# Optimization with polynomials and fixed-order robust controllers: a design example

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**Abstract**

With the help of an inverted pendulum example, we show how convex optimization over linear matrix inequalities can be used to design a robust $H_\infty$ controller of fixed order, based on a purely algebraic approach and recent results on positive polynomials.

## 1 Introduction

When designing robust control laws for linear systems, closed-loop specifications can be formulated in the frequency domain, on peak values of Bode magnitude plots of (possibly weighted) system transfer functions, this is the so-called $H_\infty$ optimization framework surveyed e.g. in [4].

In a state-space setting, $H_\infty$ design algorithms generally result in controllers of the same order as the plant. Indeed, with the help of linearizing changes of variables, it was shown that $H_\infty$ controller design (possibly combined with other performance and robustness specifications) boils down to convex optimization over linear matrix inequalities (LMIs), or semidefinite programming, only in the case when controller and open-loop plant share the same order, see [11] for a recent overview. Unfortunately, in several applications such as embedded control systems for the space and aeronautics industry, a controller of low order is a fundamental requirement. One must then resort to time-consuming controller reduction techniques, not always with the guarantee that closed-loop performance is preserved.

In [5], results on positive polynomials and strictly positive real transfer functions were used to come up with an LMI formulation of fixed-order robust controller design, in the algebraic, or

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polynomial framework initiated in [3]. As explained in [5], the key ingredient in the design procedure resides in the choice of a central polynomial, or desired nominal closed-loop characteristic polynomial around which design is carried out. This approach was recently extended in [6] to $H_\infty$ controller design.

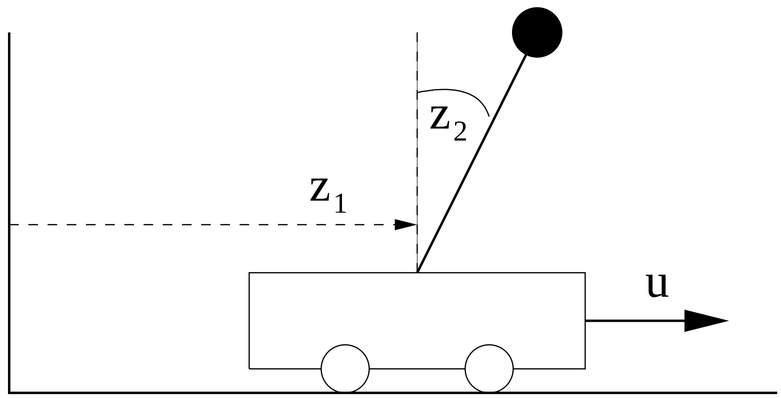


Figure 1: Inverted pendulum system.

In this note we apply the methodology proposed in [5] and [6] to design a fixed-order robust controller minimizing $H_\infty$ performance criteria for an inverted pendulum system described in [9] and depicted in figure 1.

Our objective is to convince the reader that algebraic control techniques can be combined with convex optimization, and especially semidefinite programming, to provide computationally tractable solutions to potentially difficult control problems such as fixed-order robust controller design.

# 2 $H_\infty$ design

Formally, the continuous-time $H_\infty$ design problem for scalar transfer functions can be stated as follows.

**Problem 1** *Given a set of polynomials $n_i^k(s)$, $d_i^k(s)$ for $i = 1,2,\ldots$, $k = 1,2,\ldots$, as well as a set of positive real numbers $\gamma^k$, seek polynomials $x_i(s)$ of given degrees such that*

$$\left\| \frac{\sum_i n_i^k(s)x_i(s)}{\sum_i d_i^k(s)x_i(s)} \right\|_\infty < \gamma^k, \quad k = 1,2,\ldots \tag{1}$$

In the above inequalities

$$\|S\|_\infty = \sup_{s=j\omega,\, \omega\in\mathbb{R}} |S(s)|$$

denotes the peak value of the magnitude of rational transfer function $S$ evaluated along the imaginary axis.

