Application of Generalized Polynomials to the Decoupling of Linear Multivariable Systems

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Abstract. In this paper, properties of generalized polynomials and generalized polynomial matrices are applied to the problem of decoupling with Λ-stability of linear square multivariable systems, i.e. decoupling of linear systems with closed-loop poles in a region Λ of the complex plane. We present first some extensions of well-known results about generalized polynomials, which are basically defined as rational functions with poles in a symmetric region of the extended complex plane. Then, an application of the concepts to the problem of decoupling with Λ-stability of linear square multivariable systems is presented. The conditions for this problem to have a solution are stated in terms of the row and global zero structure of the system out of the region Λ.

Keywords : Generalized Polynomials, Canonical Forms, Linear Systems, Decoupling, Stability.

1 Introduction

Algebraic concepts are fundamental in practically any branch of engineering, and control theory is by no means the exception. The analysis and design of dynamic control systems relies to a deep extent on the notions and properties from algebraic objects. For instance, polynomials, proper rational functions, proper and stable rational functions, matrices with elements from these sets, and canonical forms related to these matrices are a key tool in system description, analysis and design. These rational functions can be regarded as particular cases of what is known as generalized polynomials.

There exist many contributions in the literature about generalized polynomials and its applications to control theory. The concept of rational functions with poles in a prescribed region of the complex plane was originally studied in [1], where it was shown that this set is a principal ideal domain. In [2] it is shown that the set $\mathbb{R}_S$ of all transfer functions with denominators in a prescribed set S is a principal ideal domain, and system invariants under feedback and cascade control are characterized by the free $\mathbb{R}_S$-module generated by the columns of the system transfer function. Generalized polynomials are introduced in [3] as rational functions without poles in a certain region of the finite complex plane, and they are used to describe the set of all transfer functions for particular signals using a set of admissible controllers. In [4] it is shown that the ring of proper and stable rational functions is a Euclidean domain, and the Euclidean division

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process in this ring is characterized. The algebra of proper and Ω-stable rational functions and matrices is examined in [5], where Ω is a symmetric region of the finite complex plane $\mathcal{C}$, containing $\mathcal{C}^+ := \{s \in \mathcal{C}, \mathrm{Re}(s) \geq 0\}$, and excluding at least one negative real number.

Generalized polynomials are usually limited in the previous references to proper rational functions with poles (or with no poles) in a region of the finite complex plane. We present in this paper an extension of this concept, by including the point at infinity, and considering a region Λ of the extended complex plane. Algebraic properties of generalized polynomials are studied, and extensions of well-known results are presented. With a suitable definition of a degree function, it is shown that the set of Λ-generalized polynomials is a Euclidean ring, and canonical forms for Λ-generalized polynomial matrices are derived. We consider here only the properties of generalized polynomials that are needed to study the problem of decoupling with Λ-stability of linear square multivariable systems. Even though most of the properties and results presented here about generalized polynomials and generalized polynomial matrices can be considered as extensions of standard results, we believe our approach is more comprehensive and provides a nice interpretation of concepts and results in terms of poles and zeros of the functions belonging or not to the region Λ. For instance, concepts like the degree of a generalized polynomial, or the characterization of a unit in the ring of generalized polynomial, have a nice interpretation in terms of the zeros of the function out of the region Λ. Also, a wider set of rational functions like polynomials and nonproper rational functions can be considered as particular cases of Λ-generalized polynomials.

Afterwards, we present an application of the notions and properties of Λ-generalized polynomials to the problem of decoupling with Λ-stability of linear square multivariable systems, i.e. decoupling of linear systems with closed-loop poles in a region Λ of the complex plane. Roughly speaking, decoupling of dynamic systems implies independence between the inputs and the outputs of the system, and from the practical point of view it is of interest to achieve decoupling because it is often desirable to control the outputs of the system independently. We are interested in the row-by-row decoupling of linear multivariable systems by static state feedback. This problem has been extensively studied since the 1960's, and it has been solved for the case of regular state feedback (see for instance [6,7]), which is the kind of state feedback that has to be applied to decouple systems with the same number of inputs and outputs (square systems). The regular decoupling problem with stability by static state feedback has been solved in [8,9].

