České vysoké učení technické v Praze
Fakulta elektrotechnická

Czech Technical University in Prague
Faculty of Electrical Engineering

Doc. Ing. Didier Henrion, Ph.D.

Míry a lineární maticové nerovnosti v optimálním polynomiálním řízení

Measures and linear matrix inequalities in polynomial optimal control

Summary

This lecture describes the application of modern techniques of convex optimization to solve nonconvex nonlinear optimal control problems (OCPs) which may feature oscillation phenomena (chattering control) or concentration phenomena (impulsive control).

First, with the help of occupation measures, we reformulate nonconvex nonlinear OCPs as convex linear programming (LP) problems on the cone of nonnegative measures.

Second, relying on recent results merging techniques of real algebraic geometry, functional analysis (measures, problems of moments) and mathematical programming (semidefinite optimization), we propose a general methodology to solve numerically these infinite-dimensional LP on measures. We describe a hierarchy of convex semidefinite programming (SDP) or linear matrix inequality (LMI) relaxations that can be implemented to solve OCP with asymptotic convergence guarantees.

Souhrn

Tato přednáška popisuje, jak aplikovat moderní techniky konvexní optimalizace na řešení problémů optimálního řízení nelineárních a nekonvexních systémů, které mohou obsahovat jevy oscilace a koncentrace.

Nejprve pomocí měr obsazenosti vyjádříme nelineární a nekonvexní problémy optimálního řízení jako konvexní problémy lineárního programování vkuželu nezáporných měr.

Poté na základě nejnovějších výsledků, které spojily techniky reálné algebraické geometrie, funkcionální analýzy (míry, problémy momentů) a matematického programování (semi-definitní optimalizace), navrhujeme všeobecnou metodologii numerických řešení těchto nekonečněrozměrných problémů lineárního programování měr. Popisujeme hierarchii konvexních semi-definitních problémů neboli lineární maticové nerovnosti, která může být použita křešení problémů optimálního řízení se zárukou asymptotické konvergence.

Klíčová slova:

Optimální řízení, konvexní optimalizace.

Keywords:

Optimal control, Convex optimization.

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Contents

1 Introduction

This lecture is about the application of modern techniques of convex optimization to solve nonconvex nonlinear problems of optimal control which do not admit classical solutions. Optimal control is a discipline at the border between engineering and applied mathematics, which focuses on modifying optimally the behavior of dynamical systems modeled by differential equations, so as to ensure stability and performance requirements. It can be interpreted as an outgrowth of the calculus of variations, a field of functional analysis that deals with optimizing a functional subject to constraints. An elementary problem of calculus of variations is that of finding the curve of shortest length, or geodesic, connecting two points. The control can then be interpreted as the derivative, or velocity of the curve. Since the 17th century and the study of the brachistochrone curve [43], problems of calculus of variations and optimal control have been at the source of many developments of functional analysis, see e.g. [10] and [11]. A central reference on optimal control is the historical book by the Soviet school [35], and a very readable introductory account with historical perspective can be found in [26].

Generally speaking, an optimal control problem (OCP) can be formulated as a nonlinear optimization problem in a functional, or vector space (e.g. the set of continuously differentiable functions, or the set of square integrable functions). It is a central question of functional analysis to determine whether a given OCP possesses a solution in the given space. Continuity, coercivity and convexity of the functional to be optimized often ensure existence of an optimal solution. However, for many OCPs of physical relevance, these are too strong requirements, and existence of an optimal solution cannot be guaranteed if the vector space is too small. Commonly, the optimal control law can be approached arbitrary well by a sequence of admissible functions, but these are typically increasingly ill-behaved. The question of enlarging the space over which optimization is carried out for OCP has been extensively studied in the 1960s and 1970s [48, 47, 13] and it is a central theme in the excellent textbook [28]. Roughly speaking, for a sufficiently regular nonlinear functional on a sufficiently regular constraint set, the space over which optimization is carried out should be the dual (i.e. the set of linear functionals) of a Banach space. In the context of OCP, a larger enough space is the set of bounded linear functionals on the set of continuous functions on a closed time interval. This is the space of measures of bounded total variation, see [38, 17, 40].

A first key idea conveyed by this lecture is that formulating OCP in the space of measures allows to deal easily with two typical limit phenomena:

- **oscillation** effects, or **chattering controls**, which correspond to control laws with infinitely fast variations, but bounded amplitude;
- **concentration** effects, or **impulsive controls**, which correspond to control laws active on an arbitrarily small time period, resulting in a bounded discontinuity of the state trajectory.

Quite often, these phenomena are difficult to account for and to handle by classical numerical methods for OCP. Chattering or impulsive controls are however implementable physically, and desirable in practice [3, 20, 49]. It turns out that OCP problems can be formulated appropriately when lifted, or relaxed, in the space of **measures**, which can be interpreted as generalized functions. We will see that a nonlinear nonconvex OCP problem can be reformulated as a convex **linear programming** (LP) problem on the cone of nonnegative measures. Historically, the idea of reformulating nonconvex nonlinear ordinary differential equations (ODE) into convex LP, and especially linear partial differential equations (PDE) in the space of probability measures, is not new. It was Joseph Liouville in 1838 who first introduced the linear PDE involving the Jacobian of the transformation exerted by the solution of an ODE on its initial condition [27]. The idea was then largely expanded in Henri Poincaré's work on dynamical systems at the end of the 19th century, see in particular [32, Chapitre XII (Invariants intégraux)]. This work was pursued in the 20th century in [19], [29, Chapter VI (Systems with an integral invariant)] and more recently in the context of optimal transport by e.g. [36], [45] or [1]. Poincaré himself in [33, Section IV] mentions the potential of formulating nonlinear ODEs as linear PDEs, and this programme has been carried out to some extent by [8], see also [22], [18], [42], [46], [5], [37], [16] and more recently [2], [44], [12]. For recent studies of the Liouville equation in optimal control see e.g. [21] and [7].

A second key idea of the lecture is that recent progress in convex optimization, real algebraic geometry and functional analysis can be exploited to **solve numerically** LP problems in the cone of nonnegative measures, under the assumption (not very restrictive in practice) that all the data are **polynomials** (performance measure, dynamics, constraints). The main message of the book [24] is that we can formulate the infinite-dimensional LP on measures as a generalized moment problem, and we approach it via an asymptotically converging hierarchy of finite-dimensional convex linear matrix inequality (LMI) relaxations. These LMI relaxations are then modeled with our specialized software GloptiPoly 3 [14], solved with a general-purpose **semidefinite programming** (SDP) solver, e.g. SeDuMi [34].