Following [6], we define $c(s)$ as a central polynomial of the same degree $\delta$ as scalar polynomials

$$n^k(s) = \gamma(\sum_i d_i^k(s)x_i(s)) + \sum_i n_i^k(s)x_i(s), \quad d^k(s) = \gamma(\sum_i d_i^k(s)x_i(s)) - \sum_i n_i^k(s)x_i(s).$$

We also define the 2-by-2 polynomial matrices

$$N^k(s) = \begin{bmatrix} n^k(s) & 0 \\ 0 & n^k(s) \end{bmatrix}, \quad D^k(s) = \begin{bmatrix} d^k(s) & 0 \\ 0 & d^k(s) \end{bmatrix}, \quad C(s) = \begin{bmatrix} c(s) & c(s) \\ -c(s) & c(s) \end{bmatrix}$$

whose coefficient matrices corresponding to increasing powers of indeterminate $s$ are gathered in block matrices

$$N^k = \begin{bmatrix} N_0^k & N_1^k & \cdots & N_\delta^k \end{bmatrix}, \quad D^k = \begin{bmatrix} D_0^k & D_1^k & \cdots & D_\delta^k \end{bmatrix}, \quad C = \begin{bmatrix} C_0 & C_1 & \cdots & C_\delta \end{bmatrix}.$$

Note that matrices $N^k$ and $D^k$ are linear in coefficients of polynomials $x_i(s)$. With these notations, results on positive polynomial matrices are invoked in [6] to derive the following solution to problem 1.

**Theorem 1** *Given polynomials $n_i^k(s)$, $d_i^k(s)$, positive scalars $\gamma^k$ for $k = 1, 2, \ldots$ and central polynomial $c(s)$, there exist polynomials $x_i(s)$ solving $H_\infty$ specifications (1) if matrix inequalities*

$$(N^k)^T C + C^T N^k - H(P_n^k) \succ 0, \quad (D^k)^T C + C^T D^k - H(P_d^k) \succ 0, \quad k = 1, 2, \ldots \tag{2}$$

*are feasible. This is a convex LMI problem in coefficients of polynomials $x_i(s)$ and symmetric matrices $P_n^k$ and $P_d^k$.*

Central polynomial $c(s)$ plays the role of a target closed-loop characteristic polynomial around which the design is carried out. Sensible strategies for the choice of central polynomial $c(s)$ are discussed in [6], but a general rule of thumb is that open-loop stable poles must be mirrored in the central polynomial, completed by sufficiently fast additional dynamics.

Since LMI (2) is a sufficient condition to enforce $H_\infty$ specifications (1), generally these specifications will be satisfied with a certain amount of conservatism, i.e. $\gamma^k$ is always an upper bound on the actual $H_\infty$ norm in (1) achieved by feedback. Design LMIs are conservative in the sense that they are based on a convex inner approximation of the non-convex stability domain in the space of coefficients of a polynomial. In other words, if LMI (2) is infeasible, it does not mean that there exists no controller of given order ensuring the expected $H_\infty$ performance.

# 3 Design with one measurement

In this section and the next one, numerical experiments were carried out with the help of the Polynomial Toolbox 2.5 [10], the Control System Toolbox 5.2 [8], the LMI interface YALMIP 2.1 [7] and the semidefinite programming solver SeDuMi 1.05 [12] under a Matlab 6.5 environment running on a Sun Solaris Sparc Blade 150 workstation.

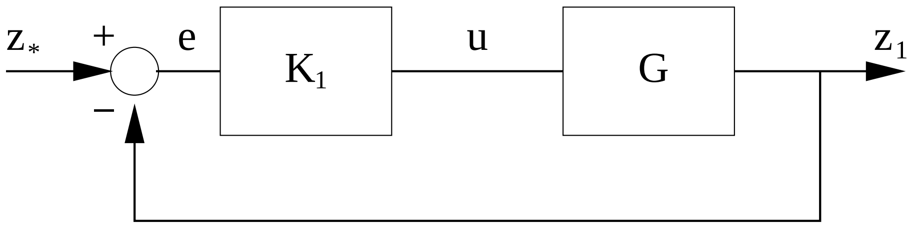


Figure 2: Feedback configuration for the inverted pendulum setup with one measurement.