In this paper, it is proved that a linear system is decouplable with Λ-stability if and only if the so-defined Λ-stable interactor of the system is a diagonal matrix, or equivalently, if and only if the row and global zero structure of the system out of the region Λ coincide. The Λ-stable interactor and the row and global zero structure of the system out of Λ are defined via the column Hermite form and the Smith form of the system transfer function over the ring of Λ-generalized polynomials. If the system is decouplable with Λ-stability, a procedure to compute a decoupling state feedback is presented. It is also shown that the conditions for the row-by-row decoupling problem and decoupling with stability can be easily obtained as particular cases of our solution.

This work is organized as follows. In section 2, Λ-generalized polynomials are introduced and some of their algebraic properties are considered. Section 3 deals with Λ-generalized polynomial matrices, and the generalized Hermite form and the Smith form of a matrix are presented. These canonical forms can be considered as the natural extensions of similar concepts for polynomial matrices. No proofs of the results presented in section 2 and section 3 are provided, since they can be considered as extensions of well-known results. An application of the properties of Λ-generalized polynomials and matrices is presented in section 4, where we study the problem of decoupling with Λ-stability of linear multivariable systems. Finally, we end with some conclusions.

2 Λ-Generalized polynomials

We will first introduce some notation. Through this work, $\mathbb{R}$ will denote the field of real numbers, $\mathcal{C}$ will stand for the complex plane and $\mathcal{C}^*$ for the extended complex plane, i.e. $\mathcal{C}^* = \mathcal{C} \cup \{\infty\}$. Also, $\mathcal{C}_-$ will denote the open left half complex plane, and $\mathcal{C}^*_-$ its extended version, i.e. $\mathcal{C}^*_- = \mathcal{C}_- \cup \{\infty\}$. The set of polynomials in the variable s with coefficients in $\mathbb{R}$ will be denoted by $\mathbb{R}$[s], and the set of rational functions over $\mathbb{R}$ will be denoted by $\mathbb{R}$(s). The point $s = w_1 \in \mathcal{C}^*$ is said to be a zero of $f(s)$ if $\lim_{s\to w_1} f(s) = 0$, and $s = w_2 \in \mathcal{C}^*$ is said to be a pole of $f(s)$ if $\lim_{s\to w_2} f(s) = \infty$. According to this definition, if zeros and poles are counted with their own multiplicities, then any rational function has the same number of poles and zeros since the point at infinity is taken into account. The rational function $f(s) = a(s)/b(s)$, where $a(s)$ and $b(s)$ are coprime polynomials, is said to be proper if deg $b(s) \geq$ deg $a(s)$, strictly proper if deg $b(s) >$ deg $a(s)$, and proper and stable if it is proper, and its poles lie in the open left half complex plane $\mathcal{C}_-$. A symmetric subset of $\mathcal{C}^*$ is a set $\Lambda \subset \mathcal{C}^*$, containing at least one point of the real axis, and with the property that if $s_1 = \alpha + j\omega \in \Lambda$, then $s_2 = \alpha - j\omega \in \Lambda$. These restrictions on Λ are basically due to the fact that we are dealing with rational functions with real coefficients. As a special case, we can also define $\Lambda = \{\infty\}$.

The main concept of this section is the following: Given a symmetric set $\Lambda \subset \mathcal{C}^*$, the set of Λ-generalized polynomials, denoted $\mathbb{R}_\Lambda(\mathrm{s})$, is defined as the set of rational functions whose poles are in Λ.

Example 2.1 In this example we specify some regions for Λ and find their corresponding set $\mathbb{R}_\Lambda(\mathrm{s})$:

- If Λ is taken as the extended complex plane $\mathcal{C}^*$, then $\mathbb{R}_\Lambda(\mathrm{s})$ is the set of rational functions $\mathbb{R}(\mathrm{s})$.
- If Λ is defined as the complex plane $\mathcal{C}$, then $\mathbb{R}_\Lambda(\mathrm{s})$ is the set of proper rational functions. In this case $f(s) \in \mathbb{R}_\Lambda(\mathrm{s})$ has to have all its poles in finite positions within the complex plane as the proper rational functions do. The set of proper rational functions will be denoted by $\mathbb{R}_{\mathrm{p}}(\mathrm{s})$.
- If $\Lambda = \{\infty\}$ then $\mathbb{R}_\Lambda(\mathrm{s})$ is the set of polynomials $\mathbb{R}[\mathrm{s}]$. This can also be seen from the fact that any polynomial can be considered as a rational function $f(s) = a(s)/b(s)$, where $b(s) = 1$, i.e. a rational function with no finite poles.
- If $\Lambda = \mathcal{C}_-^*$, then $\mathbb{R}_\Lambda(\mathrm{s})$ is the set of stable rational functions.
- If $\Lambda = \mathcal{C}_-$, then $\mathbb{R}_\Lambda(\mathrm{s})$ is the set of proper and stable rational functions, hereafter denoted $\mathbb{R}_{\mathrm{ps}}(\mathrm{s})$.

