H_2 for HIFOO

Denis Arzelier[1], Georgia Deaconu[2], Suat Gumussoy[3], Didier Henrion[4]

October 7, 2010

Abstract

HIFOO is a public-domain Matlab package initially designed for H_∞ fixed-order controller synthesis, using nonsmooth nonconvex optimization techniques. It was later on extended to multi-objective synthesis, including strong and simultaneous stabilization under H_∞ constraints. In this paper we describe a further extension of HIFOO to H_2 performance criteria, making it possible to address mixed H_2/H_∞ synthesis. We give implementation details and report our extensive benchmark results.

Keywords: fixed-order controller design, H_2 control, mixed H_2/H_∞ control, optimization.

1 Introduction

HIFOO is a public-domain Matlab package originally conceived during a stay of Michael Overton at the Czech Technical University in Prague, Czech Republic, in the summer of 2005. HIFOO relies upon HANSO, a general purpose implementation of an hybrid algorithm for nonsmooth optimization, mixing standard quasi-Newton (BFGS) and gradient sampling techniques. The acronym HIFOO (pronounce [haıfu:]) stands for H-infinity Fixed-Order Optimization, and the package is aimed at designing a stabilizing linear controller of fixed-order for a linear plant in standard state-space configuration while minimizing the H_∞ norm of the closed-loop transfer function.

The first version of HIFOO was released and presented during the IFAC Symposium on Robust Control Design in Toulouse, France in the summer of 2006, see [6], based on the theoretical achievements reported in [5]. HIFOO was later on extended to cope

[1]CNRS; LAAS; 7 avenue du colonel Roche, F-31077 Toulouse, France; Université de Toulouse; UPS, INSA, INP, ISAE; LAAS; F-31077 Toulouse, France (e-mail: denis.arzelier@laas.fr)

[2]CNRS; LAAS; 7 avenue du colonel Roche, F-31077 Toulouse, France; Université de Toulouse; UPS, INSA, INP, ISAE; LAAS; F-31077 Toulouse, France (e-mail: georgia.deaconu@laas.fr)

[3]Katholieke Universiteit Leuven, Department of Computer Science, Belgium (e-mail: suat.gumussoy@cs.kuleuven.be)

[4]CNRS; LAAS; 7 avenue du colonel Roche, F-31077 Toulouse, France; Université de Toulouse; UPS, INSA, INP, ISAE; LAAS; F-31077 Toulouse, France. Faculty of Electrical Engineering, Czech Technical University in Prague, Technická 4, CZ-16626 Prague, Czech Republic (e-mail: didier.henrion@laas.fr)

with multiple plant stabilization and multiple conflicting objectives and the second major release of HIFOO was announced during the IFAC Symposium on Robust Control Design in Haifa, Israel, in the summer of 2009, see [7].

Since then HIFOO has been used by various scholars and engineers. Benefiting from feedback from users, we feel that it is now timely to extend HIFOO to H_2 norm specifications. Indeed, H_2 optimal design, a generalization of the well-known linear quadratic regulator design, is traditionally used in modern control theory jointly with H_∞ optimal design, see [11]. In particular, the versatile framework of mixed H_2/H_∞ design described e.g. in [9] is frequently used when designing high-performance control laws for example in aerospace systems, see [2]. See also [10] for an application of the H_2 norm for smoothening H_∞ optimization.

The objective is this paper is to describe the extension of HIFOO to H_2 norm specifications in such a way that users understand the basic mechanisms underlying the package, and may be able to implement their own extensions to fit their needs for their target applications. For example, the algorithms of HIFOO can also be extended to cope with discrete-time systems, pole placement specifications or time-delay systems. On the HIFOO webpage

`www.cs.nyu.edu/overton/software/hifoo`

we are maintaining a list of publications reporting such extensions and applications in engineering. The HIFOO and HANSO packages can also be downloaded there.