The developments in these lecture notes follow our papers [23], [9] and [15], but the presentation is streamlined and therefore we hope that the technical material is more accessible.

2 Motivating examples

In this section we recall prototypical examples from optimal control which are originally not well formulated, in the sense that the optimal control law is not attained in the given space. These problems can be solved appropriately by enlarging the space, which amounts to relaxing the problem.

Our general setup for an optimal control problem (OCP) is the following:

$$\begin{array}{rll} v^* := & \inf & \displaystyle\int_{t_I}^{t_F} l(t, x(t), u(t))dt \\ & \text{s.t.} & \dot{x}(t) = f(t, x(t), u(t)), \\ & & x(t) \in X, \quad t \in [t_I, t_F], \\ & & x(t_I) \in X_I, \quad x(t_F) \in X_F \end{array} \tag{1}$$

where the infimum is with respect to a function

$$u \in V([t_I, t_F]; U)$$

called control law, or input control, mapping the time interval $T := [t_I, t_F] \subset \mathbb{R}$ to a given input constraint set $U \subset \mathbb{R}^m$, and belonging to a given vector space V. The dot denotes differentiation w.r.t. time. Also given are the smooth (i.e. infinitely-differentiable) integrand l (sometimes called Lagrangian), the smooth dynamics f, the state constraint set $X \subset \mathbb{R}^n$, the initial state constraint set $X_I \subset \mathbb{R}^n$ and the final state constraint set $X_F \subset \mathbb{R}^n$. The initial time t_I is given (typically equal to 0), and the final time t_F is either given or unknown.

A control law $u(t)$ such that the trajectory $x(t)$ satisfies all the constraints in problem (1) is called admissible. In problem (1), the dynamics are driven by a first-order ordinary differential equation (ODE)

$$\dot{x}(t) = f(t, x(t), u(t)) \tag{2}$$

meaning that the state trajectory can be obtained by integration

$$x(t) = x(t_I) + \int_{t_I}^{t} f(s, x(s), u(s))ds. \tag{3}$$

Since the vector field f is smooth, there is a unique solution (3) to ODE (2), see e.g. [6, Theorem 2.1.3].

2.1 Bolza

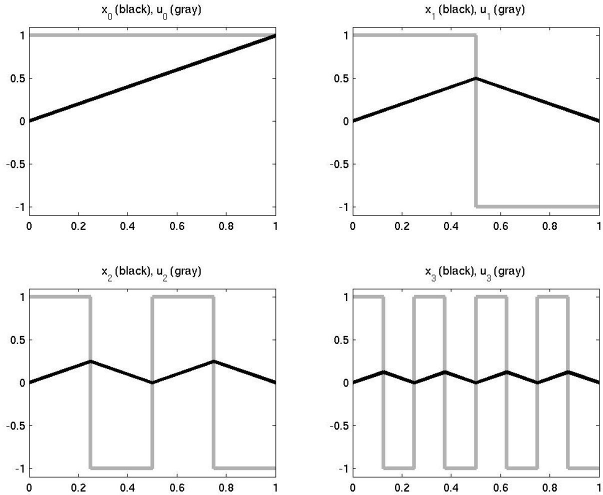

Figure 1: Sequences of state trajectories and control inputs for Bolza's example.

This is a classical example of calculus of variations illustrating that an OCP with smooth data (infinitely differentiable Lagrangian and dynamics, no state or input constraints) can have a highly oscillatory optimal solution, see [10, Example 4.8].

Consider the OCP

$$\begin{aligned} v^* := \quad & \inf \quad \int_0^1 x^4(t) + (u^2(t)-1)^2 dt \\ & \text{s.t.} \quad \dot{x}(t) = u(t), \quad t \in [0,1], \\ & \qquad\;\; x(0) = 0, \quad x(1) = 0 \end{aligned}$$

where the infimum is w.r.t. a control $u \in L^1$, Intuitively, the state trajectory $x(t)$ should remain zero, and the velocity $\dot{x} = u$ should be equal to $+1$ or -1, so that the nonnegative Lagrangian $l(t, x(t), u(k)) = x^4(t) + (u^2(t) - 1)^2$ remains zero, and hence the objective function is zero, the best we can hope. We can build a sequence of bang-bang controls $u_k(t)$ such that for each $k = 0, 1, 2, \ldots$ the corresponding state trajectory $x_k(t)$ has a sawtooth shape, see Figure 1. With such a sequence the objective function tends to $\lim_{k\to\infty}\int_0^1 l(t,x_k(t),u_k(t))dt = \int_0^1 x_k^4(t)dt = 0$ and hence $v^* = 0$. This infimum is however not attained with a control law $u(t)$ belonging to the space of Lebesgue integrable functions.

In Section 4 we will see that instead of optimizing over a control law u which is a time-dependent function, we may optimize over a relaxed control law which is a time-dependent probability measure. In other words, we enlarge our control space to the space of probability measures.

Time-dependent probability measures are called Young measures in the control literature, see e.g. [48, 47]. They have been widely used also in connection with PDE problems, see e.g. [31, 39]. Such relaxed controls, also called chattering controls, can appropriately model oscillation phenomena, see [49].

2.2 Luenberger

Consider the OCP of [28, Ex. 3, p. 125]

$$\begin{array}{rll} v^* := & \inf & \displaystyle\int_0^{t_F} |u(t)|dt \\ & \text{s.t.} & \dot{x}_1(t) = x_2(t), \\ & & \dot{x}_2(t) = -1 + u(t), \quad t \in [0, t_F] \\ & & x_1(0) = 0, \quad x_2(0) = 0, \quad x_1(t_F) = 1 \end{array}$$

which consists of selecting the thurst program $u(t)$ for a (highly simplified) vertically ascending rocket-propelled vehicle, subject only to the forces of gravity and rocket thrust in order to reach a given altitude with minimum fuel expenditure. The above infimum is w.r.t. a control $u \in L^\infty$ and also w.r.t. the final time t_F which is unspecified.

It can be shown that $v^* = \sqrt{2}$ but this value cannot be reached by a control which is a measurable function of time or state. The optimal control is actually an impulse at time $t = 0$, allowing the state to jump immediately from $x_1 = 0$ to $x_1 = 1$.

In Section 5 we will indeed see that the infimum can be achieved by considering a larger class of controls, namely measures of time, allowing for example Dirac measures. Such controls are called impulsive controls [3, 20], and they can appropriately model concentration phenomena.