With the usual definitions of addition and multiplication of rational functions, it can be seen that $\mathbb{R}_\Lambda(\mathrm{s})$ is a commutative ring (with multiplicative unit 1 and additive unit 0), and it is also an integral domain.

The next step is to show that $\mathbb{R}_\Lambda(\mathrm{s})$ is a Euclidean ring. To this end, for any nonzero element $f(s) \in \mathbb{R}_\Lambda(\mathrm{s})$, we define the function $\gamma(f)$, called the degree of $f(s)$ and denoted by $\deg_\Lambda f(s)$, as the number of zeros of $f(s)$ lying outside of Λ.

Remark 2.1 Observe that the previous definition agrees completely with the definition of the degree of an element in the rings $\mathbb{R}[\mathrm{s}]$, $\mathbb{R}_{\mathrm{p}}(\mathrm{s})$ and $\mathbb{R}_{\mathrm{ps}}(\mathrm{s})$, where:

- If $a(s) \in \mathbb{R}[\mathrm{s}]$ is a polynomial ($\Lambda = \infty$), then $\gamma(a) =$ deg $a(s)$, which corresponds to the number of finite zeros of $a(s)$.
- If $f(s) = a(s)/b(s) \in \mathbb{R}_{\mathrm{p}}(\mathrm{s})$ is a proper rational function ($\Lambda = \mathcal{C}$), then $\gamma(f) =$ deg $b(s) -$ deg $a(s)$, which corresponds to the number of infinite zeros of $f(s)$.
- If $g(s) \in \mathbb{R}_{\mathrm{ps}}(\mathrm{s})$ is a proper and stable rational function ($\Lambda = \mathcal{C}_-$), then $\gamma(g) =$ number of unstable + infinite zeros of $g(s)$, i.e. number of zeros of $g(s)$ outside of $\mathcal{C}_-$.

Through this work, the degree of the proper rational function $f(s) \in \mathbb{R}_{\mathrm{p}}(\mathrm{s})$ will be denoted by $\deg_{\mathrm{p}} f(s)$, and the degree of the proper and stable rational function $g(s) \in \mathbb{R}_{\mathrm{ps}}(\mathrm{s})$ will be denoted by $\deg_{\mathrm{ps}} g(s)$.

With this definition of $\deg_\Lambda f(s)$, it is evident that the units of the ring $\mathbb{R}_\Lambda(\mathrm{s})$ are the nonzero elements $f(s) \in \mathbb{R}_\Lambda(\mathrm{s})$ such that $\deg_\Lambda f(s) = 0$.

From the algebraic point of view, the following is a fundamental property of $\mathbb{R}_\Lambda(\mathrm{s})$.

Theorem 2.1 Given a symmetric set Λ, then $\mathbb{R}_\Lambda(\mathrm{s})$ is a Euclidean ring, with the degree function as defined previously, i.e. as the number of zeros of $f(s)$ lying outside of Λ.

Some more concepts are further introduced next.

Lemma 2.1 Let $h(s)$, $f(s) \in \mathrm{I\!R}_\Lambda(\mathrm{s})$. Then $h(s)$ divides $f(s)$ (i.e. there exists $x(s) \in \mathrm{I\!R}_\Lambda(\mathrm{s})$ such that $f(s) = h(s)x(s)$), denoted $h(s) \mid f(s)$, if and only if each zero of $h(s)$ outside of Λ is also a zero of $f(s)$, at least with the same multiplicity.

The function $h(s) \in \mathrm{I\!R}_\Lambda(\mathrm{s})$ is said to be a greatest common divisor (gcd) of $f(s)$ and $g(s) \in \mathrm{I\!R}_\Lambda(\mathrm{s})$ if

- $h(s)$ is a common divisor of $f(s)$ and $g(s)$, i.e., $h(s) \mid f(s)$ and $h(s) \mid g(s)$, and
- any other common divisor of $f(s)$ and $g(s)$ is a divisor of $h(s)$.