2 H_2 and H_2/H_∞ synthesis

2.1 H_2 synthesis

We use the standard state-space setup

$$\begin{array}{rcl} \dot{x} & = & Ax + B_1 w + B_2 u \\ z & = & C_1 x + D_{11} w + D_{12} u \\ y & = & C_2 x + D_{21} w + D_{22} u \end{array}$$

where x contains the states, u the physical (control) inputs, y the physical (measured) outputs, w the performance inputs and z the performance outputs. Without loss of generality, we assume that

$$D_{22} = 0$$

otherwise we can use a linear change of variables on the system inputs and outputs, see e.g. [11].

We want to design a controller with state-space representation

$$\begin{array}{rcl} \dot{x}_K & = & A_K x_K + B_K y \\ u & = & C_K x_K + D_K y \end{array}$$

so that the closed-loop system equations become

$$\begin{aligned}\dot{\mathbf{x}} &= \mathbf{A}(k)\mathbf{x} + \mathbf{B}(k)w\\ z &= \mathbf{C}(k)\mathbf{x} + \mathbf{D}(k)w\end{aligned}$$

in the extended state vector $\mathbf{x} = [x^T \ \ x_K^T]^T$ with matrices

$$\begin{aligned}\mathbf{A}(k) &= \begin{bmatrix} A + B_2 D_K C_2 & B_2 C_K \\ B_K C_2 & A_K \end{bmatrix}\\ \mathbf{B}(k) &= \begin{bmatrix} B_1 + B_2 D_K D_{21} \\ B_K D_{21} \end{bmatrix}\\ \mathbf{C}(k) &= \begin{bmatrix} C_1 + D_{12} D_K C_2 & B_K D_{21} \end{bmatrix}\\ \mathbf{D}(k) &= D_{11} + D_{12} D_K D_{21}\end{aligned}$$

depending affinely on the vector k containing all parameters in the controller matrices.

The H_2 norm of the closed-loop transfer function $T(s)$ between input w and output z is finite only if matrix $\mathbf{A}$ is asymptotically stable and if $\mathbf{D}$ is zero (no direct feedthrough). This enforces the following affine constraint on the D_K controller matrix:

$$D_{11} + D_{12} D_K D_{21} = 0. \tag{1}$$

We use the singular value decomposition to rewrite this affine constraint in an explicit parametric vector form, therefore reducing the number of parameters in controller vector k. If the above system of equations has no solution, then there is no controller achieving a finite H_2 norm.

In order to use the quasi-Newton optimization algorithms of HANSO, we must provide a function evaluating the H_2 norm in closed-loop and its gradient, given controller parameters. Formulas can already be found in the technical literature [8], but they are reproduced here for the reader's convenience. The (square of the) norm of the transfer function $T(s)$ is given by

$$f(k) = \|T(s)\|_2^2 = \text{trace}\,(\mathbf{C}X(k)\mathbf{C}^T) = \text{trace}\,(\mathbf{B}^T Y(k)\mathbf{B})$$

where matrices $X(k)$ and $Y(k)$ solve the Lyapunov equations

$$\begin{aligned}\mathbf{A}^T(k)X(k) + X(k)\mathbf{A}(k) + \mathbf{C}^T(k)\mathbf{C}(k) &= 0,\\ \mathbf{A}(k)Y(k) + \mathbf{A}^T(k)Y(k) + \mathbf{B}(k)\mathbf{B}^T(k) &= 0\end{aligned} \tag{2}$$

and hence depend rationally on K. The gradient of the H_2 norm with respect to controller parameters K is given by:

$$\begin{aligned}\nabla_K f(k) &= 2(\mathbf{B}_2^T X(k) + \mathbf{D}_{12}^T \mathbf{C}(k))Y(k)\mathbf{C}_2\\ &\quad +2\mathbf{B}_2^T X(k)\mathbf{B}(k)\mathbf{D}_{21}^T\end{aligned}$$

upon defining the augmented system matrices

$$\mathbf{B}_2 = \begin{bmatrix} 0 & B_2 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{C}_2 = \begin{bmatrix} 0 & 1 \\ C_2 & 0 \end{bmatrix},$$

$$\mathbf{D}_{12} = [0 \ \ D_{12}], \quad \mathbf{D}_{21} = \begin{bmatrix} 0 \\ D_{21} \end{bmatrix}.$$

