Convergent LMI relaxations for non-convex optimization over polynomials in control

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Introduction

It is well known that most analysis and design problems in robust and nonlinear control can be formulated as global optimization problems with polynomial objective functions and constraints [1]. Typical examples include robust stability analysis for characteristic polynomials with parametric uncertainty, simultaneous stabilization of linear systems, pole assignment by static output feedback, and stability analysis for polynomial systems by Lyapunov's second approach. In some specific cases, there exist computationally efficient techniques for solving these problems. For example, vertex or extremal results such as Kharitonov's Theorem or the Edge Theorem can be used to perform robust stability analysis without optimization [2, 3, 4]. Some of these results have been extended to robust design of fixed-order or fixed-structure controllers, for example, PID design [5]. In the same vein, static state-feedback design, or design of a controller of the same order as the plant, can be formulated as a convex linear matrix inequality (LMI) optimization problem [6], for which polynomial-time interior point methods are available.

Polynomial optimization problems arising from control problems are often highly non-convex, with several local optima, and are difficult to solve [1]. Although general purpose global optimization algorithms can be applied, the computational cost is often an exponential function of the number of decision variables. To overcome the use of computationally intensive algorithms, researchers have focused on the development of relaxation or simplification techniques relying on convex optimization. A convex relaxation of a non-convex problem is obtained by removing non-convex constraints or replacing them with necessary (but generally not equivalent) convex constraints, hence simplifying and enlarging the set over which the optimization is carried out. In the last decade, semidefinite programming, or optimization over LMIs, has established itself as a popular convex relaxation technique in the systems and control community [7].

Conservatism is the price one has to pay when simplifying a non-convex problem. For example, convex sufficient stability conditions are frequently used instead of non-convex necessary and sufficient stability conditions when performing robust design. Generally speaking, there is a trade-off between the amount of conservatism and the computational cost when solving a non-convex problem. Due to the amount of conservatism inherent in LMI techniques, which is difficult to measure accurately for practical control problems [8], there has recently been a surge of interest in approaches that gradually increase computational complexity. Most of these approaches are based on sufficient conditions for the positivity of multivariable polynomials. For example, positivity of polynomials is replaced with the stronger "sum of squares" constraint, which has an LMI formulation [9]. An alternative approach based on the theory of moments has been developed in [10, 11].

For non-convex problems, the relaxation technique described in [10, 11] enables the user to systematically construct an increasing sequence of convex LMI relaxations, whose optima are guaranteed to converge monotonically to the global optimum of the original non-convex global optimization problem. A Matlab [12] implementation of the relaxation technique GloptiPoly has been recently developed as an open-source freeware based on the LMI solver SeDuMi [13]. An alternative software is SOSTOOLS [14], which also uses SeDuMi to solve sums-of-squares optimization programs over polynomials, based on the theory described in [9]. Numerical experiments suggest that for most small- and medium-size problems in the technical literature on global optimization, the global optimum is reached with LMI relaxations of medium size, at a relatively low computational cost. Moreover, global optimality can sometimes be proved by using sufficient rank conditions and numerical linear algebra techniques [15, 16].

The objective of this article is to show how GloptiPoly can solve challenging non-convex optimization problems in robust and nonlinear control. First, we describe several non-convex optimization problems arising in control system analysis and design. These problems involve multivariable polynomial objective functions and constraints. We then review the theoretical background behind GloptiPoly. Next, an example is presented to illustrate the successive LMI relaxations and general features of the GloptiPoly software. Finally, we apply GloptiPoly to several control-related problems.

Examples of control problems

In this section we enumerate standard control problems that can be formulated into optimization problems with multivariable polynomial objective functions and constraints. These problems are of the form

$$\begin{aligned} \mathbb{P} : p^* = \quad & \min_x \quad g_0(x), \\ & \text{s.t.} \quad\; g_k(x) \geq 0, \quad k = 1, \ldots, m, \end{aligned} \tag{1}$$

where $g_k : \mathbb{R}^n \to \mathbb{R}$ is a real-valued polynomial of $\mathbb{R}[x_1, \ldots, x_n]$ for all $k = 1, \ldots, m$.

Problem I: Robust stability analysis

As explained in [2], analyzing the stability of linear systems affected by parametric uncertainty amounts to checking robust stability of uncertain polynomials. Consider the characteristic polynomial

$$s^3 + (1 + q_1 + q_2)s^2 + (q_1 + q_2 + 3)s + (1 + 6q_1 + 6q_2 + 2q_1q_2),$$

where the uncertain parameters q_1 and q_2 satisfy $q_1 \in [1.4 - 1.1k,\ 1.4 + 1.1k]$ and $q_2 \in [0.85 - 0.85k, 0.85 + 0.85k]$, where $k > 0$. The uncertain parameters q_1 and q_2 can represent physical quantities such as masses and lengths whose exact values are not known, or controller parameters that might vary due to implementation errors. We are interested in computing the largest value of k for which the polynomial is robustly stable. See [2] for physical examples such as a crane and four-wheel car steering that lead to uncertain characteristic polynomials.

Due to the nonlinear dependence of the polynomial coefficients on the uncertain parameters, specifically, the q_1q_2 monomial in the constant coefficient, we cannot apply Kharitonov's Theorem or the Edge Theorem [2] to ascertain robust stability. Alternatively, recall that the Hurwitz stability criterion states that the third-degree monic polynomial $s^3 + a_2s^2 + a_1s + a_0$ is stable if and only if $a_2a_1 - a_0 > 0$ and $a_0 > 0$. As a result, the maximum stability bound $k^\star = \min(k_1,\ k_2)$ is given by the solutions of the optimization problems

$$
\begin{aligned}
k_1 \quad = \quad \min \quad & k, \\
\text{s.t.} \quad & 2 - 2q_1 - 2q_2 + q_1^2 + q_2^2 \leq 0, \\
& 1.4 - 1.1k \leq q_1 \leq 1.4 + 1.1k, \\
& 0.85 - 0.85k \leq q_2 \leq 0.85 + 0.85k
\end{aligned} \tag{2}
$$

and

$$
\begin{aligned}
k_2 \quad = \quad \min \quad & k, \\
\text{s.t.} \quad & 1 + 6q_1 + 6q_2 + 2q_1q_2 \leq 0, \\
& 1.4 - 1.1k \leq q_1 \leq 1.4 + 1.1k, \\
& 0.85 - 0.85k \leq q_2 \leq 0.85 + 0.85k,
\end{aligned} \tag{3}
$$

both of which are of the form (1). Problems (2) and (3) yield the smallest value of k that violates the positivity of the expressions $a_2a_1 - a_0$ and a_0, respectively. These problems were originally described in [17, Ex. 1] as test cases for global optimization software.

Problem II: Simultaneous stabilization

As shown in [18], the problem of simultaneously stabilizing the three plants

$$
\frac{2-s}{(s^2-1)(s+2)}, \quad \frac{2-s}{s^2(s+2)}, \quad \frac{2-s}{(s^2+1)(s+2)}
$$

with a single stable second-order compensator of the form

$$
\frac{a(s+b)^2}{(s+d)^2}
$$

is equivalent to solving the Liénard-Chipart polynomial inequalities in the controller parameters a, b, and d given by

$$
\begin{aligned}
&a, b, d > 0,\\
&ab^2 - d^2 > 0,\\
&-ab + a + d^2 - d - 1 > 0,\\
&ab - ad - 2a + d^3 + 4d^2 + 4d > 0,\\
&ab^3 - ab^2d - 4ab^2 + 2abd + 4ab + 2bd^3 + 5bd^2 + 2bd - d^3 - 4d^2 - 4d > 0,\\
&ab - 2a - bd^2 - 4bd - 4b + 2d^2 + 3d - 2 > 0.
\end{aligned}
$$

To find a feasible solution to these polynomial inequalities, we can choose an arbitrary polynomial criterion to obtain a problem $\mathbb{P}$ of the form (1).

