two decades a considerable amount of time has been spent analyzing the question of whether some properties of a system (mainly asymptotic stability) are preserved under the presence of unknown perturbations. Several important findings have appeared in the open literature, leading to procedures for designing the so-called robust controllers, see [9, 17] and references therein.

However these design procedures usually do not take directly into account the presence of control saturation. These physically motivated bounds on system inputs are consequences of technological limitations and/or safety requirements. They have always been a common feature in practical control problems. This justifies the recently renewed interest in the study of linear systems subject to input saturations [3, 15]. Significant results have lately emerged in the scope of global [14] and semi-global stabilization [1, 13]. They inherently require stability assumptions on the open-loop system.

This paper aims at studying linear systems that are not only uncertain but also subject to input saturation. Relaxing open-loop stability assumptions, we focus on a robust local stabilization approach. That is to say, we simultaneously seek a stabilizing feedback law and the associated domain of stability. A new approach for robust saturating controller design is proposed by combining a polytopic representation of saturation nonlinearities and standard quadratic stabilization results. With this formulation, our design algorithm is a readily implementable iterative procedure based on LMI relaxations.

The paper is organized as follows. In Section 2 we introduce the concept of robust local stabilizability and we pose the problem to be addressed. In Section 3 we establish the correspondence, in a given subset of the state space, between the nonlinear saturated system and a polytopic representation. Standard facts on robust stabilization are also recalled. These results are combined

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in Section 4 to address the robust controller design problem via nonlinear matrix inequalities. Several LMI relaxation formulations are proposed and a heuristic iterative algorithm is derived. It is illustrated in Section 5 on two numerical examples borrowed from the control literature. Finally, we draw some concluding remarks.

2 Problem Statement and Motivations

2.1 Uncertain Saturated Linear System

We consider the continuous-time system

$$\begin{array}{rcl} \dot{x} & = & A(F(t))x + B(F(t))u \\ y & = & C(F(t))x + D(F(t))u \end{array} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^m$ is the control input, $y \in \mathbb{R}^p$ is the measurement output and $F(t)$ is a time-varying parameter uncertainty matrix affecting entries of system matrices. System (1) is subject to the following assumptions.

Assumption 1 Uncertainty matrix $F(t)$ is norm-bounded [9, 17] and enters system matrices as follows

$$\begin{bmatrix} A(F(t)) & B(F(t)) \\ C(F(t)) & D(F(t)) \end{bmatrix} = \begin{bmatrix} A_0 & B_0 \\ C_0 & D_0 \end{bmatrix} + \begin{bmatrix} D_1 \\ D_2 \end{bmatrix} F(t)[E_1 \quad E_2]$$

where $F(t) \in \mathcal{F} = \{F \in \mathbb{R}^{q \times r} \; : \; \|F\|_2 \leq 1\}$ for all t.

Assumption 2 Control vector components u_i are bounded. For given scalars $\omega_i > 0$, they verify

$$|u_i| \leq \omega_i, \quad i = 1, \ldots, m. \tag{2}$$

2.2 Closed-loop System

We assume that only partial state information is available through output y. Our output feedback law is generated by a strictly proper full order linear controller

$$\begin{array}{rcl} \dot{x}_c & = & A_c x_c + B_c u_c \\ y_c & = & C_c x_c \end{array} \tag{3}$$

where $x_c \in \mathbb{R}^n$ is the controller state, $u_c \in \mathbb{R}^p$ is the controller input, $y_c \in \mathbb{R}^m$ is the controller output. A convenient way to proceed is to gather all controller parameters into a matrix

$$\Theta = \begin{bmatrix} A_c & B_c \\ C_c & 0 \end{bmatrix}. \tag{4}$$

Let us define the saturation function of a control channel u_i as

$$\mathrm{sat}(u_i) = \begin{cases} \omega_i & \text{if } u_i > \omega_i \\ u_i & \text{if } |u_i| \leq \omega_i \\ -\omega_i & \text{if } u_i < -\omega_i \end{cases}$$

From Assumption 1, the closed-loop relations between system (1) and controller (3) are

$$\begin{array}{rcl} u_c & = & y \\ u & = & \mathrm{sat}(y_c) \; = \; \mathrm{sat}(C_c x_c). \end{array} \tag{5}$$