3 Occupation measures for optimal control

Let us use the following notations. Let $I_A(\cdot)$ denote the indicator function of a set A, i.e., a function equal to 1 on A and 0 elsewhere. Let λ denote the Lebesgue measure on $X \subset \mathbb{R}^n$, i.e. the standard n-dimensional volume. Let $\operatorname{spt}\mu$ denote the support of a measure μ, that is, the closed set of all points x such that $\mu(A) > 0$ for every neighborhood A of x. For a set X let $M(X)$ denote the Banach space of signed Borel measures supported on X, so that a measure $\mu \in M(X)$ can be interpreted as a function that takes any subset of X and returns a number in $\mathbb{R}$. Alternatively, elements of $M(X)$ can be interpreted as linear functionals acting on the Banach space of continuous functions $C(X)$, that is, as elements of the dual space $C(X)'$. The action of a measure $\mu \in M(X)$ on a test function $v \in C(X)$ can be modeled with the duality pairing

$$\langle v,\ \mu \rangle \ := \int_X v(x)\, d\mu(x).$$

Nonnegative measures $\mu \in M(X)$ are such that $\mu(A) \geq 0$ for all Borel subsets $A \in X$. The set of all nonnegative measures is a cone, since it is closed under addition and multiplication of a nonnegative constant. Nonnegative measures $\mu \in M(X)$ which are normalized such that $\mu(X) = 1$ are called probability measures, and the set of probability measures on X is denoted by $P(X)$.

3.1 Occupation measures of ODEs

Admissible trajectories in OCP (1) are absolutely continuous functions of time $x(.)$ with values in X, satisfying initial and final constraints, and such that there exists a measurable control $u(.)$ with values in U:

$$\begin{aligned}
&x(t) = x_I + \int_T f(t, x(t), u(t))dt \in W^{1,1}(T; X), \quad T := [t_I, t_F],\\
&x_I := x(t_I) \in X_I, \quad x_F := x(t_F) \in X_F,\\
&u(t) \in L^1(T; U).
\end{aligned}$$

Given an initial condition $x_I \in X_I$ and an admissible trajectory $x(.)$, with its corresponding control $u(.)$, define the **occupation measure**

$$\mu(A \times B \times C \,|\, x_I) := \int_A I_{B\times C}(x(t), u(t))\, dt$$

for all subsets $A \times B \times C$ in the Borel σ-algebra of subsets of $T \times X \times U$. This measures the time spent by the triplet $(t, x(t), u(t))$ in a given subset $A \times B \times C$, all along the trajectory.

Define further the linear operator $\mathcal{L} : C^1(T \times X) \to C(T \times X \times U)$ by

$$v \mapsto \mathcal{L}v := \frac{\partial v}{\partial t} + \sum_{i=1}^{n} \frac{\partial v}{\partial x_i} f_i(t, x, u) = \frac{\partial v}{\partial t} + (\text{grad})^T v \cdot f$$

and its adjoint operator $\mathcal{L}' : C(T \times X \times U)' \to C^1(T \times X)'$ by the relation

$$\langle v,\, \mathcal{L}'\mu \rangle := \langle \mathcal{L}v,\, \mu \rangle = \int_{T \times X \times U} \mathcal{L}v(t, x, u)\, \mu(dt, dx, du)$$

for all $\mu \in M(T \times X \times U) = C(T \times X \times U)'$ and $v \in C^1(T \times X)$. This operator can also be expressed as

$$\mu \mapsto \mathcal{L}'\mu = -\frac{\partial \mu}{\partial t} - \sum_{i=1}^{n} \frac{\partial (f_i \mu)}{\partial x_i} = -\frac{\partial \mu}{\partial t} - \text{div}\, f\mu$$

where the derivatives of measures are understood in the sense of distributions (i.e., via their action on appropriate test functions), and the change of sign comes from the integration by parts formula.

Given a test function $v \in C^1(T \times X)$, it follows from the above definition of the occupation measure μ that

$$\begin{aligned} v(t_F, x_F) &= v(t_I, x_I) + \textstyle\int_T \dot{v}(t, x(t))\, dt \\ &= v(t_I, x_I) + \textstyle\int_T \mathcal{L}v(t, x(t), u(t))\, dt \\ &= v(t_I, x_I) + \textstyle\int_{T \times X \times U} \mathcal{L}v(t, x, u)\, \mu(dt, dx, du \,|\, x_I). \end{aligned} \tag{4}$$

Now consider that the initial state is not a single point but that its distribution in space is modeled by a probability measure $\mu_I \in P(X)$, and that to each initial state x_I an admissible control function $u(\cdot \,|\, x_I) \in L^1(T; U)$ is assigned in such a way that $x(\cdot \mid x_I)$ is admissible. Then we can define the average occupation measure $\mu \in M(T \times X \times U)$ by

$$\mu(A \times B \times C) := \int_X \mu(A \times B \times C \,|\, x_I)\, \mu_I(dx_I),$$

and the final measure $\mu_F \in P(X_F)$ by

$$\mu_F(B) := \int_X I_B(x(t_F \,|\, x_I))\, \mu_I(dx_I).$$

It follows by integrating (4) with respect to μ_I that

$$\int_{X_F} v(t_F, x)\, \mu_F(dx) = \int_{X_I} v(t_I, x)\, \mu_I(dx) + \int_{T\times X\times U} \mathcal{L}v(t, x, u)\, \mu(dt, dx, du) \tag{5}$$

or more concisely

$$\langle v(t_F, \cdot),\ \mu_T\rangle \ = \ \langle v(t_I, \cdot),\ \mu_I\rangle + \langle \mathcal{L}v,\ \mu\rangle \quad \forall\, v \in C^1(T \times X), \tag{6}$$

which is a linear equation linking the nonnegative measures μ_I, μ_T and μ.

To summarize, we have reformulated a nonlinear controlled ODE

$$\dot{x}(t) = f(t, x(t), u(t)), \quad u(t) \in U, \quad t \in T \tag{7}$$

as a linear equation (6) on occupation measures. Instead of working in the space of state trajectories, we **linearized** the problem in the space of occupation measures.