Problem III: Minimum distance to a surface

Computing the minimum distance from a point to a surface is an important problem in robust and nonlinear control [19]. Consider, for example, the l_2 parametric stability margin of the linear system $\dot{x} = A(q)x$ whose state matrix $A(q)$ depends on a vector of uncertain parameters q. Assuming the matrix $A(0)$ is Hurwitz, the l_2 parametric stability margin is given by $\sqrt{\min(\rho_R,\ \rho_I)}$, where ρ_R is given by

$$
\begin{aligned}
\rho_R \;=\; &\min \quad \|q\|_2^2,\\
&\text{s.t.} \quad \det A(q) = 0,
\end{aligned}
$$

and ρ_I is given by

$$
\begin{aligned}
\rho_I \;=\; &\min \quad \|q\|_2^2,\\
&\text{s.t.} \quad H_{n-1}(q) = 0,
\end{aligned}
$$

where $H_{n-1}(q)$ is the next-to-the-last Hurwitz determinant of the matrix $A(q)$.

Both of the above problems are minimum distance problems. With the state matrix

$$A(q) = \begin{bmatrix} q_1 & 1+2q_1 & q_2 \\ 0 & q_1+q_2 & 1-q_1 \\ -1+2q_2 & -3 & -3-3q_1 \end{bmatrix}$$

studied in [19, Ex. 1], these problems are formulated as

$$\begin{aligned} \rho_R &= \min \quad q_1^2+q_2^2, \\ & \qquad \text{s.t.} \quad -3q_1^3-7q_1^2q_2-2q_1q_2^2-2q_2^3-4q_1^2+q_2^2+2q_1+2q_2-1=0 \end{aligned}$$

and

$$\begin{aligned} \rho_I &= \min \quad q_1^2+q_2^2, \\ & \qquad \text{s.t.} \quad -8q_1^3-4q_1^2q_2-2q_1q_2^2-28q_1^2+q_1q_2-3q_2^2-22q_1-7q_2+8=0. \end{aligned}$$

Clearly, both ρ_R and ρ_I are solutions of a global optimization problem $\mathbb{P}$ of the form (1).

Problem IV: Pole assignment by static output feedback

Assigning the poles of the linear system

$$\begin{aligned} \dot{x} &= Ax+Bu, \\ y &= Cx \end{aligned}$$

by static output feedback

$$u = Ky$$

amounts to solving a set of multivariable polynomial equations. Indeed, a complex number λ is a closed-loop pole if and only if it satisfies the equation

$$\det(\lambda I - A - BKC) = 0,$$

which is a polynomial function of the entries of the feedback matrix K. To find a feasible solution to these polynomial equations, we can choose an arbitrary polynomial criterion to obtain a problem $\mathbb{P}$ in the form (1).

For example, consider the linearized longitudinal motion of a helicopter at the airspeed of 135 knots introduced in [20] and considered in [21], whose state-space representation is given by

$$\dot{x} = \begin{bmatrix} -0.0366 & 0.0271 & 0.0188 & -0.4555 \\ 0.0482 & -1.0100 & 0.0024 & -4.0208 \\ 0.1002 & 0.3681 & -0.7070 & 1.4200 \\ 0 & 0 & 1.0000 & 0 \end{bmatrix} x + \begin{bmatrix} 0.4422 & 0.1761 \\ 3.5446 & -7.5920 \\ -5.5200 & 4.4900 \\ 0 & 0 \end{bmatrix} u,$$

$$y = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} x.$$

State variables correspond to horizontal velocity (knots), vertical velocity (knots), pitch rate (deg/s), pitch angle (deg), respectively. The inputs are collective pitch control and longitudinal cyclic pitch control. We wish to assign the poles to be -0.1, -0.2, -0.3, and -0.4.

Problem V: Nonlinear stability analysis

To prove asymptotic stability of the dynamical system $\dot{x} = f(x)$, where $f(x)$ is a polynomial function of the state vector x, we seek a Lyapunov function $V(x)$ such that $V(x) > 0$ and $\dot{V}(x) < 0$ for all nonzero x. When $V(x)$ is a given positive polynomial, checking $\dot{V}(x) < 0$ amounts to checking negativity of the polynomial $\dot{V}(x)$ for nonzero values of x.

Suppose we want to prove asymptotic stability of the nonlinear system described in

[22, Ex. 2.5] and studied in [9, Ex. 7.1] (beware of the typos there) given by

$$\begin{aligned}\dot{x_1} &= -x_1^3 - x_2x_3 - x_1 - x_1x_3^2,\\ \dot{x_2} &= -x_1x_3 + 2x_1^2 - x_2,\\ \dot{x_3} &= -x_3 + 2x_1^3.\end{aligned}$$

We choose

$$V(x) = (x_1^2 + x_2^2 + x_3^2)/2$$

as a candidate Lyapunov function. To check negativity of its derivative, we must solve the optimization problem

$$\max \dot{V}(x) = \max x_1(-x_1^3 - x_2x_3 - x_1 - x_1x_3^2) + x_2(-x_1x_3 + 2x_1^2 - x_2) + x_3(-x_3 + 2x_1^3),$$

which is again a problem $\mathbb{P}$ in the form (1).

Theoretical background

GloptiPoly is based on the theory of positive polynomials and moments described in [10, 11], which we briefly summarize below.

Consider the multivariate polynomial optimization problem $\mathbb{P}$ given in (1). Equality constraints are allowed by means of two opposite inequalities, so that (1) describes all optimization problems that involve polynomials. In particular, this formulation encompasses non-convex quadratic problems as well as discrete optimization problems, such as 0-1 nonlinear programming problems. Denote by $\mathbb{K}$ the feasible set of $\mathbb{P}$, that is,

$$\mathbb{K} = \{x \in \mathbb{R}^n : g_k(x) \geq 0, \quad k = 1, \ldots, m\}. \tag{4}$$

The idea behind the methodology in GloptiPoly is to construct a sequence of convex semidefinite programming (SDP) or LMI relaxations of $\mathbb{P}$ of increasing size and whose sequence of optimal values converges to the global optimal value $p^* = \inf \mathbb{P}$.

The original idea can be traced back to the pioneering *reformulation linearization technique* (RLT) of [23, 24], where additional redundant constraints (products of the original ones) are introduced and linearized in a higher space (lifting) by introducing additional variables such as $y_{ij} = x_i x_j$ to obtain a linear programming (LP) relaxation. Convergence has been proved for 0-1 nonlinear programs [11]. Later, Shor [25, 26] proposed a lifting procedure to reduce any polynomial programming problem to a quadratic one, and then used a semidefinite relaxation to obtain a lower bound for p^* [27]. The strikingly good approximation of Goemans and Williamson for the MAX-CUT problem [28], obtained from a simple LMI relaxation, excited the curiosity of researchers studying LMI relaxations. However, with the exception of the LP-relaxations of Sherali-Adams and the conceptual "lift and project" method of Lovász-Schrijver, both for 0-1 problems, no proof of convergence was provided.

The proof of convergence of the LMI relaxations defined in [10, 11] and used in GloptiPoly is based on recent results in real algebraic geometry concerning the representation of polynomials that are strictly positive on a semi-algebraic set; see [9] for a related approach. It turns out that the primal and dual LMI relaxations of GloptiPoly correspond to the dual theories of *moments* and *positive polynomials*.

Indeed, while the primal relaxations aim at finding the moments of a probability measure with mass concentrated on some global minimizers of $\mathbb{P}$, the dual relaxations aim at representing the polynomial $g_0(x) - p^*$, which is positive on the semi-algebraic feasible set $\mathbb{K}$ of $\mathbb{P}$, as a linear combination of the g_i's with polynomial weights that are sums of squares, as in Putinar's representation of polynomials that are strictly positive on a semi-algebraic set [29].