In order to combine system (1) and controller (3), we define the extended state vector

$$z = \begin{bmatrix} x \\ x_c \end{bmatrix}$$

and corresponding extended matrices

$$\begin{array}{lll} \mathbf{A}_0 = \begin{bmatrix} A_0 & 0 \\ B_c C_0 & A_c \end{bmatrix} & \mathbf{B}_0 = \begin{bmatrix} B_0 \\ B_c D_0 \end{bmatrix} & \mathbf{K} = [0 \quad C_c] \\ \mathbf{D} = \begin{bmatrix} D_1 \\ B_c D_2 \end{bmatrix} & \mathbf{E}_1 = [E_1 \quad 0] & \mathbf{E}_2 = E_2 \\ \mathbf{A}(F(t)) = \mathbf{A}_0 + \mathbf{D}F(t)\mathbf{E}_1 & \mathbf{B}(F(t)) = \mathbf{B}_0 + \mathbf{D}F(t)\mathbf{E}_2. & \end{array}$$

Using relations (5), closed-loop system (1-3) becomes an uncertain nonlinear system

$$\dot{z} = \mathbf{A}(F(t))z + \mathbf{B}(F(t))\mathrm{sat}(\mathbf{K}z). \tag{6}$$

2.3 Local Quadratic Stabilization

Let $\mathbf{K}_i$ stand for the i-th row of matrix $\mathbf{K}$. Given a positive vector $\gamma_0 \in \mathbb{R}^m$, we define the symmetric polyhedron

$$\mathcal{S}_{\gamma_0} = \{z : |\mathbf{K}_i z| \leq \omega_i / \gamma_{0_i}, \ i = 1, \ldots, m\}.$$

If control u does not saturate, that is if $|y_{c_i}| \leq \omega_i \ \forall i = 1, \ldots, m$, or equivalently $z \in \mathcal{S}_1$, then nonlinear system (6) admits the following linear representation

$$\dot{z} = (\mathbf{A}(F(t)) + \mathbf{B}(F(t))\mathbf{K})z. \tag{7}$$

The above model is linear inside $\mathcal{S}_1$ but that does not imply that any trajectory of closed-loop saturated system (6) initiating in this set is a trajectory of linear system (7), see [5]. Therefore it is relevant to characterize a domain of stability $\mathcal{D}_0$ for system (6), such that for any initial condition in $\mathcal{D}_0$ the system converges asymptotically to the origin. Even when a stabilizing feedback is known for system (6) it may not be possible to determine analytically the region of attraction of the origin [15]. Hence the set $\mathcal{D}_0$ appears as an interesting approximation of this region.

When $\mathcal{D}_0$ is an a priori given arbitrary large bounded set, to find a stabilizing feedback control is referred to as the semi-global stabilization [1, 13]. When $\mathcal{D}_0$ is the whole state space, the approach is called the global stabilization [14]. Both methods require stability assumptions on the open-loop system. In the sequel, no particular assumption is made about open-loop stability and no a priori information is available regarding $\mathcal{D}_0$. We aim at finding a stabilizing feedback control together with a safe set $\mathcal{D}_0$ of initial conditions. This approach is referred to as local stabilization [5, 15].

On the other hand, it is now recognized that the concept of quadratic stabilizability introduced by Barmish [2] plays a key role for the robust stabilization of uncertain systems by linear feedback. It relies upon the existence of a unique Lyapunov function for all admissible uncertainties. In our case, without any open-loop stability requirements and with the presence of saturation in feedback (5), the study of local stabilization of system (1) requires for quadratic stabilizability to be studied locally.

Definition 1 *System (1) under Assumptions 1 and 2 is* locally quadratically stabilizable *by output feedback if there exist a dynamic controller (3), a positive definite symmetric Lyapunov matrix P and a set $\mathcal{D}_0$ such that the inequality*

$$[\mathbf{A}_{F(t)}z + \mathbf{B}_{F(t)}\mathrm{sat}(\mathbf{K}z)]'Pz + z'P[\mathbf{A}_{F(t)}z + \mathbf{B}_{F(t)}\mathrm{sat}(\mathbf{K}z)] < 0 \tag{8}$$

holds for all non-zero $z \in \mathcal{D}_0$ and all $F(t) \in \mathcal{F}$.

Note that this definition slightly differs from the one given in [2]. Here inequality (8) must hold for all admissible uncertainties $F(t)$ but *only* in some domain $\mathcal{D}_0$ of the state space. This restriction originates the concept of local quadratic stabilizability.

If $\mathcal{E}_P = \{z : z'Pz \leq 1\}$ stands for the ellipsoid shaped by Lyapunov matrix P, a natural choice of stability domain is the Lyapunov level set

$$\mathcal{D}_0 = \rho^{-\frac{1}{2}}\mathcal{E}_P = \{z : z'Pz \leq \rho^{-1}\} \tag{9}$$

for a suitable positive value of ρ. It is well-known that ellipsoid (9) is positively invariant and contractive [5, 15] whenever condition (8) holds. Since every homothetic set $\varrho\mathcal{D}_0$ for $\varrho \leq 1$ is also positively invariant and contractive, the Lyapunov level set $\mathcal{D}_0$ should be as large as possible. Based on these considerations, the problem we address in this paper is as follows.