We consider the inverted pendulum setup of figure 1, described in [9, §7], whose linearized transfer function from input $u$ (dragging force) to output $z_1$ (card position) is given by

$$G(s) = G_1(s) = \frac{b_1(s)}{a(s)} = \frac{2(s^2-10)}{s^2(s^2-20)}. \tag{3}$$

It can be seen that this open-loop plant is non-minimum phase (one unstable zero at $\sqrt{10}$), unstable (one pole at $\sqrt{20}$), and contains a double integrator (two poles at 0).

We want to find a single-input-single-output (SISO) robust linear control law $u(s) = K_1(s)e(s)$ with

$$K_1(s) = \frac{y_1(s)}{x(s)} \tag{4}$$

such that pendulum position $z$ tracks a reference position $z_*$, according to the feedback scheme depicted in figure 2.

In the $H_\infty$ setting, closed-loop robustness is ensured if the sensitivity function

$$S_{11}(s) = \frac{1}{1+G_1(s)K_1(s)}$$

and the complementary sensitivity function (closed-loop transfer function)

$$T_{11}(s) = 1 - S_{11}(s) = \frac{G_1(s)K_1(s)}{1+G_1(s)K_1(s)}$$

have sufficiently small $H_\infty$ norms [4]. According to [1], typical acceptable values are

$$\|S_{11}(s)\|_\infty \leq 1.4, \quad \|T_{11}(s)\|_\infty \leq 1.2. \tag{5}$$

We formulate our SISO $H_\infty$ design problem as follows: given plant polynomials $a(s)$ and $b_1(s)$ of degree $n$ find controller polynomials $x(s)$ and $y_1(s)$ of given degree $m$ such that

$$\|S_{11}(s)\|_\infty = \|\frac{a(s)x(s)}{a(s)x(s)+b_1(s)y_1(s)}\|_\infty \leq M_{S_{11}}$$

where $M_{S_{11}}$ is a given upper bound. This formulation fits the general framework of problem 1. Note that we do not enforce an upper bound on the $H_\infty$ norm of $T_{11}(s)$. Practice reveals that in this case, it will be minimized by side-effect.

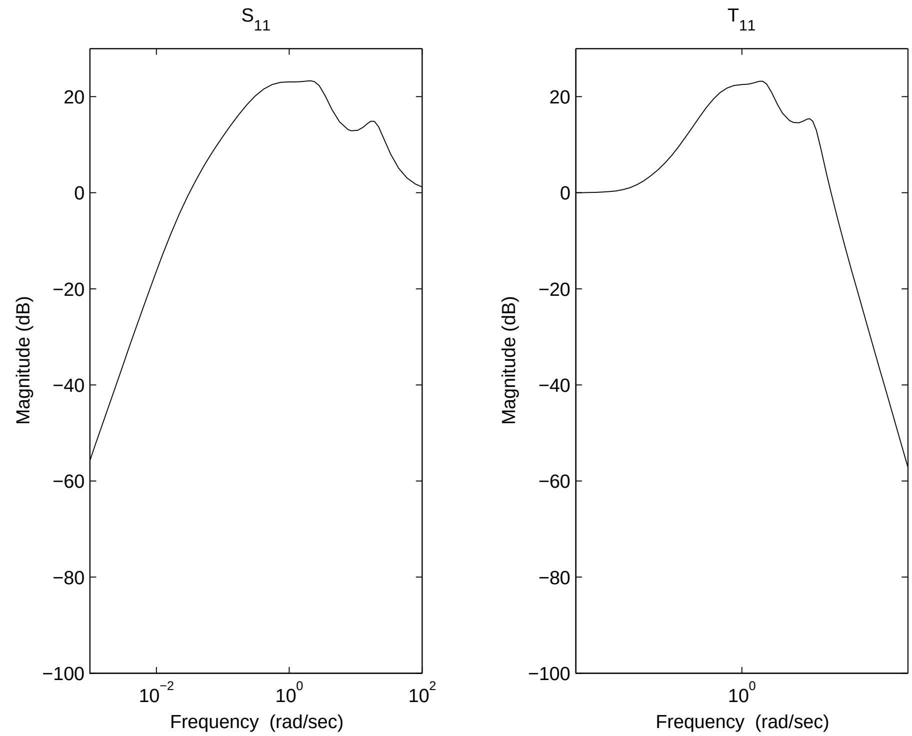


Figure 3: Magnitude Bode plots of sensitivity function $S_{11}$ and complementary sensitivity function $T_{11}$ for the inverted pendulum with one-measurement feedback.