Primal relaxations

In brief, the primal LMI relaxations $\{\mathbb{Q}_i\}$ of $\mathbb{P}$ are relaxations of the moment problem

$$p^* = \min_{\mu}\{\int g_0 \, d\mu \; : \; \mu(\mathbb{K}) = 1, \; \mu(\mathbb{K}^c) = 0\}, \tag{5}$$

which is equivalent to $\mathbb{P}$, where $\mathbb{K}^c$ denotes the complement of $\mathbb{K}$. In other words, the minimum is taken over all probability measures μ on the feasible set $\mathbb{K}$ of $\mathbb{P}$. For $\alpha \in \mathbb{N}^n$, $g_0(x) = \sum_\alpha (g_0)_\alpha x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ is a polynomial, and thus the criterion is a finite linear combination $\sum_\alpha (g_0)_\alpha y_\alpha$ of *moments*

$$y_\alpha = \int x_1^{\alpha_1} \cdots x_n^{\alpha_n} \, d\mu$$

of the probability measure μ. The relaxations of (5) are obtained by replacing the constraint that μ has its support in $\mathbb{K}$ ($\mu(\mathbb{K}^c) = 0$) with progressively stronger semidefinite programming conditions on its moments.

For instance, let $2v_k - 1$ or $2v_k$ be the degree of the polynomial g_k in the definition (4) of the set $\mathbb{K}$, and let $v = \max_k v_k$. Then, the relaxation of order i includes the constraints

$$\int f^2 g_k \, d\mu \geq 0, \quad k = 1, \ldots, m, \tag{6}$$

for all polynomials $f \in \mathbb{R}[x_1, \ldots, x_n]$ of degree at most $i - v$. Inequalities (6) translate into equivalent LMI constraints on the moments $\{y_\alpha\}$ of μ of order $|\alpha| \leq 2i$. Of course, the larger the order i, the larger the size of the associated LMI constraints.

Dual relaxations

On the other hand, the LMI relaxations $\{\mathbb{Q}_i^*\}$ that are dual to $\{\mathbb{Q}_i\}$ solve the optimization problems

$$\max_{p_i, \{q_k\}} \{p_i \ : \ g_0(x) - p_i = q_0 + \sum_{k=1}^{m} g_k(x) q_k(x)\}, \tag{7}$$

where the unknowns $\{q_k\}$ are polynomials in x, all sums of squares. Both the number of variables and the number of constraints in the relaxation $\mathbb{Q}_i^*$ depend on the maximum degree $2i$ allowed in the right-hand-side of (7). Requiring that all of the polynomials q_k are sums-of-squares of degree at most $2(i - v_k)$ (with $v_0 = 0$) translates easily into equivalent LMI conditions and yields the dual LMI of $\mathbb{Q}_i$. The increasing numbers of variables and constraints in the relaxations reflect that the degree $2i$ must be large enough in (7) for p_i to be as close as desired to p^* (and often to be exactly equal to p^*). For more details on these dual points of view, the reader is referred to [10, 11] and the references therein.

We consider mild technical assumptions on the feasible set $\mathbb{K}$, which are satisfied, for example, when $\mathbb{K}$ is a polytope, or when the level set $g_k(x) \geq 0$ is compact for some index k. Such assumptions can always be satisfied by enforcing a sufficiently large feasibility radius on the decision variables, that is, by introducing the additional Euclidean norm constraint $\|x\|^2 \leq R^2$ for sufficiently large R. Then it was proved in [10] that $\inf \mathbb{Q}_i$ converges to $\inf \mathbb{P}$ as i tends to infinity. In other words, letting p_i^* denote the optimum obtained by solving the LMI relaxation $\mathbb{Q}_i$ of order i, we obtain a monotone sequence of optimal values p_i^* converging asymptotically to the globally optimal value p^* of the original optimization problem in (1), that is, $p_i^* \uparrow p^*$ as $i \to \infty$. This monotonicity means that the sequence is designed to do better, or at least not worse, at each step. Moreover, our computational experiments on global optimization benchmark examples [15] reveal that in practice p_i^* is very close to p^* for relatively small values of i. In addition, in many cases the exact optimal value p^* is obtained at some particular relaxation $\mathbb{Q}_i$, that is, $p^* = p_i^*$ for some relatively small i.

In GloptiPoly we have implemented a numerical linear algebra algorithm that detects global optimality, for example, to determine whether the LMI relaxation $\mathbb{Q}_i$ provides the optimal value $p_i^* = p^*$, and another algorithm to extract global minimizers. Roughly speaking, detecting global optimality amounts to checking successive ranks of moment matrices, whereas global minimizer extraction amounts to computing a Cholesky factor of the moment matrix and solving an eigenvalue problem. All of these tasks can be carried out efficiently with standard algorithms of numerical linear algebra [16]. For a comprehensive treatment of the methodology and the theory lying behind GloptiPoly, see [10, 11].

Numerical Examples

We now consider two examples to illustrate GloptiPoly by constructing successive LMI relaxations. In particular, GloptiPoly is able to transcend the difficulties normally associated with non-convex optimization. By emulating these examples, the reader should be able to build up LMI relaxations for more general polynomial optimization problems.

Example 1

Consider the non-convex optimization problem

$$
\begin{aligned}
\max \quad & x_2, \\
\text{s.t.} \quad & 3 + 2x_2 - x_1^2 - x_2^2 \geq 0, \\
& -x_1 - x_2 - x_1 x_2 \geq 0, \\
& 1 + x_1 x_2 \geq 0,
\end{aligned}
$$

where the linear objective function $x \mapsto x_2$ is maximized over a non-convex feasible set delimited by circular and hyperbolic arcs. The feasible region is shown in Figure 1.

The first LMI relaxation $\mathbb{Q}_1$ is

$$
\begin{aligned}
\max \quad & y_{01}, \\
\text{s.t.} \quad & \begin{bmatrix} 1 & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{bmatrix} \succeq 0, \\
& 3 + 2y_{01} - y_{20} - y_{02} \geq 0, \\
& -y_{10} - y_{01} - y_{11} \geq 0, \\
& 1 + y_{11} \geq 0
\end{aligned}
$$

with optimal value $p_1 = 2$. In this relaxation, the 3×3 positive semidefinite matrix is a moment matrix of order up to 2. Problem constraints are linearized with the help of these moment variables.

In Figure 2 we show the projection of the feasibility set of LMI relaxation $\mathbb{Q}_1$ onto the plane y_{10}, y_{01} of first-order moments. This convex feasibility set inscribes the original non-convex feasible set. We can see that the optimum of the LMI relaxation is achieved at a point that is infeasible for the non-convex problem.

The second LMI relaxation $\mathbb{Q}_2$ is

$$
\begin{array}{ll}
\max & y_{01} \\
\text{s.t.} & \begin{bmatrix} 1 & y_{10} & y_{01} & y_{20} & y_{11} & y_{02} \\ y_{10} & y_{20} & y_{11} & y_{30} & y_{21} & y_{12} \\ y_{01} & y_{11} & y_{02} & y_{21} & y_{12} & y_{03} \\ y_{20} & y_{30} & y_{21} & y_{40} & y_{31} & y_{22} \\ y_{11} & y_{21} & y_{12} & y_{31} & y_{22} & y_{13} \\ y_{02} & y_{12} & y_{03} & y_{22} & y_{13} & y_{04} \end{bmatrix} \succeq 0, \\
& \begin{bmatrix} 3+2y_{01}-y_{20}-y_{02} & 3y_{10}+2y_{11}-y_{30}-y_{12} & 3y_{01}+2y_{02}-y_{21}-y_{03} \\ 3y_{10}+2y_{11}-y_{30}-y_{12} & 3y_{20}+2y_{21}-y_{40}-y_{22} & 3y_{11}+2y_{12}-y_{31}-y_{13} \\ 3y_{01}+2y_{02}-y_{21}-y_{03} & 3y_{11}+2y_{12}-y_{31}-y_{13} & 3y_{02}+2y_{03}-y_{22}-y_{04} \end{bmatrix} \succeq 0, \\
& \begin{bmatrix} -y_{10}-y_{01}-y_{11} & -y_{20}-y_{11}-y_{21} & -y_{11}-y_{02}-y_{12} \\ -y_{20}-y_{11}-y_{21} & -y_{30}-y_{21}-y_{31} & -y_{21}-y_{12}-y_{22} \\ -y_{11}-y_{02}-y_{12} & -y_{21}-y_{12}-y_{22} & -y_{12}-y_{03}-y_{13} \end{bmatrix} \succeq 0, \\
& \begin{bmatrix} 1+y_{11} & y_{10}+y_{21} & y_{01}+y_{12} \\ y_{10}+y_{21} & y_{20}+y_{31} & y_{11}+y_{22} \\ y_{01}+y_{12} & y_{11}+y_{22} & y_{02}+y_{13} \end{bmatrix} \succeq 0
\end{array}
$$

with optimal value $p_2 = 1.6180$, which is the global optimum p^* within numerical accuracy. In addition, first order moments $(y^*_{10}, y^*_{01}) = (-0.6180, 1.6180)$ provide an optimal solution of the original problem. This problem features a 6×6 moment matrix corresponding to moments of order up to 4. The three 3×3 LMI constraints are the LMI formulation of (6).