Local Quadratic Stabilization (LQS) Problem *Given system (1) under Assumptions 1 and 2, find a controller matrix Θ and the largest possible Lyapunov level set $\mathcal{D}_0$ such that condition (8) holds for uncertain saturated system (6) in $\mathcal{D}_0$.*

In this form, the LQS problem is very hard, if not impossible, to solve. This paper does not pretend to give a general solution of this problem, but rather presents approximation techniques that provide a tractable heuristic design procedure.

3 Equivalent Polytopic Representation

When trying to tackle the LQS problem, the difficulties stem from two different points: actuator saturation and presence of uncertainties. In the sequel, we show that these two problems can be approached independently and respectively by a polytopic model of saturation nonlinearities (Section 3.1) and an LMI formulation of robust stabilization (Section 3.2), possibly at the expense of some conservatism. The results exposed in this section are quite standard. They are reformulated here for the sake of clarity.

3.1 Polytopic Model of Saturation Nonlinearities

System (6) can be written

$$\dot{z} = \mathbf{A}(F(t))z + \mathbf{B}(F(t))\mathbf{G}_{\gamma(z)}\mathbf{K}z.$$

where $\mathbf{G}_{\gamma(z)}$ is a diagonal matrix whose components are

$$\gamma_i(z) = \begin{cases} \omega_i/\mathbf{K}_i z & \text{if } \mathbf{K}_i z > \omega_i \\ 1 & \text{if } |\mathbf{K}_i z| \leq \omega_i \\ -\omega_i/\mathbf{K}_i z & \text{if } \mathbf{K}_i z < -\omega_i. \end{cases}$$

for $i = 1, \ldots, m$. Note that $\gamma_i(z)$ lies in the interval $]0, 1]$ for any vector z. When γ_i approaches 0 there is almost no feedback from input u_i, whereas $\gamma_i = 1$ simply means that u_i does not saturate.

Recalling our formulation of the LQS problem, the control objective consists in constraining the domain of evolution of the state of system (6) to the Lyapunov level set $\mathcal{D}_0$. Recall that $\mathcal{D}_0$ is compact, positively invariant and contractive. Therefore for any $z \in \mathcal{D}_0$, components of vector $\gamma(z)$ admit a lower bound

$$\gamma_{0_i} = \min \{\gamma_i(z) : \forall z \in \mathcal{D}_0\}$$

such that $0 < \gamma_{0_i} \leq \gamma_i(z) \leq 1$. Given such a vector γ_0 we define the vertex matrices

$$\mathbf{A}(F(t)) + \mathbf{B}(F(t))\mathbf{G}^j_{\gamma_0}\mathbf{K} \tag{10}$$

where the $\mathbf{G}^j_{\gamma_0}$ are the 2^m vertex diagonal matrices whose elements can take the value 1 or γ_{0_i} [15]. In the polyhedron

$$\mathcal{S}_{\gamma_0} \supset \mathcal{D}_0 \tag{11}$$

we use difference inclusions results [15] to describe system (6) by its polytopic representation

$$\dot{z} = [\sum_{j=1}^{2^m} \lambda_j \cdot (\mathbf{A}(F(t)) + \mathbf{B}(F(t))\mathbf{G}^j_{\gamma_0}\mathbf{K})]z \tag{12}$$

with $\lambda_j \geq 0$ for $j = 1, \ldots, 2^m$ and $\sum_{j=1}^{2^m} \lambda_j = 1$. For any $z \in \mathcal{S}_{\gamma_0}$, the state transition matrix in (12) belongs to a convex hull of matrices whose vertices are given in (10). In view of inclusion relation (11), the following result has been shown

Lemma 1 *In set $\mathcal{D}_0$, trajectories of nonlinear system (6) can be represented by trajectories of polytopic system (12)*

Note however that the converse to Lemma 1 is not true: some trajectories of polytopic system (12) do not belong to the set of trajectories of nonlinear system (6). As a result, some conservatism may be introduced when replacing representation (6) by representation (12).

3.2 LMI Approach to Robust Stabilization

In this section we propose an LMI approach to the design of a controller (3) that stabilizes system (1) in presence of the uncertainties described in Assumption 1, but without considering actuator constraints of Assumption 2.