As an academic example for which the H_2 optimal controller can be computed analytically, consider the system

$$\begin{aligned}\dot{x} &= -1 + w + u \\ z &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\ y &= x\end{aligned}$$

with a static controller

$$u = ky$$

with k a real scalar to be found. Closed-loop system matrices are

$$\mathbf{A} = -1 + k, \ \ \mathbf{B} = 1, \ \ \mathbf{C} = \begin{bmatrix} 1 \\ k \end{bmatrix}.$$

The first Lyapunov equation in (2) reads

$$2(-1 + k)X(k) + 1 + k^2 = 0$$

so the square of the H_2 norm is equal to

$$f(k) = \frac{1 + k^2}{2(1 - k)}.$$

For the gradient computation, we have to solve the second Lyapunov equation in (2)

$$2(-1 + k)Y(k) + 1 = 0$$

and hence

$$\nabla f(k) = \frac{1 + 2k - k^2}{2(1 - k)^2}.$$

This gradient vanishes at two points, one of which violating the closed-loop stability condition $-1 + k < 0$. The other point yields the optimal feedback gain

$$k^* = 1 - \sqrt{2} \approx -0.4142$$

see Figure 1.

Using HIFOO with the input sequence

```
P=struct('A',-1,'B1',1,'B2',1,...
'C1',[1;0],'C2',1,'D11',[0;0],...
'D12',[0;1],'D21',0,'D22',0);
options.prtlevel=2;
K=hifoo(P,'t',options)
```

we generate the 3 sequences of optimized H_2 norms displayed on Figure 2, yielding an optimal H_2 norm of 0.6436 consistent with the analytic global minimum $\sqrt{\sqrt{2} - 1}$. Note the use of the optional third input parameter specifying a verbose printing level. Note also that the sequences generated on your own computer may differ since random starting points are used.

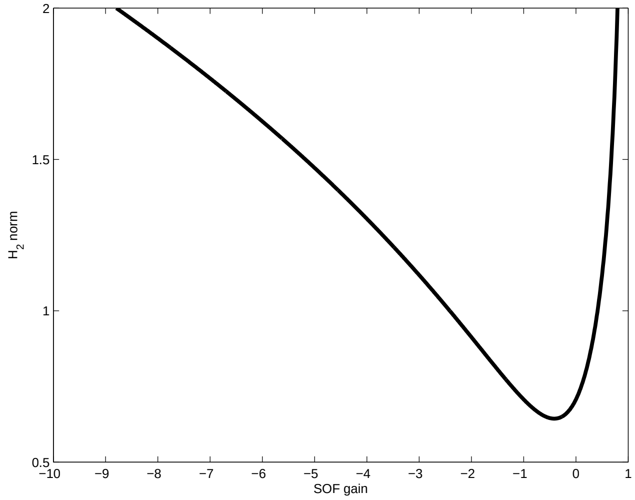

Figure 1: H_2 norm as a function of feedback gain.

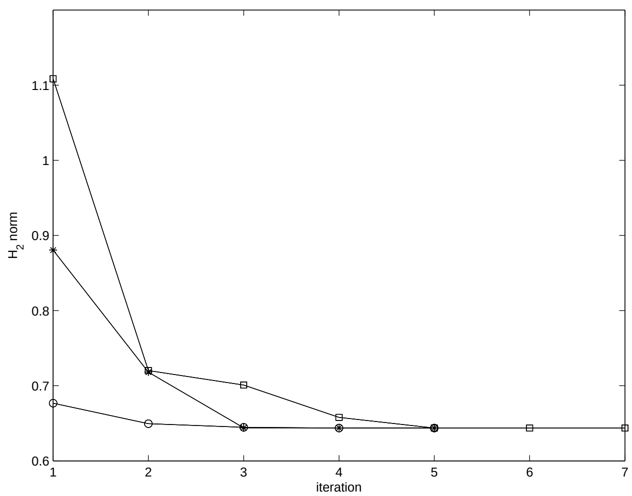

Figure 2: H_2 norm sequences optimized within HIFOO.