3.2 Liouville's PDE

Denoting δ_t the Dirac measure at a point t and $\otimes$ the product of measures, we can write $\langle \mu_I,\ v(t_I, \cdot)\rangle = \langle \delta_{t_I} \otimes \mu_I,\ v\rangle$ and $\langle \mu_F,\ v(t_F, \cdot)\rangle = \langle \delta_{t_F} \otimes \mu_F,\ v\rangle$. Then, equation (6) can be rewritten equivalently using the adjoint $\mathcal{L}'$ as

$$\langle \mathcal{L}'\mu,\ v\rangle = \langle \delta_{t_F} \otimes \mu_F, v\rangle - \langle \delta_{t_I} \otimes \mu_I,\ v\rangle \quad \forall\, v \in C^1(T \times X),$$

and since this equation is required to hold for all test functions v, we obtain a linear partial differential equation (PDE) on measures

$$\frac{\partial \mu}{\partial t} + \operatorname{div} f\mu = \delta_{t_I} \otimes \mu_I - \delta_{t_F} \otimes \mu_F \tag{8}$$

where the derivatives should be understood in the sense of distributions. This equation is classical in fluid mechanics and statistical physics, and it is referred to as the conservation of mass equation, or continuity equation, or advection equation, or **Liouville's equation**.

Each family of admissible trajectories starting from a given initial distribution $\mu_I \in P(X_I)$ satisfies Liouville's equation (8). The converse may not hold in general although the two formulations can be considered equivalent, at least from a practical viewpoint. Let us briefly elaborate more on this point now.

3.3 Relaxed control

We can disintegrate the occupation measure as

$$\mu(dt, dx, du) = \omega(du \mid t, x)\xi(dx \mid t)dt \tag{9}$$

where $\xi(dx \mid t) \in P(X)$ is the state stochastic kernel, a probability measure on X for each $t \in T$, which models the state interpreted as a random variable, and $\omega(du|t, x) \in P(U)$ is the control stochastic kernel, a probability measure on U for each $t \in T$ and $x \in X$, which models the control interprepted as a random variable. Instead of a control function $u(.) \in L^1(T; U)$, we have a **relaxed control measure**

$$\omega \in P(U) \tag{10}$$

parametrized in time $t \in T$ and space $x \in X$. Such parametrized probability measures are called Young measures in the calculus of variations and PDE literature, see e.g. [48, 31, 39]. Equation (5) can then be written equivalently as

$$\begin{array}{l}
\int_{X_F} v(t_F, x)\mu_F(dx) - \int_{X_I} v(t_I, x)\mu_I(dx) = \\
\quad \int_{T\times X\times U} \mathcal{L}v(t, x, u)\mu(dt, dx, du) = \\
\quad \int_{T\times X\times U} \left(\frac{\partial v(t,x)}{\partial t} + (\mathrm{grad}\, v(t, x))^T f(t, x, u)\right) \omega(du \mid t, x)\xi(dx \mid t)dt = \\
\quad \int_{T\times X} \left(\frac{\partial v(t,x)}{\partial t} + (\mathrm{grad}\, v(t, x))^T \left[\int_U f(t, x, u)\omega(du \mid t, x)\right]\right) \xi(dx \mid t)dt
\end{array}$$

and it can be shown that measure ξ and hence occupation measure μ satisfying Liouville's equation (8) is generated by a family of absolutely continuous trajectories of the ODE

$$\dot{x}(t) = \int_U f(t, x, u)\omega(du \mid t, x). \tag{11}$$

4 Measuring the control

To summarize our developments so far, we can say that if the set of allowable controls is enlarged from functions $L^1(T; U)$ to measures $P(U)$, controlled ODE (7) becomes a measure-controlled, or relaxed ODE (11). Solutions of this ODE are captured by occupation measures solving Liouville's PDE (8).

Our control, originally chosen as a measurable function (of time and space), is therefore relaxed to a probability measure (parametrized in time and space).

Notice that the space of probability measures is larger than any Lebesgue space, since for the particular choice of a time-dependent Dirac measure

$$\omega(du \mid t, x) = \delta_{u(t,x)} \in P(U)$$

we retrieve a classical control law which is a function of time and space.

Note finally that the vector field in ODE (11) is not Lipschitz in general[1] and uniqueness of the trajectories does not hold in general. All possible trajectories are however superposed and captured by the measure solution to Liouville's equation (8)

4.1 OCP as an LP

The linearization applied to ODE (2) can be applied to the Lagrangian of OCP (1) as well, since if $u(.)$ is an admissible control, $x(.)$ the resulting state trajectory and $\mu(.)$ the corresponding occupation measure, it holds

$$\int_T l(t, x(t), u(t))dt = \int_{T\times X\times U} l(t, x, u)\mu(dt, dx, du) = \langle l,\ \mu\rangle.$$

In our original OCP (1), we can relax the dynamics (7) to (11) to obtain the relaxed OCP

$$\begin{array}{rll} \bar{v}^* := & \inf & \int_T \int_U l(t, x(t), u)\omega(du \mid t, x)dt \\ & \text{s.t.} & \dot{x}(t) = \int_U f(t, x(t), u)\omega(du \mid t, x), \\ & & x(t) \in X, \quad t \in T, \\ & & x(t_I) \in X_I, \quad x(t_F) \in X_F \end{array} \tag{12}$$

where the infimum is with respect to a measure $\omega \in P(X)$ parametrized in t and x. Equivalently, we obtain the following **linear programming problem on measures**:

$$\begin{array}{rll} \bar{v}^* := & \inf & \langle l,\ \mu\rangle \\ & \text{s.t.} & \frac{\partial\mu}{\partial t} + \operatorname{div} f\mu = \delta_{t_I} \otimes \mu_I - \delta_{t_F} \otimes \mu_F \\ & & \operatorname{spt} \mu \subset T \times X \times U \\ & & \operatorname{spt} \mu_I \subset X_I, \quad \operatorname{spt} \mu_F \subset X_F \end{array} \tag{13}$$

where the infimum is w.r.t. nonnegative measures μ, μ_I and μ_F. Measures μ_I and μ_F may be given or free, depending on the problem. Occupation measure μ encodes the system trajectories and the optimal control, recall (9). Problem

[1] Choose for example $f(t, x, u) = u$ and $\omega(du \mid t, x) = \delta_{u(t,x)=\sqrt{x(t)}}$.

(13) is an LP in the cone of nonnegative measures. Since the supports of our measures are assumed to be compact, we can use the weak-* topology to prove that the infimum in LP (13) is attained, i.e. it is a minimum, see e.g. [28].

Note finally that $\bar{v}^* \leq v^*$, and the inequality can be strict in some contrived cases, typically in the presence of overly stringent state constraints, see [15, Appendix A] for more details and examples. For most of the practically relevant cases, there is no gap between the original OCP (1) and the relaxed OCP (13), i.e. $\bar{v}^* = v^*$.