Lemma 2 *System (1) under Assumption 1 is quadratically stabilizable by controller (3) if and only if there exist symmetric matrices R and S solutions to the LMI set*

$$\begin{gathered}
\mathcal{N}_R' \begin{bmatrix} AR + RA' + D_1 D_1' & RE_1' \\ E_1 R & -I_r \end{bmatrix} \mathcal{N}_R < 0 \\
\mathcal{N}_S' \begin{bmatrix} A'S + SA + E_1'E_1 & SD_1 \\ D_1'S & -I_q \end{bmatrix} \mathcal{N}_S < 0 \\
\begin{bmatrix} R & I_n \\ I_n & S \end{bmatrix} \geq 0
\end{gathered} \tag{13}$$

where $\mathcal{N}_R$ and $\mathcal{N}_S$ stand for the null bases of $[B' \quad E_2']$ and $[C \quad D_2]$ respectively.

Proof The frequency-domain interpretation of quadratic stability can be found for instance in Section 5.1 in [4]. Inequalities (13) arise from the LMI formulation of the existence of suboptimal controllers with $\mathcal{H}_\infty$ norm less than one [7]. $\square$

Suppose that LMI set (13) is feasible. A closed-loop Lyapunov matrix

$$P = \begin{bmatrix} S & N \\ N' & T \end{bmatrix} \tag{14}$$

is then obtained as the unique solution to linear equation

$$\begin{bmatrix} S & I \\ N' & 0 \end{bmatrix} = P \begin{bmatrix} I & R \\ 0 & M' \end{bmatrix}$$

for M, N full rank matrices such that $MN' = I - RS$. Given such a P and with the shorthands

$$\mathbf{M} = \left[\begin{array}{cc|c|c} 0 & I & 0 & 0 \\ C & 0 & D_2 & 0 \end{array}\right] \quad \mathbf{N}_P = \left[\begin{array}{cc|c|c} N' & T & 0 & 0 \\ B'S & B'N & 0 & E_2' \end{array}\right] \quad \Psi_P = \left[\begin{array}{cc|c|c} A'S + SA & A'N & SD_1 & E_1' \\ N'A & 0 & N'D_1 & 0 \\ \hline D_1'S & D_1'N & -I & 0 \\ \hline E_1 & 0 & 0 & -I \end{array}\right]$$

the Bounded Real Lemma inequality is used in [7] to show the following result.

Lemma 3 *Suppose that LMI set (13) is feasible and yields a Lyapunov matrix P as in (14). Then, provided P is fixed, the following LMI in Θ*

$$\Psi_P + \mathbf{M}'\Theta'\mathbf{N}_P + \mathbf{N}_P{}'\Theta\mathbf{M} < 0 \tag{15}$$

parameterizes controllers (3) that quadratically stabilize system (1) under Assumption 1.

Since Lyapunov matrix P is not allowed to vary in LMI (15), it must be noted that the above lemma does not necessarily provide a convex parameterization of the whole set of quadratically stabilizing controllers for system (1). Different convex sets of stabilizing controllers may be described for different choices of Lyapunov matrices. This is an additional potential source of conservatism.

4 Solution to the LQS Problem

In the preceding section, we have shown that saturation nonlinearities and robust stabilization can be dealt with using a polytopic representation and an LMI formulation, respectively. Our objective now is to combine both approaches to solve the LQS problem. The following theorem is our main result.

Theorem 1 *Suppose that LMI set (13) is feasible. If there exist a matrix P as in (14), a matrix Θ as in (4), a vector γ_0 and a scalar ρ such that*

$$\begin{gathered} \Psi_P + \mathbf{M}'\Theta'\mathbf{\Gamma}^j_{\gamma_0}\mathbf{N}_P + \mathbf{N}_P{}'\mathbf{\Gamma}^j_{\gamma_0}\Theta\mathbf{M} < 0, \quad j = 1, \ldots, 2^m \\ \begin{bmatrix} P & \gamma_{0i}\mathbf{K}_i' \\ \gamma_{0i}\mathbf{K}_i & \rho\omega_i^2 \end{bmatrix} \geq 0, \quad 0 \leq \gamma_{0i} \leq 1, \quad i = 1, \ldots, m \end{gathered} \tag{16}$$

where $\mathbf{\Gamma}^j_{\gamma_0} = \mathrm{blockdiag}(I, \mathbf{G}^j_{\gamma_0})$, then controller matrix Θ and Lyapunov level set $\mathcal{D}_0 = \rho^{-\frac{1}{2}}\mathcal{E}_P$ solve the LQS problem.