2.2 Mixed H_2/H_∞ synthesis

One interesting feature of adding H_2 performance in the HIFOO package is the possibility to address the general mixed H_2/H_∞ synthesis problem depicted on Figure 3 where the open-loop plant is denoted by P and the controller is denoted by K. A minimal state-space realization of the plant is given by

$$P(s) := \left[\begin{array}{c|ccc} A & B_\infty & B_2 & B \\ \hline C_\infty & D_\infty & \mathbf{0} & D_{\infty u} \\ C_2 & \mathbf{0} & \mathbf{0} & D_{2u} \\ C & D_{y\infty} & \mathbf{0} & \mathbf{0} \end{array}\right].$$

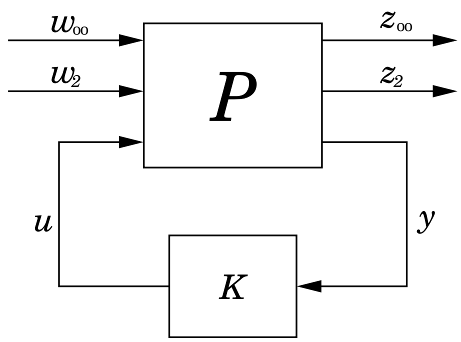

Figure 3: Standard feedback configuration for mixed H_2/H_∞ synthesis.

The optimization problem reads

$$\begin{array}{ll} \min_K & \|P_2(s)\|_2 \\ \text{s.t.} & \|P_\infty(s)\|_\infty \leq \gamma_\infty \end{array}$$

where $P_2(s)$ is the transfer function between H_2 performance signals w_2 and z_2, and $P_\infty(s)$ is the transfer function between H_∞ signals w_∞ and z_∞:

$$\begin{aligned} P_2(s) &:= \left[\begin{array}{c|cc} A & B_2 & B \\ \hline C_2 & \mathbf{0} & D_{2u} \\ C & \mathbf{0} & \mathbf{0} \end{array}\right] \\ P_\infty(s) &:= \left[\begin{array}{c|cc} A & B_\infty & B \\ \hline C_\infty & D_\infty & D_{\infty u} \\ C & D_{y\infty} & \mathbf{0} \end{array}\right]. \end{aligned} \tag{3}$$

An academic example for which the global optimal solution has been calculated in [3] is used as an illustration for the mixed H_2/H_∞ synthesis problem. Data for the model are given by

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad B_2 = \mathbf{1}_2 \qquad D_{2u} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$C_\infty = \begin{bmatrix} 0 & 1 \end{bmatrix} \quad B_\infty = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad D_{\infty u} = 0$$

$$D_\infty = 0 \qquad D_{y\infty} = 0 \qquad D_{y2} = \mathbf{0}_{1\times 2}.$$

The analytical solution may be found by solving the following mathematical programming problem as in [3]

$$\begin{array}{ll} \min\limits_{k} & J(k) \\ \text{s.t.} & \\ & k < 0 \\ & f(k) \leq \gamma_\infty. \end{array} \tag{4}$$

For a non redundant mixed H_2/H_∞ ($1 < \gamma_\infty < \frac{3}{\sqrt{5}}$), the global optimal solution is

$$k^* = -\sqrt{2 - 2\sqrt{1 - 1/\gamma^2}}$$
$$\|P_2\|_2 = \alpha^* = \sqrt{\frac{4-3\sqrt{1-1/\gamma^2}}{\sqrt{2-2\sqrt{1-1/\gamma^2}}}}. \tag{5}$$