4.2 Bolza again

Let us come back to Bolza's example of Section 2.1. We saw that the infimum can be approached by a control sequence switching increasingly quickly between -1 and $+1$, so the idea is to relax the ODE (2) with the following differential equation

$$\dot{x}(t) = \int_U f(t, x(t), u)\omega(du \mid t) \tag{14}$$

where $\omega(du|t)$ is a probability measure supported on U and parametrized in t. State trajectories are then obtained by integration w.r.t. time and control

$$x(t) = x(t_I) + \int_{t_I}^{t} \int_U f(s, x(s), u)\omega(du \mid s)ds.$$

Here for the Bolza example we choose

$$\omega(du \mid t) = \frac{1}{2}(\delta_{u=-1} + \delta_{u=+1})$$

a time-independent weighted sum of two Dirac measures at $u = -1$ and $u = +1$. The relaxed state trajectory is then equal to

$$\begin{array}{rcl} x(t) & = & \frac{1}{2}\left(\int_0^t f(s, x(s), -1)ds + \int_0^t f(s, x(s), +1)ds\right) \\ & = & \frac{1}{2}\left(-\int_0^t ds + \int_0^t ds\right) = 0 \end{array}$$

and the relaxed objective function is equal to

$$\begin{array}{rcl} \int_0^1 \int_U l(t, x(t), u)d\omega(u \mid t)dt & = & \frac{1}{2}\left(\int_0^1 l(t, x(t), -1) + \int_0^1 l(t, x(t), +1)\right) \\ & = & \int_0^1 x^4(t)dt = 0 \end{array}$$

so that the infimum $v^* = 0$ is reached.

5 Control as a measure

In Section 4 we relaxed the control to a probability measure supported on a compact set U. This allows to account for oscillation and chattering effects. If the support is not bounded, we cannot apply these techniques. In particular, we cannot deal with impulsive controls, and effects of concentration in time, since the time marginal of the occupation measure is the Lebesgue measure, recall (9).

To be able to handle unbounded controls and concentration effects, we restrict the class of OCP problems (1) to problems of the form

$$\begin{array}{rcl} v^* & = & \inf \quad \int_T l(t,x(t))dt + \int_T L(t)u(t)dt \\ & & \text{s.t.} \quad \dot{x}(t) = f(t,x(t)) + F(t)u(t), \\ & & \qquad x(t) \in X, \quad t \in T = [t_I, t_F] \\ & & \qquad x(t_I) \in X_I, \quad x(t_F) \in X_F \end{array} \tag{15}$$

where both the Lagrangian and the dynamics are affine in the control, with l, L, f and F smooth functions. Indeed, as in Section 4, the idea is to relax both the dynamics and the Lagrangian so that the OCP becomes:

$$\begin{array}{rcl} v^* & = & \inf \quad \int_T l(t,x(t))dt + \int_T L(t)dw(t) \\ & & \text{s.t.} \quad dx(t) = f(t,x(t))dt + F(t)dw(t), \\ & & \qquad x(t) \in X, \quad t \in T = [t_I, t_F] \\ & & \qquad x(t_I) \in X_I, \quad x(t_F) \in X_F \end{array} \tag{16}$$

where the infimum is now over a (vector) function of bounded variation $w \in BV(T;\mathbb{R}^m)$. The control is the weak derivative of w, a (vector) signed measure of time

$$w(dt) \in M(T;\mathbb{R}^m)$$

which has finite total variation, see e.g. [41] and [38, §4].

In the relaxation of the previous section, the control was a measured variable (10) modeled by a probability measure. Here **the control is a signed measure of time** which is not necessarily nonnegative and/or with unit mass. The affine dependence on w of the Lagrangian and the dynamics in the above formulation is justified: a measure can be interpreted as a linear functional, so it should act linearly. The dynamics should be understood as follows:

$$x(t) = x(t_I) + \int_{t_I}^{t} f(s,x(s))ds + \int_{t_I}^{t} F(s)w(ds). \tag{17}$$

and this allows possible jumps in state-space, i.e. discontinuities of $x(.)$. Since the state constraint set X is bounded, only finite jumps are allowed.

Note that in the absence of impulses, the distributional differential is the traditional differential, and the dynamics are classical differential equations with controls

$$w(dt) = u(t)dt$$

which are absolutely continuous with respect to the Lebesgue measure.

Because distributional derivatives of functions of bounded variation on compact supports can be identified with measures [38, §50], the dynamics in problem (16) may be interpreted as a measure differential equation. As $X \subset \mathbb{R}^n$ is assumed to be compact, by one of the Riesz representation theorems [17, §36.6], these measures can be put in duality correspondence with all continuous functions $v(t, x)$ supported on $T \times X$. We will use these test functions to define linear relations between the measures.

By Lebesgue's decomposition theorem [17, §33.3], we can split the control measures $w(dt)$ into two parts: their absolutely continuous parts with density $u : T \to \mathbb{R}^m$ (with respect to the Lebesgue measure) and their purely singular parts with jump amplitude vectors $u_{t_j} \in \mathbb{R}^m$ supported at impulsive jump instants t_j, $j \in J$, with J a subset of Lebesgue measure zero of T, not necessarily countable[2]. We write

$$w(dt) = u(t)dt + \sum_{j \in J} F(t_j)u_{t_j}\delta_{t=t_j}$$

to model jumps in state-space

$$x^+(t_j) = x^-(t_j) + F(t_j)u_{t_j}, \quad \forall\, j \in J.$$

5.1 OCP as an LP

Now, given an initial state $x_I \in X_I$ and given a control $w(.) \in BV(T; \mathbb{R}^m)$, denote by $x(.) \in BV(T; X)$ the corresponding admissible trajectory. Then for smooth test functions $v \in C(T \times X)$, it holds

$$\begin{aligned}\textstyle\int_T dv(t, x(t)) = v(t_F, x(t_F)) - v(t_I, x(t_I)) = \int_T \big(\frac{\partial v}{\partial t} + (\text{grad } v)^T f\big)\, dt \\ + \textstyle\int_T (\text{grad } f)^T F u dt + \sum_{j \in J} v(t_j, x^+(t_j)) - v(t_j, x^-(t_j)).\end{aligned} \tag{18}$$

[2]We suspect however that for the control problems studied in this paper, subset J can be assumed countable without loss of generality.