Proof According to Lemma 2, existence of matrices R and S solutions to LMI system (13) guarantees that system (1) is quadratically stabilizable by an unconstrained controller (3). In view of Lemmas 1 and 3, saturated closed-loop system (6) admits polytopic representation (12) thus LMI (15) must hold for all vertices defined in (10). Hence the first set of matrix inequalities in controller matrix Θ and diagonal matrices $\mathbf{\Gamma}^j_{\gamma_0}$, which parameterizes saturated quadratically stabilizing controllers. Since polytopic representation (12) is only valid in polyhedron $\mathcal{S}_{\gamma_0}$, the second set of inequalities are LMIs corresponding to the inclusion $\mathcal{D}_0 \subset \mathcal{S}_{\gamma_0}$, see Section 7.2.3 in [4]. □

The main difficulty regarding application of Theorem 1 resides in the fact that matrix inequalities (16) are multilinear in the three unknowns P, Θ and γ_0. For instance, if Lyapunov matrix P is given, $\mathbf{\Gamma}^j_{\gamma_0}$ acts as a scaling matrix for controller matrix Θ and equations (16) are only bilinear matrix inequalities (BMIs, see [11]). Further, when two matrix variables are fixed, equations (16) become linear matrix inequalities in the third variable.

Multilinear matrix inequalities are common feature of robust control problems. Since they are generally not convex, their numerical properties are usually hard to characterize. Nevertheless, there exist relatively efficient methods to deal with such nonlinear problems. Relaxation techniques are frequently used and reported to be useful in practice, see for instance [10]. In the sequel, we propose some relaxation schemes that aim at addressing the LQS problem using Theorem 1 and standard numerical tools.

5 Algorithm

Our relaxation schemes are based on LMIs, for which powerful computer tools are available. Recall that nonlinear matrix inequalities (16) feature a product of three unknown matrix variables. Thus, in order to get an LMI two variables must be fixed while varying the remaining one. We propose the following LMI relaxations (LMIR) to multilinear matrix inequalities (16).

LMIR 1	Given P and Θ, solve for γ_0 and ρ the problem	$\min \sum_{i=1}^m \gamma_{0i}$	s.t. LMI set (16).
LMIR 2	Given P and γ_0, solve for Θ and ρ the problem	$\min \rho$	s.t. LMI set (16).
LMIR 3	Given Θ and γ_0, solve for P and ρ the problem	$\min \operatorname{trace} P$	s.t. LMI set (16).
LMIR 4	Given Θ and γ_0, solve for P and ρ the problem	$\min \log\det P^{-1}$	s.t. LMI set (16).

Meanings of the proposed LMI relaxation schemes are as follows. As stated in the LQS problem formulation, the set $\mathcal{D}_0 = \rho^{-\frac{1}{2}}\mathcal{E}_P$ must be as large as possible. Since the volume of $\mathcal{D}_0$ is proportional to the determinant of $(\rho P)^{-\frac{1}{2}}$, maximizing $\mathcal{D}_0$ can be achieved either by minimizing the scalar factor ρ (LMIR 2), or by minimizing $\operatorname{trace} P$ (LMIR 3), or also by minimizing the convex criterion $\log\det P^{-1}$ (LMIR 4). Moreover, in view of inclusion relation (11), components of vector γ_0 must also be minimized so as to enlarge polyhedron $\mathcal{S}_{\gamma_0}$ and allow maximum saturation within $\mathcal{D}_0$, hence LMIR 1. Our algorithm will alternate between these four relaxation procedures.

Design Algorithm

Step 1 Solve LMI set (13) for R and S and build closed-loop Lyapunov matrix P as in (14).

Step 2 Solve LMI set (15) for controller matrix Θ. Let $\gamma_{0_i} = 1$ for all $i = 1, \ldots, m$.

Step 3 Solve LMIR 1, 2, 3 or 4.

Step 4 If some condition on the volume of $\mathcal{D}_0 = \rho^{-\frac{1}{2}}\mathcal{E}_P$ is fulfilled, then stop. Otherwise go to Step 3.

In the above algorithm, the key point is the selection of relaxation procedure in Step 3. Apparently, it does not seem to be possible to know beforehand which sequence features the best rate of convergence. Moreover, the stopping criterion in Step 3 depends on the algorithm behavior. Clearly, further iterations always increase the volume of $\mathcal{D}_0$. As there are no convergence properties available for this type of relaxation algorithms, one can, for instance, decide to stop the process if there is no significant variation of the volume of $\mathcal{D}_0$. Our implementation of the design algorithm allows the user to select the relaxation procedure of his choice at each step or, alternatively, to stop the iterative process.