In Figure 3 we show the projection of the feasibility set of the LMI relaxation $\mathbb{Q}_2$

onto the plane y_{10}, y_{01} of first-order moments. By construction, the feasibility set of the LMI relaxation $\mathbb{Q}_2$ is included in the feasibility set of the LMI relaxation $\mathbb{Q}_1$. Compared to Figure 3, we can see that the feasibility set of the LMI relaxation $\mathbb{Q}_2$ is exactly the convex hull of the original non-convex feasible set, and the global optimum is now attained because the criterion $x \mapsto x_2$ is linear in the first-order moments.

Example 2

Consider the optimization problem

$$\begin{array}{ll} \max & x_1^2 + x_2^2, \\ \text{s.t.} & 3 + 2x_2 - x_1^2 - x_2^2 \geq 0, \\ & -x_1 - x_2 - x_1 x_2 \geq 0, \\ & -1 - 4x_2 - 4x_1 x_2 \geq 0, \end{array}$$

where the convex objective function $x \mapsto \|x\|^2$, the squared Euclidean norm of x, is maximized over the non-connected set shown in Figure 4. This problem admits various local optima.

The first LMI relaxation $\mathbb{Q}_1$ given by

$$\begin{array}{ll} \max & y_{20} + y_{02}, \\ \text{s.t.} & \begin{bmatrix} 1 & y_{10} & y_{01} \\ y_{10} & y_{20} & y_{11} \\ y_{01} & y_{11} & y_{02} \end{bmatrix} \succeq 0, \\ & 3 + 2y_{01} - y_{20} - y_{02} \geq 0, \\ & -y_{10} - y_{01} - y_{11} \geq 0, \\ & -1 - 4y_{02} - 4y_{11} \geq 0 \end{array}$$

yields without any problem splitting the global optimum $p^\star = p_1 = 8.3492$ attained at $(y^*_{10}, y^*_{01}) = (-1.0935, 2.6746)$. The optimum is achieved on the boundary of the convex hull of the non-convex non-connected feasible set.

If we now wish to *minimize* (instead of maximize) $\|x\|^2$, the first LMI relaxation yields the global optimum $p^\star = p_1 = 0.059176$ attained at $(y^*_{10}, y^*_{01}) = (0.0535, -0.2372)$. From Figure 4 we can see that the global optimum is *not* achieved on the boundary of the convex hull of the feasible set.

Using GloptiPoly

In this section we use GloptiPoly to construct and solve successive LMI relaxations of a non-convex optimization problem. Only the most important features are summarized here; for a more detailed description the reader is referred to the user's guide [15].

GloptiPoly version 2.2 requires Matlab version 5.3 or higher [12], together with the freeware solver SeDuMi version 1.05 [13]. Optionally, a user-friendly interface based on the Matlab Symbolic Math Toolbox 2.1, the Matlab gateway to the kernel of Maple V [30], can be used jointly with GloptiPoly to define the optimization problems symbolically with a Maple-like syntax. In the sequel we assume that GloptiPoly has been installed according to the instructions found in

`www.laas.fr/∼henrion/software/gloptipoly`

Successive LMI relaxations

We consider non-convex quadratic problem [31, Pb. 3.5]

$$\begin{aligned}
\min \quad & -2x_1 + x_2 - x_3, \\
\text{s.t.} \quad & x_1(4x_1 - 4x_2 + 4x_3 - 20) + x_2(2x_2 - 2x_3 + 9) + \\
& \qquad + x_3(2x_3 - 13) + 24 \geq 0, \\
& x_1 + x_2 + x_3 \leq 4, \quad 3x_2 + x_3 \leq 6, \\
& 0 \leq x_1 \leq 2, \quad 0 \leq x_2, \quad 0 \leq x_3 \leq 3.
\end{aligned}$$

To define this problem with GloptiPoly we use the Matlab script

```
>> P = defipoly({'min -2*x1+x2-x3',...
['x1*(4*x1-4*x2+4*x3-20)+x2*(2*x2-2*x3+9)'...
'+x3*(2*x3-13)+24>=0'],...
'x1+x2+x3<=4','3*x2+x3<=6',...
'0<=x1','x1<=2','0<=x2','0<=x3','x3<=3'},...
'x1,x2,x3');
```

As previously mentioned, the function `defipoly` uses features of the Symbolic Math Toolbox. However, it is always possible, but more tedious, to enter problems into GloptiPoly without symbolic computations, see [15].

To solve the first LMI relaxation of the quadratic problem, we type

```
>> output = gloptipoly(P)

output =

    status: 0

      crit: -6.0000

       sol: {}
```

Field `status = 0` indicates that it is not possible to detect global optimality with this LMI relaxation. Hence `crit = -6.0000` is a lower bound on the global optimum.

Next we try to solve the second, third, and fourth LMI relaxations of the quadratic problem with the instructions

```
>> output = gloptipoly(P,2)

output =

    status: 0

      crit: -5.6923

       sol: {}

>> output = gloptipoly(P,3)

output =

    status: 0

      crit: -4.0685

       sol: {}

>> output = gloptipoly(P,4)

output =

    status: 1

      crit: -4.0000

       sol: {[3x1 double]  [3x1 double]}

>> output.sol{:}

ans =                  ans =

    2.0000                 0.5000

    0.0000                 0.0000

    0.0000                 3.0000
```

Both the second and third LMI relaxations return tighter lower bounds on the global optimum. Eventually, global optimality is reached at the fourth LMI relaxation, as certified by `status = 1`. GloptiPoly also returns two globally optimal solutions $x_1 = 2$, $x_2 = 0$,

$x_3 = 0$ and $x_1 = 0.5$, $x_2 = 0$, $x_3 = 3$ leading to `crit = -4.0000`.

Features

As shown by the above numerical examples, GloptiPoly is designed to solve an LMI relaxation of a given order, and thus can be invoked iteratively with increasing orders until the global optimum is reached. Asymptotic convergence of the optimal values of the LMI relaxations to the global optimal value of the original problem is ensured under mild technical assumptions satisfied in many practical optimization problems.

General features of GloptiPoly are

- Certificate of global optimality
- Automatic extraction of globally optimal solutions
- 0-1 or ± 1 integer constraints on some of the decision variables (combinatorial optimization problems)
- Generation of input and output data in SeDuMi's format
- Generation of moment matrices associated with LMI relaxations
- User-defined scaling of decision variables
- Exploitation of sparsity of polynomial data.

Solving Control Problems

We now use GloptiPoly to solve the control examples introduced in the first section. All of the numerical computations were carried out with Matlab 6.5 and SeDuMi 1.05

running under SunOS 5.8 on a Sun Blade 100 workstation with 640 Mb of RAM. For each LMI relaxation problem, we indicate the number n of decision variables (length of SeDuMi dual vector y) and the number m of constraints (length of SeDuMi primal vector x).