A gcd of a set of more than 2 functions can be defined in an analogous manner.

Let $h(s) \in \mathrm{I\!R}_\Lambda(\mathrm{s})$ be a greatest common divisor of $f(s)$ and $g(s) \in \mathrm{I\!R}_\Lambda(\mathrm{s})$. Then $h_1(s) := h(s)u(s)$ is also a gcd of $f(s)$ and $g(s)$, where $u(s)$ is a unit of $\mathrm{I\!R}_\Lambda(\mathrm{s})$. Thus, a greatest common divisor of two generalized polynomials is unique up to multiplication by units of the ring $\mathrm{I\!R}_\Lambda(\mathrm{s})$.

Remark 2.2 The relevant information in determining the gcd of some functions $f_i(s)$ is the set of zeros outside of Λ which are common to all these functions. Indeed, let $h(s)$ be a gcd of a finite set of Λ-generalized polynomials $f_1(s)$, $f_2(s)$, $\ldots$, $f_k(s)$. From Lemma 2.1 it can be seen that $\deg_\Lambda\, h(s) = n$, where n is the number of zeros outside of Λ which are common to these functions $f_1(s)$, $f_2(s)$, $\ldots$, $f_k(s)$.

According to the previous remark, a gcd of a set of functions can be obtained analytically by inspection of the zeros of these functions. A gcd can also be obtained in an algorithmic way. The algorithm to obtain a gcd for polynomials is well known in the literature (see, for instance [10]) and it is based on the principle of polynomial division. The extension of this procedure to $\mathrm{I\!R}_\Lambda(\mathrm{s})$, which is not considered in this paper, can be readily obtained since Euclidean division also applies to Λ-generalized polynomials.

Let $h(s)$ be a gcd of $f(s)$ and $g(s) \in \mathrm{I\!R}_\Lambda(\mathrm{s})$. Then $f(s)$ and $g(s)$ are called coprime if $h(s)$ is a unit of $\mathrm{I\!R}_\Lambda(\mathrm{s})$. The following result is immediate from Remark 2.2.

Lemma 2.2 The Λ-generalized polynomials $f(s)$ and $g(s) \in \mathrm{I\!R}_\Lambda(\mathrm{s})$ are coprime if they do not have any common zeros outside of Λ.

3 Λ-Generalized polynomial matrices

Matrices whose entries are Λ-generalized polynomials will be considered in this section. The set of matrices of dimensions $m \times n$ whose entries belong to $\mathrm{I\!R}_\Lambda(\mathrm{s})$ will be denoted by $\mathrm{I\!R}_\Lambda^{\mathrm{m\times n}}(\mathrm{s})$. With the usual definitions of addition and multiplication of matrices, the set of square matrices which belong to $\mathrm{I\!R}_\Lambda^{\mathrm{m\times m}}(\mathrm{s})$ is readily seen to be a non commutative ring with respect to these operations.

A square nonsingular matrix $U(s) \in \mathrm{I\!R}_\Lambda^{\mathrm{m\times m}}(\mathrm{s})$ is said to be a unit of the ring $\mathrm{I\!R}_\Lambda^{\mathrm{m\times m}}(\mathrm{s})$ if there exists $V(s) \in \mathrm{I\!R}_\Lambda^{\mathrm{m\times m}}(\mathrm{s})$ such that $U(s)V(s) = V(s)U(s) = I_m$. Units of $\mathrm{I\!R}_\Lambda^{\mathrm{m\times m}}(\mathrm{s})$ will be called $\mathrm{I\!R}_\Lambda(\mathrm{s})$-unimodular matrices. Then, the units of $\mathrm{I\!R}_\Lambda^{\mathrm{m\times m}}(\mathrm{s})$ for polynomial, proper, and proper and stable rational matrices will be called, respectively, $\mathrm{I\!R}[\mathrm{s}]$-unimodular, $\mathrm{I\!R}_{\mathrm{p}}(\mathrm{s})$-unimodular and $\mathrm{I\!R}_{\mathrm{ps}}(\mathrm{s})$-unimodular matrices.