6 Numerical Examples

The design algorithm is illustrated on two examples. In both cases, our linear feedback law significantly improves previously obtained results. LMIR 1, 2 and 3 were implemented using the MATLAB[1] LMI Toolbox [8]. LMIR 4 was implemented using the software MAXDET [16].

6.1 First Example

Consider the system (1) given in [12] with nominal system matrices

$$A_0 = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3 \end{bmatrix} \quad B_0 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix} \quad C_0 = I \quad D_0 = 0.$$

Suppose that $A(F(t))$ and $B(F(t))$ are affected by additive uncertainties of norm less than or equal to 0.1 [12]. The uncertainty matrices were chosen as follows

$$D_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad E_1 = \frac{1}{20}[1 \quad 1] \quad D_2 = 0 \quad E_2 = 1.$$

In addition to that, actuators are constrained as in (2) by saturation levels $\omega_1 = 5$ and $\omega_2 = 2$. The goal of the authors in [12] was to derive the largest set of initial conditions for which the optimal control state feedback law

$$u = -\begin{bmatrix} 0.7283 & 0.0338 \\ 0.0135 & 1.3583 \end{bmatrix} x$$

ensures robust stabilization even when saturation occurs. The largest initial state set they obtained was $\mathcal{B} = \{x : \|x\|_2 \leq 47.59\}$. With the relaxation sequence shown in Table 1, our design algorithm yields controller matrices

$$A_c = \begin{bmatrix} -171.2 & 27.06 \\ -68.00 & -626.8 \end{bmatrix} \quad B_c = \begin{bmatrix} -598.2 & 5.539 \\ -4.567 & 149.8 \end{bmatrix} \quad C_c = \begin{bmatrix} 0.146 & 0.088 \\ -6.821 & -5.670 \end{bmatrix}.$$

LMIR	2	3	1	3	1	3	1	3	1	3
log vol $\mathcal{D}_0$	-1.358	0.5217	7.943	9.803	9.804	10.04	10.05	11.02	11.04	11.61

Table 1: Relaxation Sequence for Volume Maximization

[1] MATLAB is a trademark of the MathWorks, Inc.

Assuming that the controller initial state is set to zero ($x_c = 0$), we obtain the initial state set $\mathcal{D}_0 = \{x : x'Sx \leq 1\}$ where

$$S = 10^{-6} \begin{bmatrix} 290.5 & -7.966 \\ -7.966 & 0.500 \end{bmatrix}$$

is the upper left Lyapunov matrix given in (14). As shown in Figure 1, our feedback law considerably enlarges the set of safe initial conditions in comparison with [12]. Simulations have been carried out for different constant values of uncertainty matrix $F(t)$ and initial condition $x(0) = [70 \quad 1740]'$. As we can see in Figures 2, 3 and 4, despite control saturation, the closed-loop system remains asymptotically stable.