Problem I: Robust stability analysis

Since the first optimization problem (2) described in problem I is convex and quadratic, the first-order LMI relaxation in GloptiPoly necessarily returns the global optimum $k_1 = 0.3636$ ($n = 9$, $m = 21$, CPU time = 1.24 sec) using the script

```
>> P = defipoly({'min k','2-2*q1-2*q2+q1^2+q2^2<=0','1.4-1.1*k<=q1',...
   'q1<=1.4+1.1*k','0.85-0.85*k<=q2','q2<=0.85+0.85*k'},...
   'q1,q2,k');
>> output = gloptipoly(P)
output =
    status: 1
      crit: 0.3636
       sol: {[3x1 double]}
>> output.sol{:}'
ans =
    1.0001    1.0000    0.3636
```

The second problem (3) in Problem I is non-convex because of the bilinear term $2q_1q_2$. LMI relaxations of orders up to three lead to strict lower bounds or problems with no finite optimal values because the corresponding relaxations are too loose. The fourth-order LMI relaxation yields $k_2 = 1.2380$ as the optimal criterion ($n = 164$, $m = 3225$, CPU time = 5.68 sec) using the script

```
>> P = defipoly({'min k','1+6*q1+6*q2+2*q1*q2<=0','1.4-1.1*k<=q1',...
   'q1<=1.4+1.1*k','0.85-0.85*k<=q2','q2<=0.85+0.85*k'},...
   'q1,q2,k');
>> output = gloptipoly(P,4)
output =
    status: 1
      crit: 1.2380
       sol: {[3x1 double]}
>> output.sol{:}'
ans =
    0.0382   -0.2023    1.2380
```

The robust stability bound is then $k^\star = \min(k_1,\ k_2) = 0.3636$. Figure 5 shows the dominant pole of the system when $k = 0.3636$ and the parameters q_1 and q_2 vary within their admissible range. We can see that robust stability is indeed preserved.

Problem II: Simultaneous stabilization

In the simultaneous stabilization inequalities of Problem II we scale the parameter a by a factor of 100 and the parameter d by a factor of 10 according to the feasibility intervals proposed in [18]. We then build successive LMI relaxations of the non-convex constraint set. Since the relaxed LMI feasibility sets form an increasing sequence, it is hoped that for a sufficiently high relaxation order GloptiPoly returns a vector $[a\,b\,d]$, which is also feasible for the original problem.

Although LMI relaxations of order up to three are not stringent enough, the tighter fourth-order LMI relaxation ($n = 164$, $m = 3225$, CPU time = 7.61 sec) given in the script

```
>> P = defipoly({'a>=0','b>=0','d>=0','a*b^2-d^2>=0',...
   '-a*b+a+d^2-d-1>=0','a*b-a*d-2*a+d^3+4*d^2+4*d>=0',...
   ['a*b^3-a*b^2*d-4*a*b^2+2*a*b*d+4*a*b+2*b*d^3+' ...
    '5*b*d^2+2*b*d-d^3-4*d^2-4*d>=0'],...
   'a*b-2*a-b*d^2-4*b*d-4*b+2*d^2+3*d-2>=0'}, 'a,b,d');
>> pars.scaling = [100 1 10];
>> [output,sedumi] = gloptipoly(P,4,pars);
>> output
output =
    status: 1
      crit: []
       sol: {[3x1 double]}
>> output.sol{:}'
ans =
   70.3392    1.2035   10.0938
```

yields the parameters $a = 70.3392$, $b = 1.2035$, and $d = 10.0938$, which solve the simultaneous stabilization problem. The three closed-loop polynomials are (marginally) stable. In Figure 6 we plot the function $\min_k g_k(x)$ as a function of parameter $x = [a \quad b \quad d]$

for the value $d = 10.0938$ returned by GloptiPoly. We can see that the region for which the 8 polynomials $g_k(x)$ are all non-negative is quite small and contains the point $x = [70.3392 \quad 1.2035 \quad 10.0938]$. Note that for this simultaneous stabilization problem, without a proper scaling of parameters, GloptiPoly failed for numerical reasons. Improving numerical behavior of GloptiPoly is one of our current research directions.

Problem III: Minimum distance to a surface

When applying third-order LMI relaxations for ρ_I and ρ_R ($n = 27$, $m = 106$, CPU time = 0.94 and 1.01 sec respectively) we obtain the global optima $\rho_R = 0.2083$ and $\rho_I = 0.06857$, respectively, hence an l_2 stability margin of 0.2619. Figures 7 and 8 show the curves $\det A(q) = 0$ and $H_{n-1}(q) = 0$ together with circles measuring the distance with respect to the origin.

Problem IV: Pole assignment by static output feedback

Finding a 2-by-2 static output feedback matrix assigning the 4 poles amounts to solving a system of four polynomial equations of degree two with four unknowns. With the second LMI relaxation ($n = 69$, $m = 285$, CPU 1.22 sec), GloptiPoly finds the solution

$$K = \begin{bmatrix} -6.6217 & -0.7715 \\ -8.3705 & -1.0393 \end{bmatrix}.$$

Problem V: Nonlinear stability analysis

Consider global maximization of the Lyapunov derivative polynomial. As described in [10], if the global optimum is zero, then the Cholesky factorization of the matrix obtained by solving the dual of the LMI relaxation provides a decomposition of the polynomial $-\dot{V}(x)$ as a sum-of-squares, which implies global negativity of $\dot{V}(x)$.

Running GloptiPoly to solve the second LMI relaxation ($n = 34$, $m = 100$, CPU time 1.15 sec), we reach an almost zero global optimum. The Cholesky decomposition of the dual SeDuMi matrix returned in the output parameter `sedumi.x` yields the sum-of-squares decomposition (up to numerical round-off errors)

$$-\dot{V}(x) = (x_1^2 - x_1x_3 - x_2)^2 + (-0.9240x_1 + 0.3823x_3)^2 + (0.3823x_1 + 0.9240x_3)^2.$$

Conclusion

Numerical experiments reported in this article show that the general LMI relaxation methodology developed in [10] and implemented in the Matlab/SeDuMi freeware GloptiPoly [15] can help solve various non-convex robust control problems. Conservatism can be reduced at the cost of a limited amount of additional computation.

GloptiPoly is a general-purpose software with a user-friendly interface. It can solve a wide range of non-convex polynomial global optimization problems. An important and crucial feature of GloptiPoly is that *no* expert tuning is required, so that it can handle distinct problems coming from different branches of engineering and applied mathematics. This fact is illustrated by the extensive numerical examples provided in [15]. GloptiPoly can be used as black-box software, and thus cannot be considered a competitor to highly specialized codes that solve, for example, sparse polynomial systems of equations or large-scale combinatorial optimization problems.

Similarly, we do not claim that GloptiPoly outperforms all of the numerical algorithms specifically designed to solve the control problems described in this article. For example, the robust stability analysis problem can be solved using the Mapping Theorem [2, 3, 4], whereas the simultaneous stabilization problem can be solved by gridding over the parametrization of PID or first-order stabilizing controllers provided in [5]. The approach described in this article is sufficiently general to address these problems in a unifying way, and can be considered an alternative to specific, tailored algorithms.

It is well known that problems involving polynomial bases with monomials of increasing powers are naturally badly conditioned. If lower and upper bounds on the optimization variables are available as problem data, it may be a good idea to scale all of the intervals to $[-1,\ 1]$. Alternative bases such as Chebyshev polynomials might also prove useful.

Extensions to complex-valued polynomials might also prove useful, namely, to compute upper bounds on the complex structured singular value, to assign complex poles by static output feedback, and to tackle stability problems over n-D polynomials.

A suitable generalization of the theory of moments to polynomial matrices would result in a systematic approach for building a hierarchy of LMI relaxations for bilinear matrix inequality (BMI) problems. Since BMI problems are ubiquitous in control theory [32], this generalization would further widen the spectrum of potential applications. However, at this point, no efficient software has been proposed for solving BMI problem. The present version of GloptiPoly cannot handle BMI problems.

Finally, let us mention that the next version of GloptiPoly will be able to solve the minmax optimization problem

$$\sup_{z\in\Delta} \quad \min_{x\in\mathbb{K}} \quad \sum_{\alpha} g_{0_\alpha}(z)x^\alpha,$$

where Δ is a polytope and $z \mapsto g_{0_\alpha}(z)$ are linear mappings for all α. This problem has applications in robust analysis and design problems. Minmax problems will be included in the next release of GloptiPoly.