For $\gamma = 1.2$, HIFOO gives the global optimal solution

$$\begin{aligned} &k^* = -0.9458 \\ &\|P_2\|_2 = 1.5735 \quad \|P_\infty\|_\infty = 1.2 \end{aligned} \tag{6}$$

with the input sequence

```
P2=struct('A',[0 1;-1 0],'B1',eye(2),'B2',[0;1],...
'C1',[1 0;0 0],'C2',[0 1],'D11',zeros(2,2),...
'D12',[0;1],'D21',[0 0],'D22',0);

Pinf = struct('A',[0 1;-1 0],'B1',[1;0],...
'B2',[0;1],'C1',[0 1],'C2',[0 1],'D11',0,...
'D12',0,'D21',0,'D22',0);

K=hifoo({P2,Pinf},'th',[Inf,1.2])
```

3 Implementation details

3.1 Implementing H_2 norm into HIFOO

In the main HIFOO function `hifoomain.m`, we added an option `'t'` for H_2 norm specification, without affecting the existing features. Proceeding this way, the H_2 norm can enter the objective function or a performance constraint. The H_2 synthesis works in the same way as the H_∞ synthesis, using a stabilization phase followed by an optimization phase.

A typical call of HIFOO for H2 static output feedback design is as follows:

```
K = hifoo('AC1','t')
```

where `AC1` refers to a problem of the COMPlib database, see [6, 7].

We added the function `htwo.m` computing the H_2 norm and its gradient, given the controller parameters. We had to pay special attention to the linear system of equations arising from constraint (1). When the user also specifies the controller structure, we have added this constraint to the existing H_2 constraints. To do this we had to change the way the controller structure was treated by HIFOO.

The above formulae for the computation of the H_2 norm and its gradient are given for $D_{22} = 0$, i.e. zero feedthrough matrix. When this matrix is nonzero we can use the same functions for synthesis, with some precautions. By considering the shifted output $\tilde{y} = y - D_{22}u$, we recover the initial case with zero feedthrough matrix. We compute the controller matrices $\hat{A}_K$, $\hat{B}_K$, $\hat{C}_K$, $\hat{D}_K$ based on the shifted output and then we apply a transformation on the controller matrix. We obtain the final solution:

$$\begin{array}{rcl} A_K & = & \hat{A}_K - \hat{B}_K D_{22}(1 + \hat{D}_K D_{22})^{-1}\hat{C}_K \\ B_K & = & \hat{B}_K(1 - D_{22}(1 + \hat{D}_K D_{22})^{-1}\hat{D}_K) \\ C_K & = & (1 + \hat{D}_K D_{22})^{-1}\hat{C}_K \\ D_K & = & (1 + \hat{D}_K D_{22})^{-1}\hat{D}_K. \end{array} \tag{7}$$

Given the way this case is treated, multiple plant optimization only works if the plants have the same feedthrough matrix. Note however that the case of nonzero feedthrough matrix and imposed controller structure cannot be treated by the current version of the program but could be the object of further development.

3.2 Numerical linear algebra

For H_∞ norm optimization, HIFOO calls Matlab's function `eig` to check stability, returns `inf` if unstable, and otherwise calls the Control System Toolbox function `norm`, which proceeds by bisection on successive computations of spectra of Hamiltonian matrices. This latter function relies heavily on system matrix scaling, on SLICOT routines, and its is regularly updated and improved by The MathWorks Inc. We observe experimentally that calling `eig` once before calling `norm` is negligible (less than 5%) in terms of total computational cost.

For H_2 norm optimization, HIFOO calls `eig` to check stability, returns `inf` if unstable, and otherwise calls Matlab's `lyap` function to compute the norm and its gradient. Experimentally, we observe that the time spent by `eig` to check stability is approximately 20% of the time spent to solve the two Lyapunov functions.