We are going to express the above temporal integration (18) along the trajectory in terms of spatial integration with respect to **occupation measures**. For this purpose, and consistently with Section 4, we assume that the initial state is modeled by a probability measure $\mu_I \in P(X_I)$ and we define:

- The time-state occupation measure

$$\mu(A \times B) := \int_{X_I} \left(\int_A I_B(x(t \mid x_I))\, dt \right) \mu_I(dx_I).$$

  This measure can be disintegrated into

$$\mu(dt, dx) = \xi(dx \mid t, x_I)\mu_I(dx_I)dt$$

  where $\xi(dx|t, x_I)$ is the state stochastic kernel, a distribution of $x \in X$, conditional on $t \in T$ and $x_I \in X_I$. In our case, when the initial state x_I and the control $w(.)$ are given, the state stochastic kernel is well defined along continuous arcs of the trajectory as

$$\xi(B \mid t, x_I) = I_B(x(t)) = \delta_{x(t)}(B)$$

  for all $t \in T \setminus J$ and $B \subset X$. On the other hand, at every jump instant $t_j \in J$, we let

$$\xi(B \mid t_j, x_I) = \frac{\lambda(B \cap [x^-(t_j), x^+(t_j)])}{\lambda([x^-(t_j), x^+(t_j)])},$$

  for all $t_j \in J$ and $B \subset X$. This means that the state is uniformly distributed along the segment linking the state before and after the jump, the above denominator ensuring that $\xi(\cdot \mid t, x_I)$ has unit mass for all $t \in T$, $x_I \in X_I$.

- The control-state occupation measure

$$\nu(A \times B) := \int_{X_I} \left(\int_A \xi(B \mid t, x_I)\, dw(t) \right) \mu_I(dx_I)$$

  for all Borel subsets $A \subset T$ and $B \subset X$.

- The final state occupation measure

$$\mu_F(B) = \int_X I_B(x(t_F \mid x_I))\mu_I(dx_I)$$

  for all Borel subsets $B \subset X_F$.

With these definitions, equation (18) may be written in terms of measures as:

$$\begin{aligned}&\textstyle\int_{X_F} v(t_F,x)\mu_F(dx) - \int_{X_I} v(t_I,x)\mu_I(dx) = \\ &\quad \textstyle\int_{T\times X}\left(\frac{\partial v}{\partial t} + (\mathrm{grad}\, v)^T f\right)\mu + \int_{T\times X}(\mathrm{grad}\, v)^T F\nu = \\ &\quad \textstyle\int_{T\times X_I}\int_X \left(\frac{\partial v}{\partial t} + (\mathrm{grad}\, v)^T(f+Fu)\right)\xi(dx\,|\,t,x_I)d\mu_I(x_I)dt \\ &\qquad \textstyle + \sum_{j\in J} v(t_j,x^+(t_j)) - v(t_j,x^-(t_j)).\end{aligned}$$

Similarly, the objective function in (16) to evaluate the trajectory and the control reads:

$$\int_T l(t,x(t))dt + \int_T L(t)dw(t) = \int_{T\times X} l\mu + \int_{T\times X} L\nu = \langle l,\ \mu\rangle + \langle L,\ \nu\rangle.$$

Therefore, the OCP (16) can be relaxed to the **linear programming problem on measures**:

$$\begin{array}{rll}\bar{v}^* = & \inf & \langle \mu,\ l\rangle + \langle \nu,\ L\rangle \\ & \text{s.t.} & \frac{\partial\mu}{\partial t} + \mathrm{div}\, f\mu + \mathrm{div}\, F\nu = \delta_{t_I}\otimes\mu_I - \delta_{t_F}\otimes\mu_F \\ & & \mathrm{spt}\,\mu \subset T\times X, \quad \mathrm{spt}\,\nu\subset T \\ & & \mathrm{spt}\,\mu_I\subset X_I, \quad \mathrm{spt}\,\mu_F\subset X_F\end{array} \tag{19}$$

with the infimum is w.r.t. nonnegative measures μ, μ_I, μ_F and signed measure ν, compare with (13). Obviously, $\bar{v}^* \leq v^*$, and it should be possible to prove that $\bar{v}^* = v^*$ under reasonable assumptions.

5.2 Decomposition of control measures

All measures in (19) are nonnegative measures, except for the signed measure ν of finite total variation which deserve special treatment for our purposes. Using the Jordan decomposition theorem [17, §34], this measure may be split into a positive part ν^+ and negative part ν^-, that is $\nu = \nu^+ - \nu^-$, both being nonnegative measures.

This decomposition has the added benefit of providing an easy expression for the L_1 norm of the control, which is sometimes to be constrained or optimized in some problems. Indeed, define the total variation control measure by

$$|\nu| = \nu^+ + \nu^-.$$

The total variation norm of the measure ν is just the mass of $|\nu|$, i.e.,

$$\|\nu\| = \int d|\nu| = \int d\nu^+ + \int d\nu^-.$$

5.3 Luenberger again

Let us get back to Luenberger's example of Section 2.1. Enlarging the control space from $L^\infty(T;\mathbb{R})$ to signed measures $M(T;\mathbb{R})$ of finite total variation, the OCP can be rewritten as

$$\begin{array}{ll} \inf & \int d\nu^+ + \int d\nu^- \\ \text{s.t.} & dx_1(t) = x_2(t)dt \\ & dx_2(t) = -dt + \nu^+(dt) - \nu^-(dt) \\ & x_1(t_I) = 0, \quad x_2(t_I) = 0, \quad x_1(t_F) = 1 \end{array}$$

where the infimum is w.r.t. two nonnegative measures ν^+, ν^- supported on $T = [t_I,\, t_F]$ with fixed initial time $t_I = 0$ and unknown final time t_F. Here the support of the measures is partly unknown. The objective function is the total variation of the signed measure $\nu = \nu^+ - \nu^-$.

In [28, Ex. 3, p. 125] it is indeed shown that the optimal control is $\nu^+ = \sqrt{2}\delta_{t=0}$ and $\nu^- = 0$, with corresponding state trajectory $x_1(t) = \sqrt{2}t - \frac{1}{2}t^2$, $x_2(t) = \sqrt{2}I_{t>0}(t) - t$.

6 LMI relaxations

In Section 4 we have seen that oscillation (chattering) effects in optimal control can be accomodated for by a control variable measured by a time- and state-dependent probability measure. In Section 5 we have seen that concentration (impulse) effects in optimal control can be accomodated for by a control which is a measure of time. In both cases, we reformulated (linearized) the nonlinear nonconvex OCP (over trajectories) into a convex LP (over occupation measures), see respectively problem (13) and problem (19). In this Section we show that recent advances in real algebraic geometry and convex optimization can provide explicit tools to **solve numerically** these LP problems.