Acknowledgments

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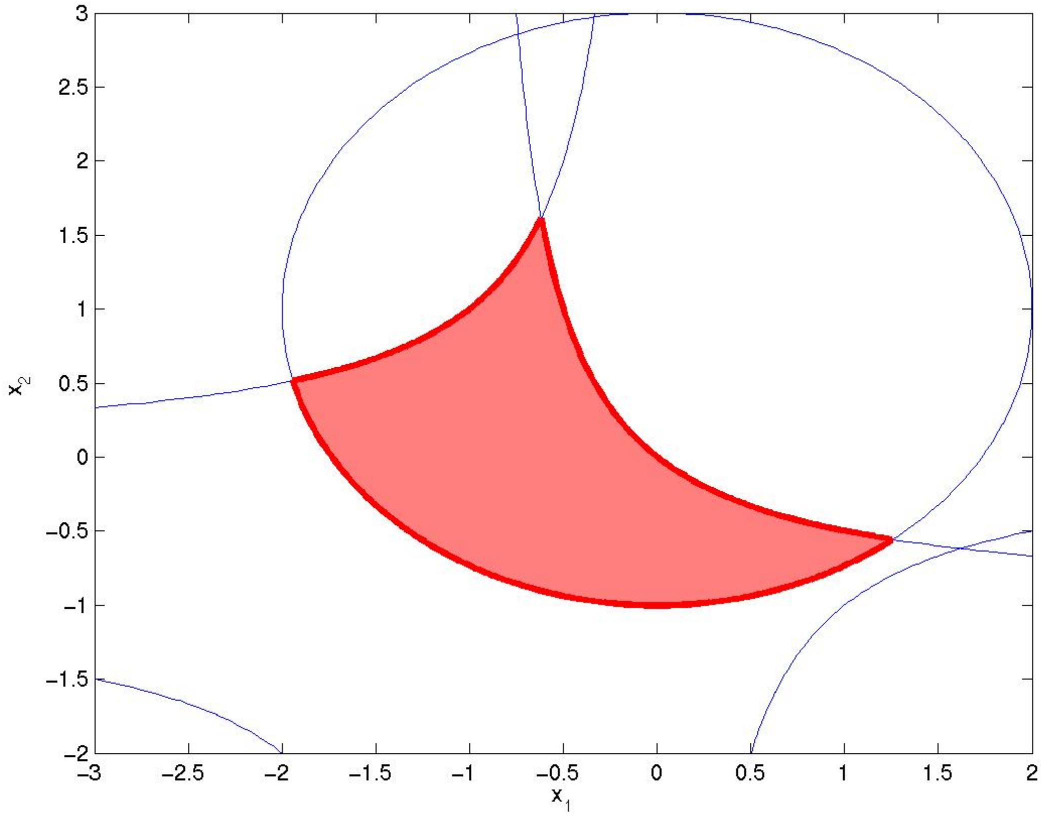

Figure 1: Feasible set for Example 1. The feasible set (shaded region) is non-convex and delimited by circular and hyperbolic arcs.

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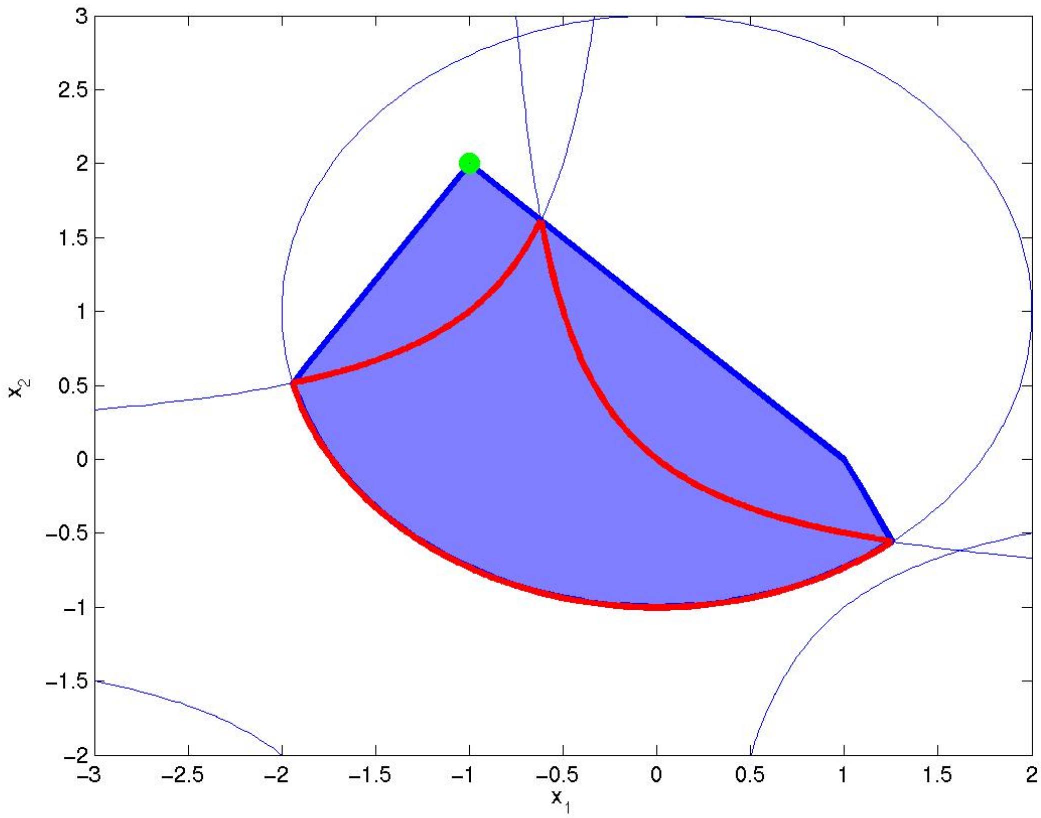

Figure 2: Feasible set of the first convex LMI relaxation for Example 1. The feasible set of the first LMI relaxation (blue region) is obtained by projecting the first-order moments onto the plane. The optimum of the first LMI relaxation is attained at the upper vertex (green dot) of the feasible set. The optimum is an upper bound on the global optimum of the original non-convex polynomial optimization problem.

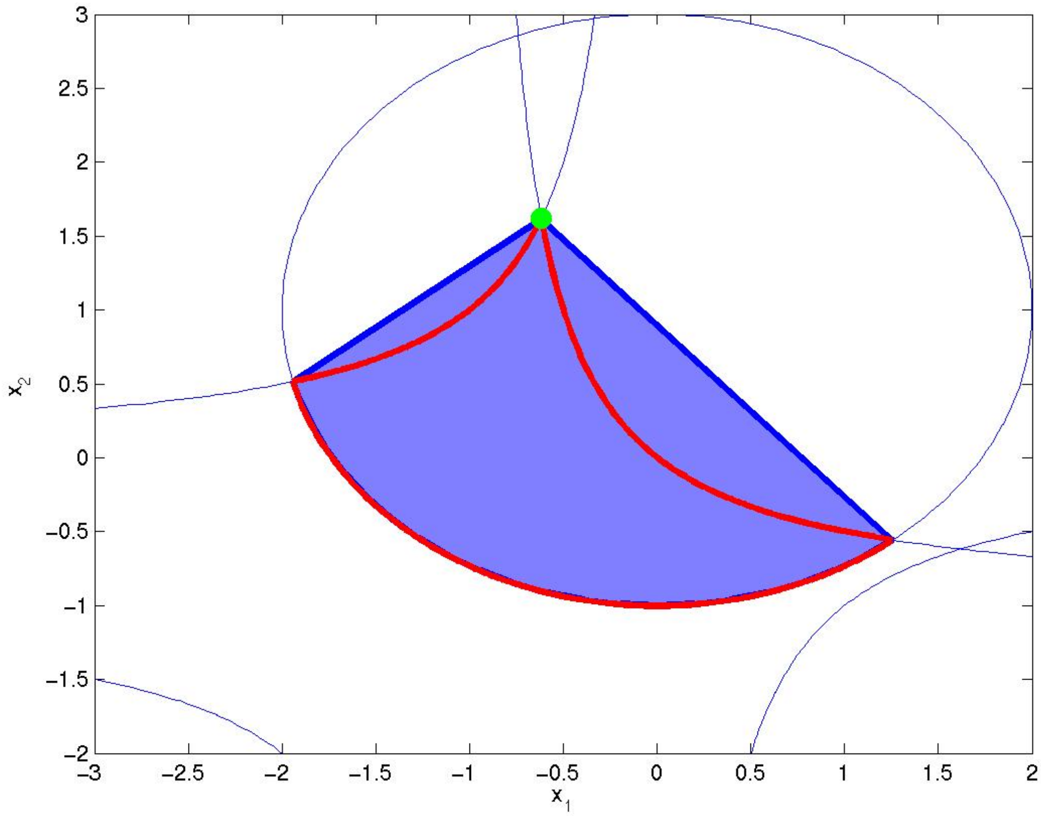

Figure 3: Feasible set of the second convex LMI relaxation for Example 1. The feasible set of the second LMI relaxation (blue region) is obtained by projecting the first-order moments onto the plane. The optimum of the second LMI relaxation is equal to the global optimum (green dot).