Suppose we are seeking a third-order $(m = 3)$ controller for our fourth-order $(n = 4)$ inverted pendulum plant. Using results of the previous section, in order to derive an LMI formulation for this design problem, we must choose a suitable central polynomial. Following rules of thumb explained in [5] and [6], we found by an error-and-trial iterative procedure the following seventh-degree $(n+m = 7)$ central polynomial

$$c(s) = (s+20)(s+10)(s+3)^2(s+1)^2(s+0.01).$$

With the upper bound $M_{S_{11}} = 20$ we solve LMI problem (2) to obtain the third degree controller polynomials

$$\begin{aligned} x(s) &= -2441.0-929.36s+17.699s^2+s^3 \\ y_1(s) &= -0.74001-32.610s+1110.4s^2+349.23s^3 \end{aligned}$$

yielding

$$\|S_{11}(s)\|_\infty = 14.610, \quad \|T_{11}(s)\|_\infty = 14.459$$

and closed-loop poles located at $-6.1977 \pm i18.301$, $-1.2428 \pm i2.4247$, $-0.86728$, $-0.48609$ and $-0.024709$. Magnitude Bode plots of $S_{11}(s)$ and $T_{11}(s)$ are represented in figure 3. The time response of pendulum position $z_1(t)$ to a step change in reference position $z_*(t)$ is represented in figure 4.

We can see that we are very far from the $H_\infty$ bounds (5) corresponding to a sensible robust design. The time response features unacceptable undershoot and overshoot that would probably exceed physical capabilities of the plant.

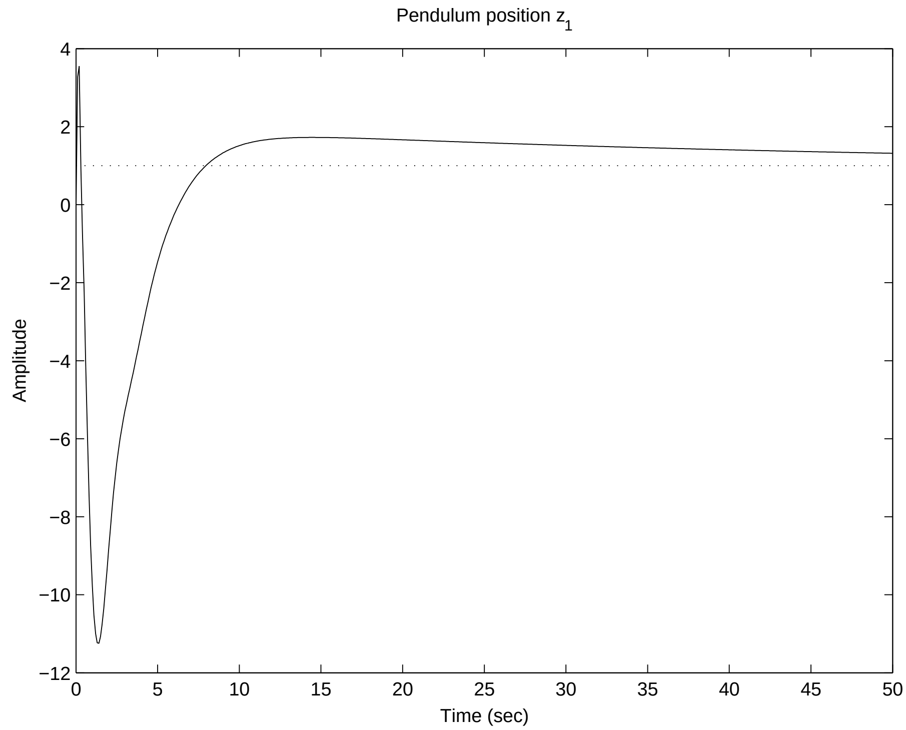


Figure 4: Step-response of the pendulum position with one-measurement feedback.