6.1 Generalized problem of moments

For this, we assume in the remainder of the lecture that all problem data in OCP (1) or (15) are polynomials: Lagrangians l, L and dynamics f, F are multivariate polynomials of t, x, u, and constraint sets

$$\begin{array}{rcl} X & = & \{x \,:\, g_j(x) \geq 0,\ j = 1, \ldots, n_X\} \\ X_I & = & \{x \,:\, g_{Ij}(x) \geq 0,\ j = 1, \ldots, n_I\} \\ X_F & = & \{x \,:\, g_{Fj}(x) \geq 0,\ j = 1, \ldots, n_F\} \\ U & = & \{u \,:\, g_{Uj}(u) \geq 0,\ j = 1, \ldots, n_U\} \end{array} \tag{20}$$

are compact basic semialgebraic sets defined by multivariate polynomials g_j, g_{Ij}, g_{Fj}, g_{Uj}. LP problem (13) or (19) can be formulated as follows:

$$\begin{array}{rll} \bar{v}^* = & \inf_\mu & \sum_k \int_{X_k} c_k \, d\mu_k \\ & \text{s.t.} & \sum_k \int_{X_k} a_{ki} \, d\mu_k = b_i, \quad \forall i \end{array} \tag{21}$$

where the unknowns are a finite set of nonnegative measures μ_k, with respective compact semialgebraic supports X_k. For notational simplificity, in the sequel we denote our variables by x, and this includes the time variable t, the state vector x and the input vector u. Notice that the test functions in the (dual to) LP problem (13) involve t and x only, whereas the measures involve t, x and u. In the (dual to) LP problem (19) the test functions and the measures involve t and x.

In LP problem (21) our measures are on our extended vector x. If we generate test functions $v(x)$ using a polynomial basis (e.g. monomials, which are dense in the set of continuous functions with compact support), all the coefficients $a(x)$, $b(x)$, $c(x)$ are polynomials, and there is an infinite but countable number of linear constraints indexed by i.

We will then manipulate each measure μ_k via its moments

$$y_{k\alpha} = \int_{X_k} x^\alpha \mu_k(dx), \quad \forall \alpha \tag{22}$$

gathered into an infinite-dimensional sequence y_k indexed by a vector of integers α, where we use the multi-index notation $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots$ LP measure problem (21) becomes an LP moment problem, or generalized problem of moments, see [24]:

$$\begin{array}{ll} \inf_y & \sum_k \sum_\alpha c_{k\alpha} y_{k\alpha} \\ \text{s.t.} & \sum_k \sum_\alpha a_{ki\alpha} y_{k\alpha} = b_i, \quad \forall i \end{array} \tag{23}$$

provided we can handle the representation condition (22) which links a measure with its moments. It turns out, see [24, Chapter 3] or [25], that if sets X_k are compact semialgebraic as in (20), we can use results of functional analysis and real algebraic geometry to design a hierarchy of linear matrix inequality (LMI) relaxations which is asymptotically equivalent to the generalized moment problem. This is described in the sequel.

6.2 Moment and localizing matrices

Let $\mathbb{R}_d[x]$ denote the vector space of real multivariate polynomials of total degree less than or equal to d. Each polynomial $p(x) \in \mathbb{R}_d[x]$ can be expressed in the monomial basis as $p(x) = \sum_{|\alpha| \leq d} p_\alpha x^\alpha$ and it can be identified with its vector of coefficients $p := (p_\alpha)$ whose entries are indexed by α. Given a vector of real numbers $y := (y_\alpha)$ indexed by α, we define the linear functional $L_y : \mathbb{R}_k[x] \to \mathbb{R}$ such that $L_y(p) := p^T y = \sum_\alpha p_\alpha y_\alpha$. When entries of y are moments of a measure μ, the linear functional models integration of a polynomial w.r.t. μ, i.e.,

$$L_y(p) = \int p(x)\, \mu(dx) = \sum_\alpha p_\alpha \int x^\alpha \mu(dx) = p^T y.$$

When this linear functional acts on the square of a polynomial p of degree d, it becomes a quadratic form $L_y(p^2) = p^T M_d(y) p$ in the polynomial coefficients space: and we denote by $M_d(y)$ and call the **moment matrix** of order d the matrix of the quadratic form, which is symmetric and linear in y. Finally, given a polynomial $g(x) \in \mathbb{R}[x]$ we consider the quadratic form $L_y(g\, p^2) = p^T M_d(g\, y) p$ and we denote by $M_d(g\, y)$ and call the **localizing matrix** of order d the matrix of the quadratic form, which is also symmetric and linear in y.

Under a mild assumption on the compact basic semi-algebraic set

$$X = \{x \,:\, g_j(x) \geq 0,\ j = 1, \ldots, n_X\}$$

with given $g_j(x) \in \mathbb{R}[x]$, it was shown by Putinar that a necessary and sufficient condition for a measure μ to have moments

$$y_\alpha = \int_X x^\alpha \mu(dx)$$

is that the conditions

$$M_d(y) \succeq 0, \quad M_d(g_j\, y) \succeq 0, \quad j = 1, \ldots, n_X$$

are satisfied for all orders $d \in \mathbb{N}$.

The above notation $M_d(y) \succeq 0$ means that symmetric matrix $M_d(y)$ is positive semidefinite, i.e. all its eigenvalues are nonnegative real. Since $M_d(y)$ is linear in y, the constraint $M_d(y) \succeq 0$ is a **linear matrix inequality** (LMI).

LMI problems can be solved numerically, at arbitrary accuracy, in a time which is a polynomial function of the size of the matrix constraint and number of variables, with the help of primal-dual interior-point methods for semidefinite programming (SDP), see e.g. [30] and [4].

6.3 A hierarchy of LMI relaxations

Returning to moment LP (23), we denote by

$$A_d y = b_d$$

the finite-dimensional truncation of the system of equations obtained truncate the moment sequences to moments of degree at most $2d$. We then define the finite-dimensional **LMI relaxation** of order d

$$\begin{array}{rcll} v_d^* & = & \inf_y & c^T y \\ & & \text{s.t.} & A_d y = b_d \\ & & & M_d(y) \succeq 0 \\ & & & M_d(g_{kj}\, y) \succeq 0, \quad \forall j,k. \end{array} \tag{24}$$

The hierarchy of LMI relaxations (24) generates an asymptotically converging monotonically increasing sequence of lower bounds on LP (21), i.e. $v_d^* \leq v_{d+1}^*$ for all $d = 1, 2\ldots$ and $\lim_{d\to\infty} v_d^* = \bar{v}^*$. For more details, see [24], as well as the tutorial [25].

6.4 Numerical solution

Practically speaking, we have designed a Matlab interface called GloptiPoly 3 [14] to construct the LMI relaxations (24) from the polynomial data of an LP measure problem (21). We solve then each LMI relaxation with any semidefinite programming solver, our favorite being SeDuMi [34]. From the solution of the (dual) LMI problem, we can approximate the optimal control law, with increasing accuracy when d increases. A precise description of this synthesis procedure lies however outside of the scope of these lecture notes.