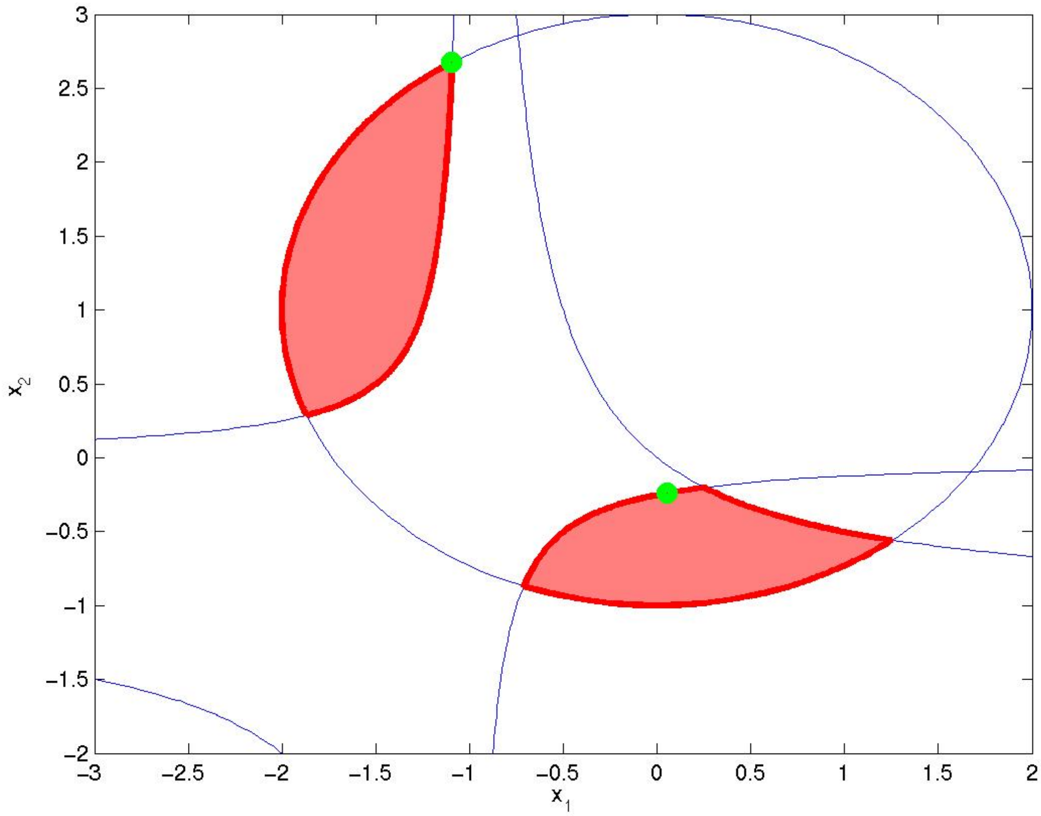

Figure 4: Feasible set for Example 2. The feasible set (shaded region) is non-convex and non-connected, delimited by circular and hyperbolic arcs. Also represented are global optima with minimum Euclidean norm (dot near the origin) and maximum Euclidean norm (dot at the top).

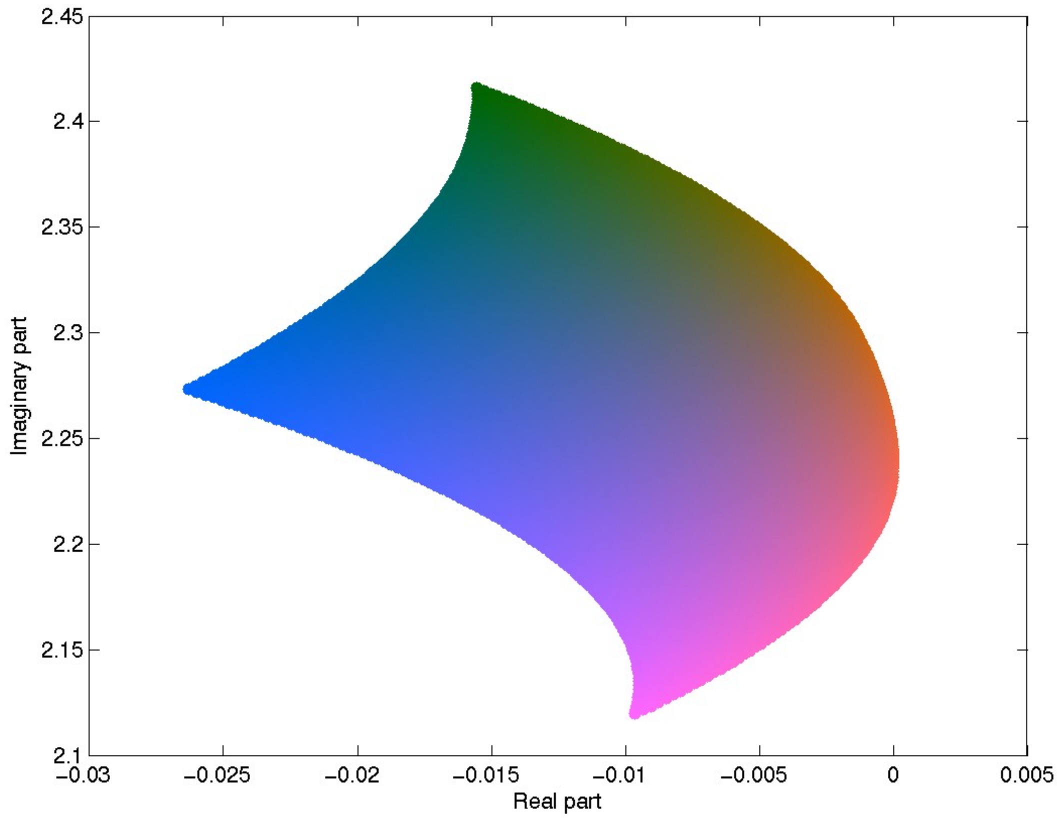

Figure 5: Robust root locus of the dominant pole for the robust stability analysis of Problem I. The dominant pole remains on the left hand side of the complex plane.