So a priori stability check is negligible for H_∞ optimization and comparatively small but not negligible for H_2 optimization. In this latter case there is some room for improvement, but since the overall objective of the HIFOO project is not performance and speed but reliability, we decided to keep the stability check for H_2 optimization.

4 Benchmarking

4.1 H_2 synthesis

We have extensively benchmarked HIFOO on problem instances studied already in [4] with an LMI/randomized algorithm. Since random starting points are used in HIFOO we kept the best results over 10 attempts each with 3 starting points, with no computation time limit. We ran the algorithms only on systems which are not open-loop stable. For comparison, we took the best results obtained in [4]. In Table 1, we use the following notations:

- $\star$: linear system (1) has a unique solution which is not stabilizing
- $\bullet$: linear system (1) has no solution
- $+$: algorithm initialized with a stabilizing controller
- $\dagger$: no stabilizing controller was found
- r: rank assumptions on problem data are violated.

Also n_x, n_u, n_y denote the number of states, inputs and outputs. In addition to H_2 norms obtained by the LMI algorithm and HIFOO, we also report for information the H_2 norm achievable by full-order controller design with HIFOO. Numerical values are reported to three significant digits for space reasons.

In some cases (e.g. `IH` and `CSE2`) we observe that the norms achieved with a full-order controller are greater than the norms achieved with a static output feedback controller. This is due to the difficulty of finding a good initial point in the full-order case. A more practical approach, not pursued here, consists in gradually increasing the order of the controller, using the lower order controller found at the previous step.

For the considered examples, HIFOO generally gives better results than the randomized/LMI method of [4]. We also report the performance achievable with a full-order controller designed with HIFOO. We could not use the H_2 optimal synthesis functions of the Control System Toolbox for Matlab as the technical assumptions (rank conditions on systems data) under which these functions are guaranteed to work are most of the time violated.

4.2 H_2 synthesis for larger order systems

For larger order systems, we compared our results with those of [1] which are also based on nonsmooth optimization (labeled NSO). In Table 2 the column n_k indicates the order of the designed controller. We observe that HIFOO yields better results, except for example `CM4` in the static output feedback case.

4.3 Mixed H_2/H_∞ synthesis

In this section we compare the results achieved with HIFOO with those of [1] in the case of the mixed H_2/H_∞ synthesis problem, depicted on Figure 3.

In Table 3 n_k is the order of the controller and γ_∞ is the level of H_∞ performance (a constraint). For problem dimensions refer to Table 2.

We observe that HIFOO returns better or similar results than the non-smooth optimization (NSO) method of [1], except for problem `CM4` in the static output feedback case.

Based on the CPU times of the NSO method gracefully provided to us by Aude Rondepierre (not reported here), we must however mention that HIFOO is typically much slower. This is not surprising however since HIFOO is Matlab interpreted, contrary to the NSO method which is compiled.

5 Conclusion

This paper documents the extension of HIFOO to H_2 performance. The resulting new version 3.0 of HIFOO has been extensively benchmarked on H_2 and H_2/H_∞ minimization problems. We illustrated that HIFOO gives better results than alternative methods for most of the considered benchmark problems.

HIFOO is an open-source public-domain software that can be downloaded at

`www.cs.nyu.edu/overton/software/hifoo`

Feedback from users is welcome and significantly helps us improve the software and our understanding of nonsmooth nonconvex optimization methods applied to systems control.

Just before the completion of this work, Pierre Apkarian informed us that several algorithms of nonsmooth optimization have now been implemented by The MathWorks Inc. and will be released in the next version of the Robust Control Toolbox for Matlab. Extensive comparison with HIFOO will therefore be an interesting further research topic.