This could be expected. Indeed, as pointed out in [9] and also [1], the open-loop system is both unstable and non-minimum phase, and the unstable zero occurs at a lower frequency than the unstable pole. It means that any stabilizing controller that can be designed for this plant will have very poor performance and robustness properties. More specifically, it can be shown that for this plant, sensitivities must necessarily satisfy $\|S_{11}(s)\|_\infty \geq 5.828$ and $\|T_{11}(s)\|_\infty \geq 5.828$ and the overshoot must exceed 241%, which is clearly unacceptable. As pointed out in [9], it is easy to gain some appreciation of the difficulty of the control problem when only the pendulum position $z_1$ is measured, by trying to balance a broom with both eyes shut.

# 4 Design with two measurements

In order to improve closed-loop performance and robustness, in this section we consider that the angle $z_2$ of the pendulum is also measured and available for feedback, along with the pendulum position $z_1$. Therefore we will have to solve a single-input-multi-output (SIMO) design problem.

Let

$$G(s) = \begin{bmatrix} G_1(s) \\ G_2(s) \end{bmatrix}$$

denote the SIMO open-loop transfer function, were $G_1(s)$ in (3) is the transfer function from $u$

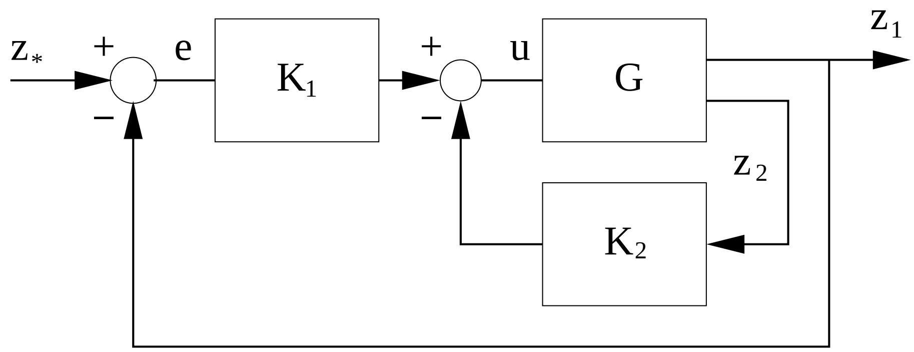


Figure 5: Feedback configuration for the inverted pendulum setup with two measurements.

to $z_1$, and

$$G_2(s) = \frac{b_2(s)}{a(s)} = \frac{-2s^2}{s^2(s^2-20)}$$

is the transfer function from $u$ to $z_2$.

As in [9] we consider the feedback configuration depicted on figure 5, where

$$u(s) = K_1(s)e(s) - K_2(s)z_2(s) = K(s)\begin{bmatrix} e(s) \\ -z_2(s) \end{bmatrix}$$

and feedback matrix

$$K(s) = \begin{bmatrix} K_1(s) & K_2(s) \end{bmatrix}$$

consists of $K_1(s)$ in (4) and

$$K_2(s) = \frac{y_2(s)}{x(s)}.$$

The 2-by-2 sensitivity functions are now given by

$$S(s) = (I + G(s)K(s))^{-1}, \quad T(s) = I - S(s).$$

Let $S_{ij}(s)$ and $T_{ij}(s)$ denote entries $(i, j)$ in $S(s)$ and $T(s)$, respectively.

We formulate our SIMO $H_\infty$ design problem as follows: given plant polynomials $a(s)$, $b_1(s)$ and $b_2(s)$ of degree $n$ find controller polynomials $x(s)$, $y_1(s)$ and $y_2(s)$ of given degree $m$ such that

$$\begin{aligned} \|T_{11}(s)\|_\infty &= \|\tfrac{b_1(s)y_1(s)}{a(s)x(s)+b_1(s)y_1(s)+b_2(s)y_2(s)}\|_\infty \le M_{T_{11}} \\ \|T_{22}(s)\|_\infty &= \|\tfrac{b_2(s)y_2(s)}{a(s)x(s)+b_1(s)y_1(s)+b_2(s)y_2(s)}\|_\infty \le M_{T_{22}} \end{aligned}$$

where $M_{T_{11}}$ and $M_{T_{22}}$ are given upper bounds. This formulation fits the general framework of problem 1. Note that we are basically interested in minimizing the $H_\infty$ norm of sensitivity functions between reference position $z^*$ and position $z_1$, so that we could also minimize $\|S_{11}\|_\infty$ instead of $\|T_{22}\|_\infty$.