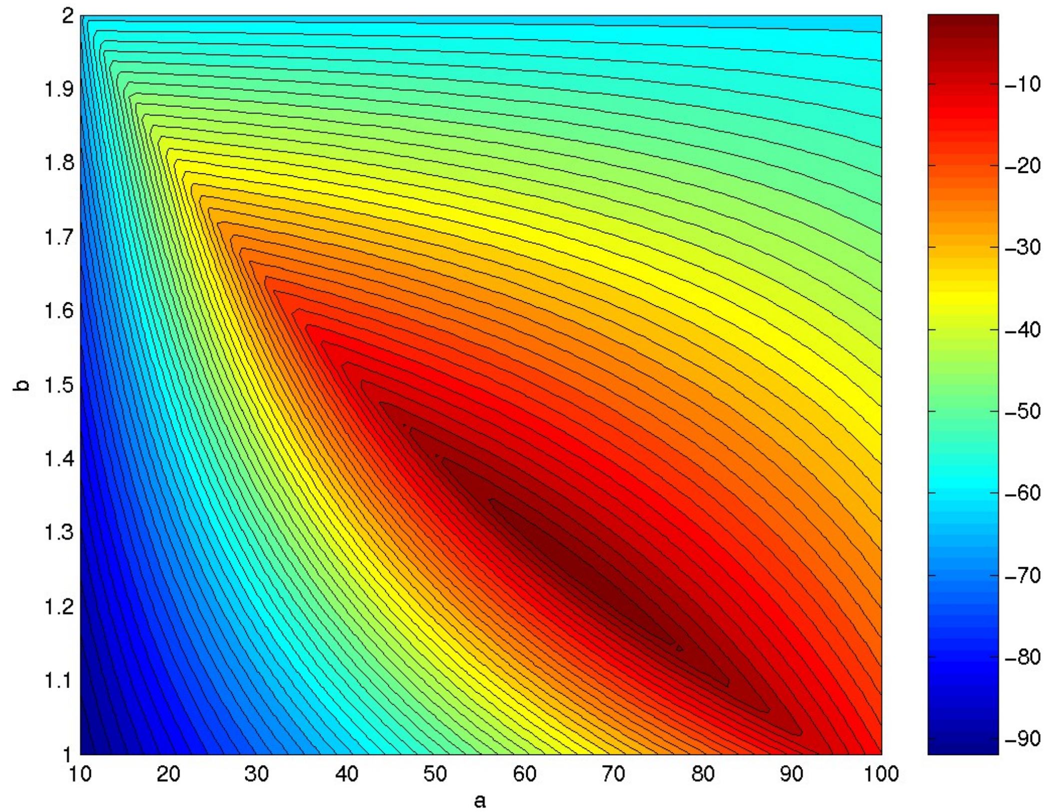

Figure 6: Value of $\min_k g_x(a, b, d)$ as a function of a and b when $d = 10.0938$ for the simultaneous stabilization problem given by Problem II. Simultaneous stabilization is ensured only for positive values (in dark red).

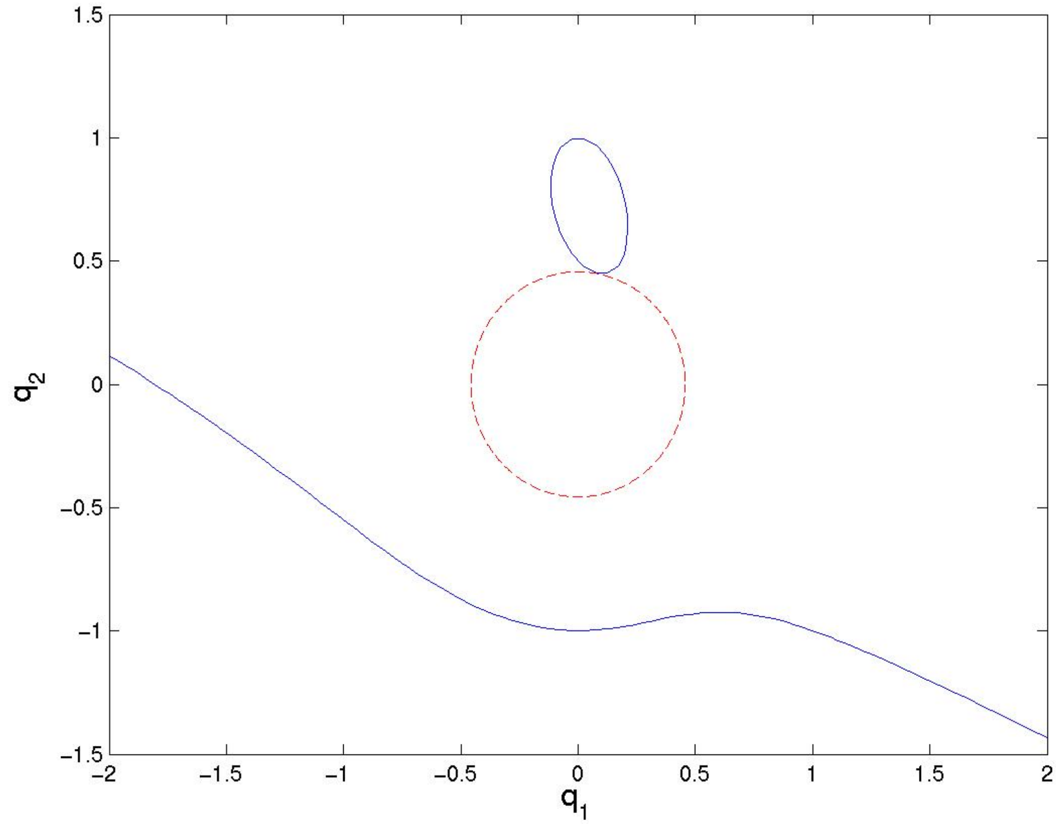

Figure 7: The minimum distance problem consists of finding the circle of minimum radius $\sqrt{\rho_R}$ (dashed line) attaining the curve $\det A(q) = 0$ (solid line). This problem typically has several local optima, such as here the circle tangent to the lower curve (local minimum) or the circle tangent to the exterior of the upper curve (local maximum).

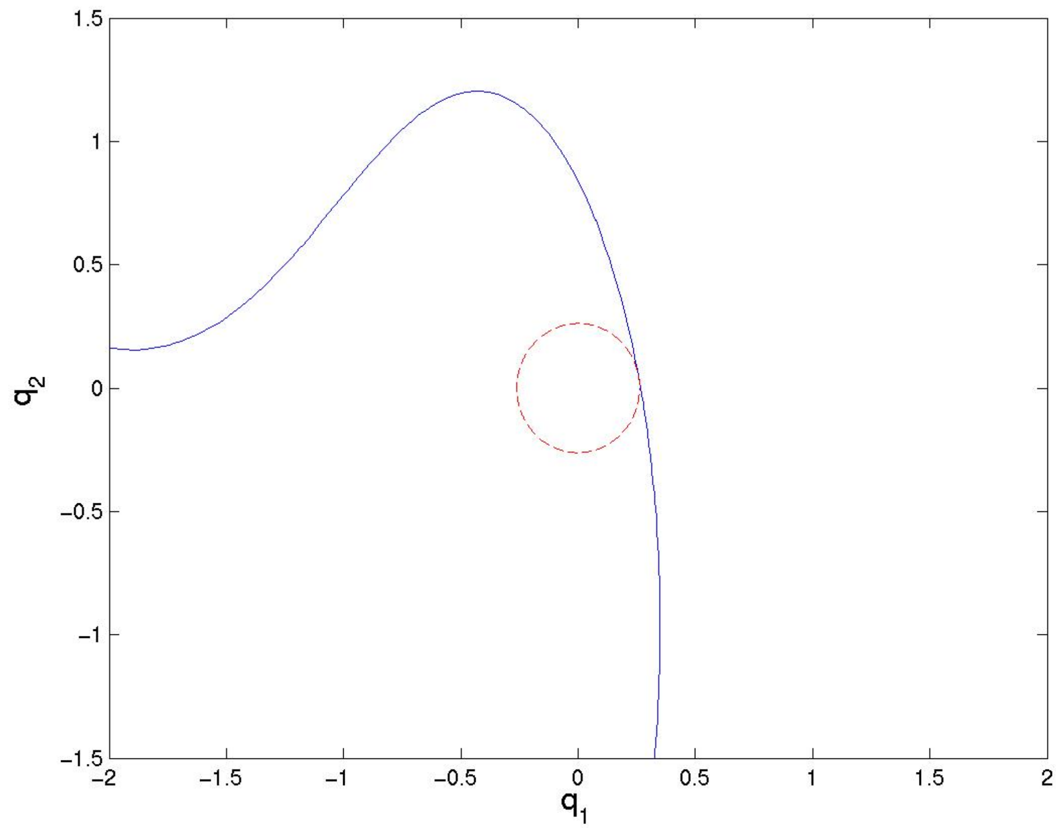

Figure 8: Curve $H_{n-1}(q) = 0$ (solid line) and circle of minimum radius $\sqrt{\rho_I}$ (dashed line). The l_2 parametric stability margin for Problem III is obtained as the minimum of $\sqrt{\rho_I}$ and $\sqrt{\rho_R}$.