Acknowledgments

This work benefited from feedback by Wim Michiels, Marc Millstone, Michael Overton and Aude Rondepierre. The research of Suat Gumussoy was supported by the Belgian Programme on Interuniversity Poles of Attraction, initiated by the Belgian State, Prime Minister's Office for Science, Technology and Culture, and of the Optimization in Engineering Centre OPTEC. The research of Didier Henrion was partly supported by project No. 103/10/0628 of the Grant Agency of the Czech Republic. Denis Arzelier gratefully acknowledges Michael Overton and the Courant Institute of Mathematical Sciences of NYU, New York City, New York, USA, for hospitality, where this work was initiated. Support for this work was provided in part by the grant DMS-0714321 from the U.S. National Science Foundation.

References

[1] P. Apkarian, D. Noll and A. Rondepierre. Mixed H_2/H_∞ control via nonsmooth optimization. SIAM Journal on Control and Optimization, 47(3):1516-1546, 2008.

[2] D. Arzelier, B. Clément, D. Peaucelle. Multi-objective H_2/H_∞/impulse-to-peak control of a space launch vehicle. European Journal of Control, Vol. 12, No. 1, 2006.

[3] D. Arzelier, D. Peaucelle. An iterative method for mixed H2/Hinfinity synthesis via static output-feedback. IEEE Conference on Decision and Control (CDC), Las Vegas, Nevada, 2002.

[4] D. Arzelier, E.N. Gryazina, D. Peaucelle, B.T. Polyak. Mixed LMI/randomized methods for static output feedback control design: stability and Performance. Technical Report LAAS-CNRS, N09640, September 2009. Available at `homepages.laas.fr/arzelier`

[5] J. V. Burke, D. Henrion, A. S. Lewis and M. L. Overton. Stabilization via nonsmooth, nonconvex optimization. IEEE Transactions on Automatic Control, 51(11):1760-1769, 2006.

[6] J. V. Burke, D. Henrion, A. S. Lewis and M. L. Overton. HIFOO - A Matlab Package for Fixed-order Controller Design and H-infinity Optimization. IFAC Symposium on Robust Control Design, Toulouse, France, 2006.

[7] S. Gumussoy, D. Henrion, M. Millstone and M.L. Overton. Multiobjective Robust Control with HIFOO 2.0. IFAC Symposium on Robust Control Design (ROCOND), Haifa, Israel, 2009.

[8] T. Rautert and E. W. Sachs. Computational design of optimal output feedback controllers, SIAM Journal on Optimization, 7(3):837-852, 1997.

[9] C. W. Scherer. Multi-objective H_2/H_∞ control. IEEE Transactions on Automatic Control, 40:1054–1062, 1995.

[10] J. Vanbiervliet, B. Vandereycken, W. Michiels, S. Vandewalle and M. Diehl. Smoothed spectral abscissa for robust stability optimization. SIAM Journal on Optimization, 20(1):156-171, 2009.

[11] K. Zhou, J. Doyle and K. Glover. Robust and Optimal Control, Prentice-Hall, 1996.

Table 1: H_2 norm achieved for SOF controller design with LMI/randomized methods and HIFOO, and full-order controller design with HIFOO.