Lemma 3.1 A square nonsingular matrix $U(s) \in \mathrm{I\!R}_\Lambda^{\mathrm{m\times m}}(\mathrm{s})$ is $\mathrm{I\!R}_\Lambda(\mathrm{s})$-unimodular if and only if $\det U(s)$ is a unit of $\mathrm{I\!R}_\Lambda(\mathrm{s})$, i.e. a nonzero rational function whose degree is zero.

Next, we will introduce the generalized Hermite form and the Smith form of a matrix, which are canonical forms defined via factorizations using $\mathrm{I\!R}_\Lambda(\mathrm{s})$-unimodular matrices.

Theorem 3.1 Let $A(s) \in \mathrm{I\!R}_\Lambda^{\mathrm{m\times n}}(\mathrm{s})$, and suppose for simplicity that rank $A(s) = m$. Then there exists a $\mathrm{I\!R}_\Lambda(\mathrm{s})$-unimodular matrix $U(s) \in \mathrm{I\!R}_\Lambda^{\mathrm{n\times n}}(\mathrm{s})$ and a matrix $H(s) \in \mathrm{I\!R}_\Lambda^{\mathrm{m\times n}}(\mathrm{s})$, called the generalized (column) Hermite form of $A(s)$ and unique up to units of $\mathbb{R}_\Lambda(s)$, such that

$$A(s)U(s) = H(s) = \begin{bmatrix} h_{11}(s) & & (0) & \vdots & \\ \vdots & \ddots & & \vdots & (0) \\ h_{1m}(s) & \dots & h_{mm}(s) & \vdots & \end{bmatrix} \tag{3.1}$$

where, for $i > j$,

$$h_{ij}(s) = 0, \quad \text{or} \quad \deg_\Lambda \, h_{ij}(s) < \deg_\Lambda \, h_{ii}(s). \tag{3.2}$$

Proof. The proof is constructive. The matrix $H(s)$ is obtained by elementary column operations on $A(s)$ over $\mathbb{R}_\Lambda(s)$ and using the division algorithm for Λ-generalized polynomials. The procedure is similar to the case of the Hermite form of polynomial matrices (see [11]).

It seems misleading to call $H(s)$ a canonical form for $A(s)$, when it happens that it is not completely unique but that it is unique up to units of $\mathbb{R}_\Lambda(s)$. We can overcome this difficulty by introducing the concept of a monic Λ-generalized polynomial. This notion, however, has to be established for every particular case of Λ-generalized polynomials. We will define it for the rings we are mainly interested in this work.

Consider the rings $\mathbb{R}[s]$, $\mathbb{R}_p(s)$, and $\mathbb{R}_{ps}(s)$, then:

- A polynomial will be taken to be monic as usually understood, i.e. $a(s) \in \mathbb{R}[s]$ is monic if the coefficient corresponding to its highest power is equal to 1.
- The proper rational function $f(s) \in \mathbb{R}_p(s)$ will be called monic if it is of the form $f(s) = \frac{1}{s^n}$, where $n = \deg_p \, f(s)$.
- The proper and stable rational function $g(s) \in \mathbb{R}_{ps}(s)$ will be called monic if it is of the form $g(s) = \frac{\epsilon(s)}{\pi^n}$ where $\epsilon(s) \in \mathbb{R}[s]$ is a monic polynomial with roots out of the open left half complex plane $\mathcal{C}_-$ (antistable polynomial), $\pi = s + \alpha$ with $-\alpha \in \mathcal{C}_-$, and $n = \deg_{ps} \, g(s)$.

The basic idea behind the previous definition is that from any Λ-generalized polynomial, we can obtain the simplest element (a monic one) by extracting units of $\mathbb{R}_\Lambda(s)$ from it. Regarding Theorem 3.1, if we further specify that $h_{ii}(s)$ in (3.1) are monic Λ-generalized polynomials, then $H(s)$ is a unique matrix. We will consider that this is the case when speaking about the Hermite form of some matrix.