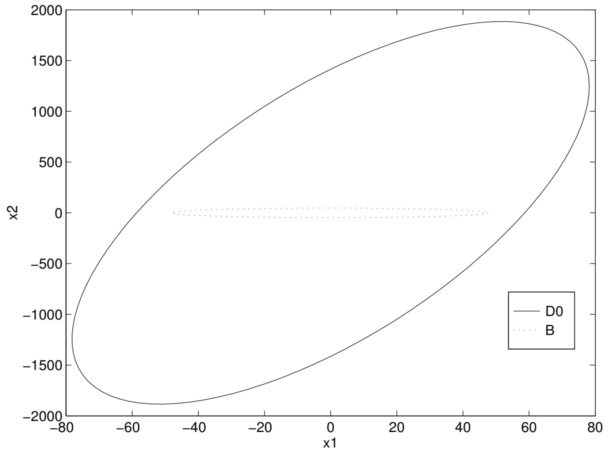

Figure 1: Initial condition sets

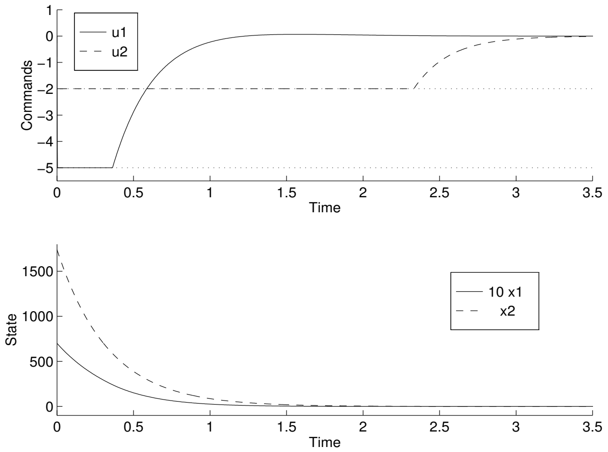

Figure 2: Simulation for $F(t) \equiv 0$

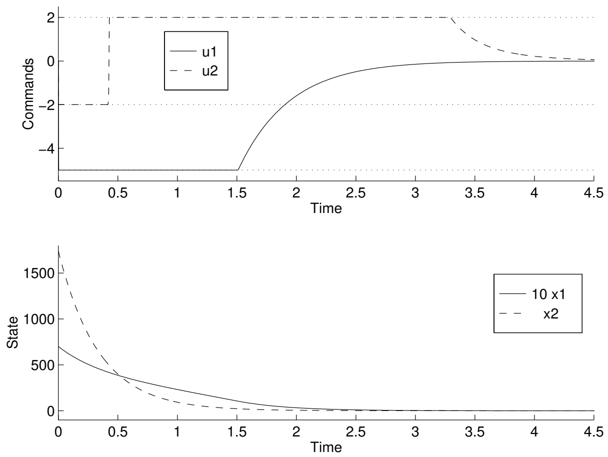

Figure 3: Simulation for $F(t) \equiv 1$

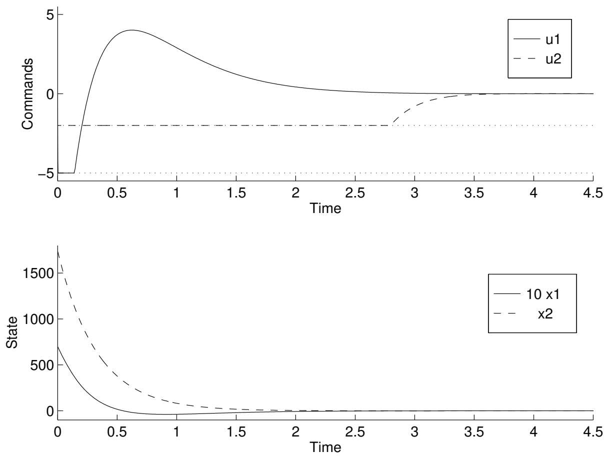

Figure 4: Simulation for $F(t) \equiv -1$

6.2 Second Example

Let us consider now the linearized equations of motion for a satellite given in [6] :

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3p^2 & 0 & 0 & 2p \\ 0 & 0 & 0 & 1 \\ 0 & -2p & 0 & 0 \end{bmatrix} x + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} u$$

where control constraints (2) are $\omega_1 = \omega_2 = 15$. Uncertain parameter p represents the period and lies within the interval $[0.5, 1.5]$. To represent uncertainty on p as in Assumption 1, we have chosen the following system matrices

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.75 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 0 \end{bmatrix} \quad B_0 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \quad C_0 = I \quad D_0 = 0$$
$$D_1 = [0 \quad 3 \quad 0 \quad 1]' \quad E_1 = [1 \quad 1 \quad 0 \quad 0.333] \quad D_2 = 0 \quad E_2 = 0.$$

In [6], the authors show that the linear state feedback

$$u = \begin{bmatrix} -19 & -13 & -2 & 0 \\ 0 & 2 & -7 & -8 \end{bmatrix} x \tag{17}$$

stabilizes the saturated linear system when $p = 1$ for any initial condition in the unit sphere, i.e. $x(0) \in \{x : \|x\|_2 \leq 1\}$. However, as stated in [6], the robust feedback (17) is not necessarily stabilizing for all admissible uncertainties.

LMIR	4	1	3	1	3	1	3
log vol $\mathcal{D}_0$	3.967	4.443	5.151	5.206	9.160	9.185	10.60

Table 2: Relaxation Sequence for Volume Maximization

With the relaxation sequence shown in Table 2, our algorithm returned controller matrices

$$\begin{aligned} A_c &= \begin{bmatrix} -77.44 & -44.16 & -10.51 & -42.03 \\ 93.04 & 53.70 & 9.254 & 46.04 \\ -74.51 & -35.87 & -36.49 & -81.96 \\ 50.30 & 25.62 & 18.98 & 46.36 \end{bmatrix} \quad B_c = \begin{bmatrix} 111.2 & -134.5 & -87.29 & -113.5 \\ -248.2 & 149.2 & -19.86 & 207.2 \\ -56.23 & -63.04 & -725.2 & 312.9 \\ -12.94 & 53.84 & 357.1 & -127.0 \end{bmatrix} \\ C_c &= \begin{bmatrix} 1.372 & 0.924 & -0.236 & -0.093 \\ 1.261 & 0.739 & 0.148 & 0.650 \end{bmatrix} \end{aligned} \tag{18}$$