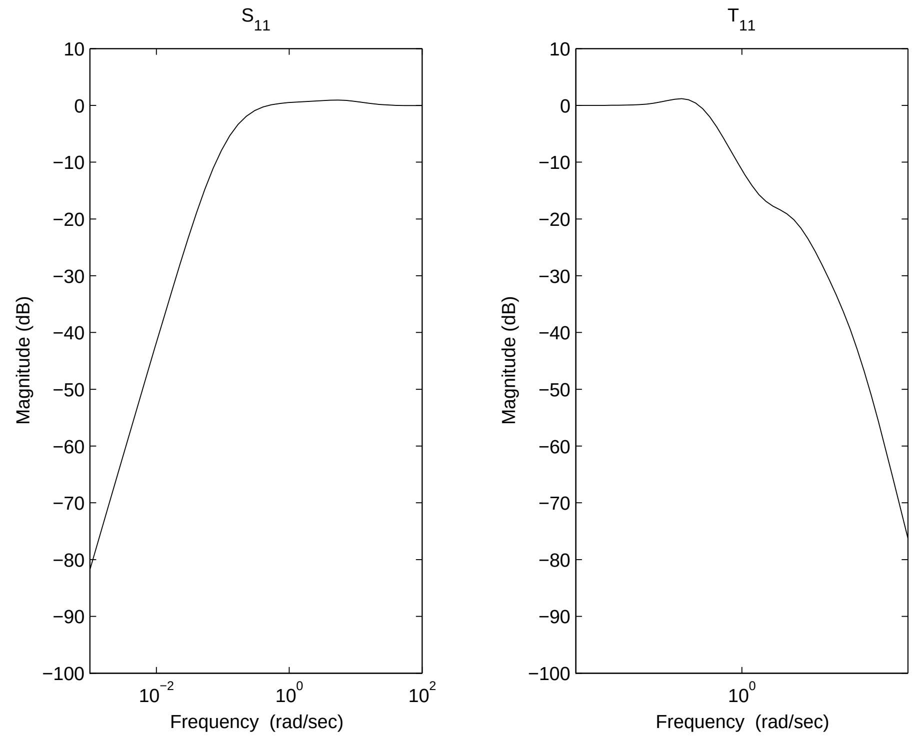


Figure 6: Magnitude Bode plots of sensitivity function $S_{11}$ and complementary sensitivity function $T_{11}$ for the inverted pendulum with two-measurement feedback.

Suppose we are seeking a second-order $(m = 2)$ controller. After various trials, we come up with the central polynomial

$$c(s) = (s+20)^2(s+1)^2(s+0.1)^2$$

and the upper bounds $M_{T_{11}} = 2$, $M_{T_{22}} = 3$. Upon solving LMI problem (2) we obtain the second degree controller polynomials

$$\begin{aligned} x(s) &= 116.35 + 145.90s + s^2 \\ y_1(s) &= -1.6205 - 33.693s - 38.843s^2 \\ y_2(s) &= -1918.5 - 2764.9s - 496.12s^2 \end{aligned}$$

yielding

$$\|S_{11}(s)\|_\infty = 1.1050, \quad \|T_{11}(s)\|_\infty = 1.1419$$

and closed-loop poles located at $-131.74$, $-6.5436 \pm i1.8276$, $-0.81886$, $-0.17771$ and $-0.073253$.

Magnitude Bode plots of $S_{11}(s)$ and $T_{11}(s)$ are represented in figure 6. The time response of pendulum position $z_1(t)$ to a step change in reference position $z_*(t)$ is represented in figure 7.

Comparing with figures 3 and 4, the improvement brought by the second measurement is apparent. Our second-order controller results in an initial undershoot of 6% and an overshoot of 13%. A comparable performance was achieved in [9] with a fourth-order controller. Further tunings would be required to improve the overall time-response, which is probably too slow. Note however that frequency domain specifications have an indirect, and not always easy to characterize, influence on the closed-loop time-response.