Theorem 3.2 Let $A(s) \in \mathbb{R}_\Lambda^{m \times n}(s)$ with rank $A(s) = r \le \min(m, n)$. Then, there exist $\mathbb{R}_\Lambda(s)$-unimodular matrices $V_1(s) \in \mathbb{R}_\Lambda^{m \times m}(s)$ and $V_2(s) \in \mathbb{R}_\Lambda^{n \times n}(s)$ such that

$$V_1(s)A(s)V_2(s) = \Psi(s) = \begin{bmatrix} \operatorname{diag}\{\psi_i(s)\}_{i=1}^r & 0 \\ 0 & 0 \end{bmatrix} \tag{3.3}$$

where $\psi_i(s)$ divides $\psi_{i+1}(s)$, i.e.

$$\psi_i(s) \mid \psi_{i+1}(s), \qquad i = 1, \dots, r-1. \tag{3.4}$$

The Λ-generalized polynomials $\psi_i(s)$ are unique up to multiplication by units of $\mathbb{R}_\Lambda(s)$. We will consider them to be monic, and they will be called the invariant factors of $A(s)$, where as we stated previously, the definition of a monic Λ-generalized polynomial depends on the ring $\mathbb{R}_\Lambda(s)$. Under this assumption the matrix $\Psi(s)$, hereafter called the Λ-generalized Smith form, or simply the Smith form of $A(s)$, is a canonical form for $A(s)$.

If we define the determinantal divisors of $A(s) \in \mathbb{R}_\Lambda^{m \times n}(s)$ as

$$\Delta_i(s) := \text{monic gcd of all } i \times i \text{ minors of } A(s), \quad i = 1, \dots, r, \qquad r = \text{rank } A(s), \tag{3.5}$$

then it can be seen that

$$\psi_i(s) = \frac{\Delta_i(s)}{\Delta_{i-1}(s)}, \qquad i = 1, \dots, r, \tag{3.6}$$

where $\Delta_0(s) := 1$.

4 Application to decoupling with Λ-stability

We tackle in this part of the paper the problem of decoupling with Λ-stability of linear multivariable square systems, which will be defined as the problem of decoupling plus the modes of the closed-loop system being located at a specific symmetric region Λ of the complex plane. The problem statement is presented in section 4.1. In section 4.2 we define the Λ-stable interactor via the column Hermite form of the system transfer function over the ring $\mathbb{R}_\Lambda(\mathrm{s})$, and show that this matrix is the part of the system that is invariant under the action of a state feedback which preserves Λ-stability. The main results and some illustrative examples are presented in section 4.3.

4.1 Problem statement

We consider in this work square linear multivariable and controllable systems described by

$$(A, B, C) \left\{ \begin{array}{l} \dot{x}(t) = Ax(t) + Bu(t) \\ y(t) = Cx(t) \end{array} \right.$$

where $x \in \mathbb{R}^{\mathrm{n}}$, $u \in \mathbb{R}^{\mathrm{p}}$ and $y \in \mathbb{R}^{\mathrm{p}}$ are, respectively, the state, input and output vectors of the system.

The system (A, B, C) is said to be row-by-row decouplable by static state feedback if there exists a state feedback

$$(F, G): \quad u(t) = Fx(t) + Gv(t)$$

where $F \in \mathbb{R}^{\mathrm{p}\times\mathrm{n}}$ and $G \in \mathbb{R}^{\mathrm{p}\times\mathrm{p}}$ are constant matrices, with G nonsingular, and $v(t)$ is a new input vector, such that the input $v_i(t)$ controls the output $y_i(t)$, $i = 1, \ldots, p$, without affecting the other outputs.

Let Λ be a symmetric subset of the complex plane $\mathcal{C}$. The system (A, B, C) is said to be row-by-row decouplable with Λ-stability by static state feedback, if it is decouplable and the eigenvalues of the matrix $(A + BF)$ are located in Λ. Observe that Λ is taken here as a subset of the complex plane $\mathcal{C}$, and not as a subset of the extended complex plane $\mathcal{C}^*$ as in section 2, since the eigenvalues of the matrix $(A + BF)$ are all finite.

From the input-output point of view, the previous formulation is equivalent to the existence of a state feedback (F, G) such that the transfer function $T_{F,G}(s)$ of the closed loop system $(A + BF, BG, C)$ is of the form

$$T_{F,G}(s) = C(sI - A - BF)^{-1}BG = \text{ diag } \{w_1(s), \ldots, w_p(s)\}$$

and the eigenvalues of the matrix $(A+BF)$ are located in Λ, which implies also that $w_i(s) \in \mathbb{R}_\Lambda(\mathrm{s})$, $i = 1, \ldots, p$.