For explicit numerical examples, please refer to [23], [24], [9] and [15].

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Doc. Ing. Didier Henrion, Ph.D.

Degrees

1994 Engineer's Degree from INSA (National Institute for Applied Sciences), Toulouse, France.

1994 Master's Degree with specialization in control from INSA, Toulouse, France.

1998 CSc. degree (Candidate of Sciences - corresponds to Ph.D degree) from the Institute of Information Theory and Automation (ÚTIA) of the Academy of Sciences of the Czech Republic (AVČR), Prague, Czech Republic.

1999 Ph.D. degree from INSA, Toulouse, France.

2007 Habilitation degree from Université Paul Sabatier (Faculty of Sciences of the University of Toulouse), Toulouse.

2008 Docent degree from the Faculty of Electrical Engineering (FEL), Czech Technical University in Prague (ČVUT), Prague, Czech Republic.

Expertise

I am interested in polynomial optimization for systems control, focusing on the development of constructive tools for addressing mathematical problems arising from systems control theory.

I am trying to explore the mathematical domains of convex optimization, real algebraic geometry, functional analysis and dynamical systems. I would like to unveil existing links between these disciplines, and exploit them to study classical problems of systems control theory from a different perspective.

Employment

1994-1996: Research assistant at Universidad Simón Bolívar, Caracas, Venezuela.

1996-1998: CSc. (Ph.D.) student at ÚTIA-AVČR, Prague.

1996-1999: Ph.D. student at Laboratory of Analysis and Architecture of Systems (LAAS) of the French National Center for Scientific Research (CNRS), Toulouse.

1999-2003: Research fellow at ÚTIA-AVČR, Prague.

2003-2012: Assistant professor at FEL-ČVUT, Prague.

2000-2010: CNRS researcher (chargé de recherche) at LAAS-CNRS, Toulouse.

2010-2012: CNRS senior researcher (directeur de recherche) at LAAS-CNRS, Toulouse.

Journal editorial boards

since 2012: Associate Editor of Asian Journal of Control

since 2011: Associate Editor of Mathematics of Control, Signals and Systems

since 2008: Associate Editor of the Czech journal Kybernetika

2005-2008: Associate Editor of the EUCA European Journal of Control

2005-2008: Associate Editor of the IEEE Transactions on Automatic Control

2004-2010: Associate Editor of the Slovak Journal of Electrical Engineering

2003-2009: Associate Editor of the IFAC journal Automatica

Selected recent grants and projects

2011-2012: Industrial project SAFE-V (Space Application Flight control Enhancement of Validation Framework). Funded by ESA-ESTEC (European Space Research and Technology Center or the European Space Agency), coordinated by EADS Astrium Space Transportation. Topic: positive polynomials, integral quadratic separators and LMIs for robustness analysis of launchers in atmospheric flight and orbital manoeuvre.

2010-2012: GAČR Research project 103/10/0628. Funded by the Grant Agency of the Czech Republic (GAČR). Project leader: Didier Henrion. Research conducted jointly with Petra Šindelarová and Sergej Čelikovský. Topic: semidefinite programming for nonlinear dynamical systems.

2009-2010: French-UK international joint project of the UK Royal Society. Project leader: Michal Kočvara, University of Birmingham, UK. Topic: Polynomial semidefinite programming with control applications.

2009-2010: French-Czech joint project Barrande. Project leader: Didier Henrion (LAAS-CNRS Toulouse). Research conducted jointly with Tomáš Vyhlídal and Pavel Zítek (Czech Technical University in Prague). Topic: optimisation and algebraic geometry for time-delay control systems.

2007-2008: Bilateral project No. 20219 between the Academy of Sciences of the Czech Republic and the French CNRS. Project leader: Didier Henrion (LAAS-CNRS Toulouse). Research conducted jointly with Sergej Čelikovský (Institute of Information Theory and Automation). Topic: positive polynomials and LMI optimization in robust and nonlinear control.

2006-2008: GAČR Research project 102/06/0652. Funded by the Grant Agency of the Czech Republic (GAČR). Project leader: Didier Henrion. Research conducted jointly with Michal Kočvara (2006) and Petra Šindelarová (2007-2008). Topic: software for polynomial matrix inequality optimization.

Teaching activities

since 2011: Various courses of applied mathematics (harmonic analysis, Hilbert space analysis, optimization) at ISAE-Supaéro (aerospace engineering school), Toulouse.

2009-2012: Optimization and PDEs at ENSEEIHT (electrical engineering school), Toulouse.

2008-2011: LMI, optimization and polynomial methods, Supélec, Paris, part of the HYCON-EECI Graduate School on Control. 4 courses of 21 hours each.

2005-2010: Mini-courses (10 to 16 hours) on LMIs and polynomial methods for systems control, Toulouse, France, April 2005; Leuven, Belgium, June 2006; Dortmund, Germany, October 2006; Cosenza, Calabria, Italy, March 2007; Barcelona, Catalonia, Spain, June 2007; Valladolid, Spain, March 2009; Torun, Poland, June 2009; Valencia, Spain, June 2009; Leuven, Belgium, April and May 2010.

2003-2011: LMI optimization with applications in control, FEL-ČVUT, Prague, regular doctoral course (16 hours).

2001: Polynomial methods for robust control, Mérida, Venezuela, October 2001.

Awards

2004: Bronze Medal from CNRS.

2005: David Marr Prize for best paper at Int. Conf. Computer Vision.

2011: Scientific award by Simone and Cino del Duca Foundation of Institut de France

2012: Charles Broyden prize for best paper in journal Optim. Methods and Sofware.

Sample of recent publications

D. Henrion, J. Malick. Projection methods for conic feasibility problems, applications to polynomial sum-of-squares decompositions. Optimization Methods and Software 26(1):23-46, 2011.

D. Henrion. Detecting rigid convexity of bivariate polynomials. Linear Algebra and its Applications, 432:218-1233, 2010.

D. Henrion, J. B. Lasserre, C. Savorgnan. Approximate volume and integration for basic semialgebraic sets. SIAM Review 51(4):722-743, 2009.

J. B. Lasserre, D. Henrion, C. Prieur, E. Trélat. Nonlinear optimal control via occupation measures and LMI relaxations. SIAM J. Control Opt. 47(4):1643-1666, 2008.

F. Kahl, D. Henrion. Globally optimal estimates for geometric reconstruction problems. Int. J. Computer Vision, 74(1):3-15, 2007.