and the initial state set $\mathcal{D}_0 = \{x : x'Sx \leq 1\}$ where

$$S = 10^{-3} \begin{bmatrix} 420.7 & 254.8 & -0.396 & 174.9 \\ 254.8 & 191.9 & -1.594 & 95.14 \\ -0.396 & -1.594 & 0.838 & 2.129 \\ 174.9 & 95.14 & 2.129 & 80.47 \end{bmatrix}$$

is the upper left Lyapunov matrix given in (14). The maximal eigenvalue of S is 0.6543, hence the unit sphere belongs to $\mathcal{D}_0$. With contrast to robust feedback (17), our controller ensures that the uncertain system can be stabilized in the unit sphere and even farther. Trajectories of the system controlled by feedback law (17) are shown in Figure 5 for $F(t) \equiv 0$ and an initial condition $x(0) = [-50 \quad 16 \quad -230 \quad 97]'$ that do not belong to the unit sphere. As we can see, the system is unstable. Figure 6 shows the trajectories of the system driven by controller (18) under the same conditions.

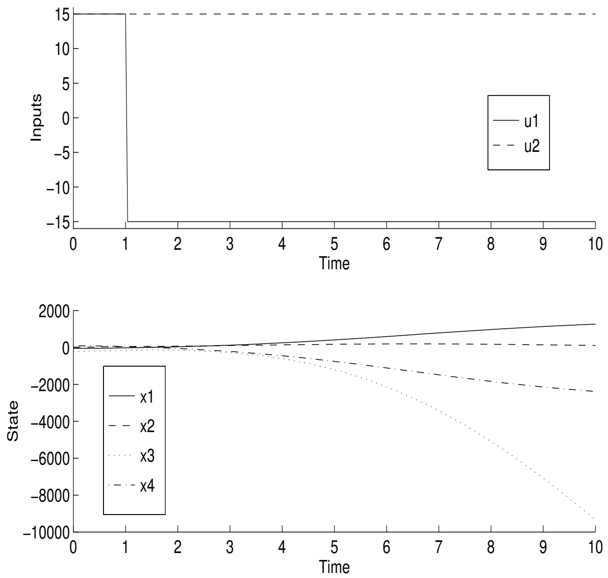

Figure 5: Simulation with feedback (17)

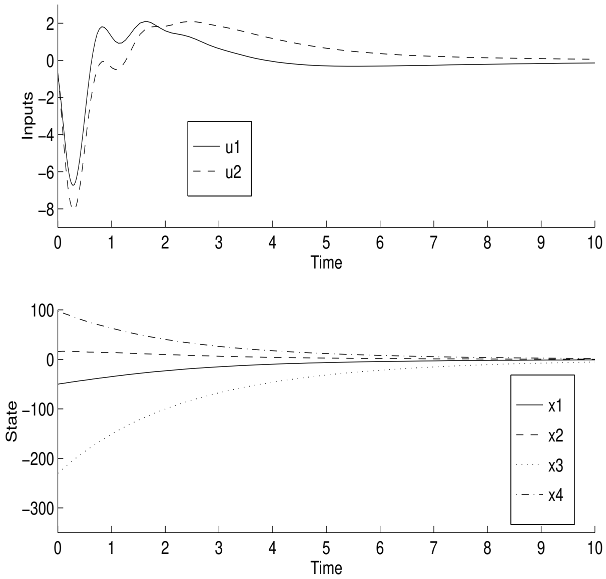

Figure 6: Simulation with controller (18)

7 Conclusion

Based on the local representation of a nonlinear saturated system by a linear polytopic model, we proposed an output feedback law that robustly stabilizes a system subject to control saturation. Our design algorithm is based on successive LMI relaxation procedures.

As for other approaches in the same vein (μ synthesis, D-K iterations), the design procedure is heuristic and its speed of convergence is unclear. The proposed solution inherently suffers from different sources of conservatism induced by the polytopic representation, the existence of a unique and constant Lyapunov matrix for the closed-loop system and the non-convex nature of the problem formulation. This is not surprising and is actually common feature in the most part of robust control problems.

However, our design algorithm can provide satisfactory solutions to robust stabilization in presence of actuator saturation, for which very few design methods have been proposed so far. Its applicability to control problems is illustrated by two numerical examples borrowed from the control literature. In addition to that, the algorithm only hinges upon standard and well-worked numerical tools. It results in an easily implementable linear feedback law and therefore should be of immediate practical use.

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