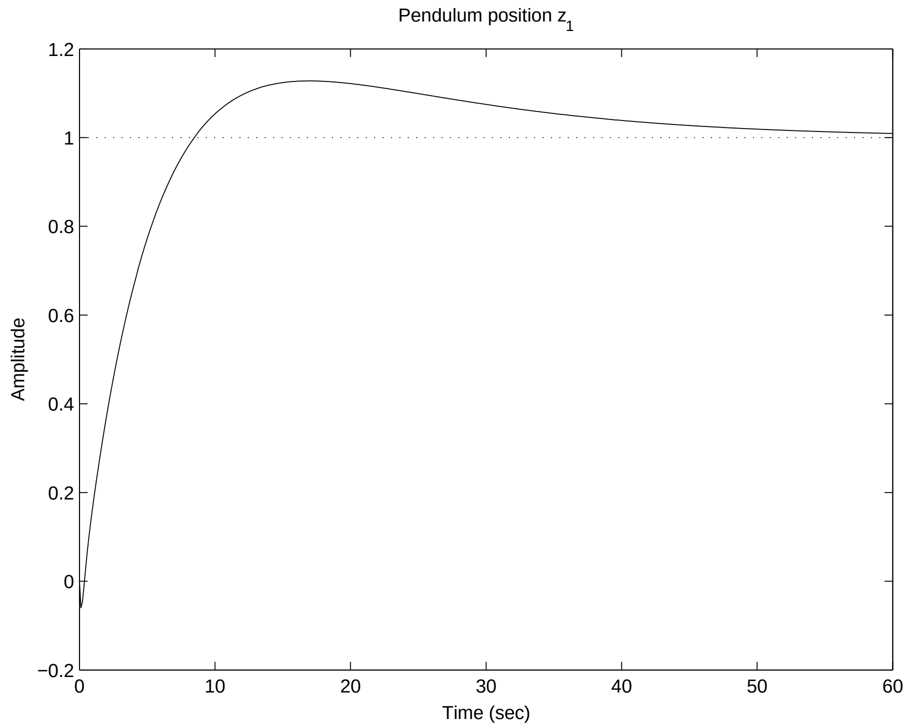


Figure 7: Step-response of the pendulum position with two-measurement feedback.

# 5 Conclusion

In this note we have applied the polynomial/LMI approach described in [5] and [6] to design a fixed-order robust controller minimizing $H_\infty$ performance criteria for an inverted pendulum system.

Due to linearity of individual numerator and denominator polynomials of the closed-loop multivariable transfer function in the SIMO case, we could apply the approach of [6] to optimize over controller polynomial coefficients. Obviously, this can be done also in the MISO case where several inputs are fed back to the system, using just one measurement. However, we do not see any obvious extension of the method to MIMO plants, where controller polynomials enter generally in a nonlinear fashion in the closed-loop numerator and denominator polynomials. The most promising extension would certainly use the Youla-Kučera parametrization of stabilizing controllers [3], where we could optimize simultaneously over the numerator and denominator polynomials of the rational Youla-Kučera parameter entering affinely in the plant transfer function. But then we have only indirect influence on the controller order.

The LMI conditions arising from polynomial positivity conditions described in [5] and [6] have a particular structure which is reminiscent of the Kalman-Yakubovich-Popov (KYP) lemma. LMI decision variables can be gathered into two categories:

- a small number of controller parameters, which are coefficients of the numerator and denominator controller polynomials;

- a large number of additional parameters, which are entries of a Lyapunov-like matrix acting as a closed-loop stability and $H_\infty$ performance certificate.

Controller parameters and additional parameters appear in a decoupled fashion. This particular stucture can be exploited to reduce significantly the number of decision variables in the LMIs, using ideas on semidefinite programming duality originally described in [13].

Another promising research direction is the study of numerical properties (computational complexity, numerical stability) of algorithms tailored to solve these structured LMI problems. As shown in [2], the Hankel or Toeplitz structure can be exploited to design fast algorithms to solve Newton steps in barrier schemes and interior-point algorithms. Numerical stability is also a concern, since it is well-known for example that Hankel matrices are exponentially ill-conditioned. Alternative polynomial bases such as Chebyshev or Bernstein polynomials may prove useful.

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