	n_x	n_u	n_y	SOF LMI	SOF HIFOO	full HIFOO
AC1	5	3	3	3.41e-7	1.46e-9	1.81e-15
AC2	5	3	3	0.0503	0.0503	0.0491
AC5	4	2	2	1470	1470	1340
AC9	10	4	5	r	1.44	1.41
AC10	55	2	2	r	27.8 (+)	†
AC11	5	2	4	3.94	3.94	3.64
AC12	4	3	4	r	0.0202	5.00e-5
AC13	28	3	4	132	132	106
AC14	40	3	4	r	⋆	7.00
AC18	10	2	2	19.7	19.7	18.6
HE1	4	2	1	0.0954	0.0954	0.0857
HE3	8	4	6	r	0.812	0.812
HE4	8	4	6	21.7	20.8	18.6
HE5	4	2	2	r	⋆	1.59
HE6	20	4	6	r	•	•
HE7	20	4	6	r	•	•
DIS2	3	2	2	1.42	1.42	1.40
DIS4	6	4	6	1.69	1.69	1.69
DIS5	4	2	2	r	⋆	1280
JE2	21	3	3	1010	961	623
JE3	24	3	6	r	•	•
REA1	4	2	3	1.82	1.82	1.50
REA2	4	2	2	1.86	1.86	1.65
REA3	12	1	3	12.1	12.1	9.91
WEC1	10	3	4	7.36	7.36	5.69
BDT2	82	4	4	r	0.795	0.655
IH	21	11	10	1.66	1.54e-4	0.203
CSE2	60	2	30	0.00890	0.00950	0.0133
PAS	5	1	3	0.00920	0.00380	0.00197
TF1	7	2	4	r	0.164	0.136
TF2	7	2	3	r	†	10.9
TF3	7	2	3	r	13.6	0.136
NN1	3	1	2	41.8	41.8	35.0
NN2	2	1	1	1.57	1.57	1.54
NN5	7	1	2	142	142	82.4
NN6	9	1	4	1350	1310	314
NN7	9	1	4	133	133	84.2
NN9	5	3	2	r	29.7	20.9
NN12	6	2	2	18.9	18.9	10.9
NN13	6	2	2	r	•	•
NN14	6	2	2	r	⋆	†
NN15	3	2	2	0.0485	0.0486	0.0480
NN16	8	4	4	0.298	0.291	0.342
NN17	3	2	1	9.46	9.46	3.87
HF2D10	5	2	3	7.12e4	7.12e4	7.06e4
HF2D11	5	2	3	8.51e4	8.51e4	8.51e4
HF2D14	5	2	4	3.74e5	3.74e5	3.73e5
HF2D15	5	2	4	2.97e5	2.97e5	2.84e5
HF2D16	5	2	4	2.85e5	2.85e5	2.84e5
HF2D17	5	2	4	3.76e5	3.76e5	3.75e5
HF2D18	5	2	2	27.8	27.8	24.3
TMD	6	2	4	r	1.36	1.32
FS	5	1	3	1.69e4	1.69e4	1.83e4

Table 2: H_2 norm achieved with HIFOO compared with the nonsmooth optimization method of [1].

	n_x	n_u	n_y	n_k	HIFOO	NSO
AC14	40	4	3	1	21.4	21.4
				10	7.00	8.10
				20	7.00	7.56
BDT2	82	4	4	0	0.791	0.794
				10	0.598	0.789
				41	0.585	0.779
HF1	130	1	2	0	0.0582	0.0582
				10	0.0581	0.0582
				25	0.0581	0.0581
CM4	240	1	2	0	61.0	0.926
				50	0.933	0.938

Table 3: Mixed H_2/H_∞ design with HIFOO compared with the nonsmooth optimization method of [1].

	n_k	γ_∞	H_2 HIFOO	H_2 NSO	H_∞ HIFOO	H_∞ NSO
AC14	1	1000	21.4	21.4	230	231
	10	1000	7.01	8.78	100	101
	1	200	21.7	21.5	200	200
	20	200	7.08	7.99	100	100
BDT2	0	10	0.790	0.804	0.908	1.06
	10	10	0.608	0.765	0.867	1.11
	0	0.8	0.919	0.791	0.943	0.800
	10	0.8	1.16	0.772	1.23	0.800
	41	0.8	1.24	0.789	2.32	0.800
HF1	0	10	0.0582	0.0582	0.460	0.461
	0	0.45	0.0588	0.0588	0.450	0.450
	10	0.45	0.0586	0.0587	0.450	0.450
	25	0.45	0.0586	0.0587	0.450	0.450
CM4	0	10	0.927	0.927	1.66	1.66
	0	1	0.986	0.984	1.00	1.00
	25	1	1.25	0.953	10.4	1.00