The eigenvalues of the matrix $(A + BF)$ are the modes of the closed-loop system $(A + BF, BG, C)$. The poles of the system are a subset of the system modes, and both sets coincide if the system is controllable and observable. The dynamics of the closed loop system $(A + BF, BG, C)$ depends on the position of the eigenvalues of the matrix $(A + BF)$, thus the importance of assigning these eigenvalues. Decoupling assures that every input is controlling every output independently; the modes location assures required system dynamics, for instance stability or faster response.

We can suppose without loss of generality that the modes of the system (the eigenvalues of matrix A), are in Λ, i.e. that (A, B, C) is Λ-stable; if not, there always exists a preliminary state feedback which places the modes in Λ, since we are considering the system (A, B, C) to be controllable. If the modes of the system are in Λ, then the entries of the system transfer function $T(s) = C(sI-A)^{-1}B$ are Λ-generalized polynomials, because they are rational functions whose poles are in Λ. Then, $T(s)$ can be considered as a Λ-generalized polynomial matrix. This consideration will be helpful when obtaining the column Hermite form and the Smith form of $T(s)$ over the ring $\mathbb{R}_\Lambda(\mathrm{s})$, and so that the conditions for decoupling with Λ-stability can be directly tested from the transfer function of the system.

4.2 State feedback with Λ-stability

Let (F, G) be a static state feedback applied on the Λ-stable system (A, B, C), and such that the closed-loop system $(A+BF, BG, C)$ is also Λ-stable. The closed-loop transfer function is given by

$$T_{F,G}(s) = C(sI - A - BF)^{-1}BG.$$

After some manipulations we obtain

$$T_{F,G}(s) = \underbrace{C(sI-A)^{-1}B}_{T(s)}\underbrace{[I - F(sI-A)^{-1}B]^{-1}G}_{R(s)}$$

where $T(s)$ is the transfer function of the system (A, B, C), and $R(s) := [I - F(sI-A)^{-1}B]^{-1}G$.

Since the closed-loop system is supposed to be Λ-stable, then $R(s)$ must be clearly a Λ-generalized polynomial matrix. Further, from

$$\begin{aligned} R^{-1}(s) &= G^{-1}[I - F(sI-A)^{-1}B] \\ &= \tfrac{1}{\det(sI-A)}\; G^{-1}[\det(sI-A)I - F\ \mathrm{Adj}(sI-A)B] \end{aligned}$$

it can be seen that $R^{-1}(s)$ is also a Λ-generalized polynomial matrix, since the roots of $\det(sI-A)$ are in Λ. Then, we have the following result.

Lemma 4.1 Let $T(s)$ be the transfer function of a Λ-stable system (A, B, C). Then, the effect of a static state feedback (F, G), G nonsingular, which preserves Λ-stability can be represented in transfer function terms as a $\mathbb{R}_\Lambda$(s)-unimodular matrix postmultiplying $T(s)$. ■

This fact establishes a natural restriction on the type of feedback we can use while trying to achieve decoupling with Λ-stability: For our purposes, the state feedback (F, G) will be said to be an admissible state feedback if its effect on the system (A, B, C) can be represented as a $\mathbb{R}_\Lambda$(s)-unimodular precompensator acting on the system transfer function $T(s)$. This can be considered as the matrix interpretation of the fact that we are neither allowed to introduce Λ-unstable poles (poles out of Λ) nor to cancel out Λ-unstable zeros in order to keep the internal Λ-stability of the closed-loop system. At this stage, it is important to consider the structural information of the system (A, B, C) which remains invariant under the action of $\mathbb{R}_\Lambda$(s)-unimodular compensation, and consequently, invariant under the action of an admissible state feedback.

4.2.1 The Λ-stable interactor

Since the action of an admissible state feedback on (A, B, C) can be represented as multiplication of $T(s)$ on the right by a $\mathbb{R}_\Lambda$(s)-unimodular matrix, the information of the system that is invariant under such a feedback is contained in the column Hermite form of $T(s)$ over the ring $\mathbb{R}_\Lambda$(s) (see Theorem 3.1), presented next.

Let $T(s)$ be the transfer function of (A, B, C). Then, there exists a $\mathbb{R}_\Lambda$(s)-unimodular matrix $U(s) \in \mathbb{R}_\Lambda^{p\times p}(s)$ and a nonsingular lower triangular matrix $\Phi_\Lambda^{-1}(s) \in \mathbb{R}_\Lambda^{p\times p}(s)$, unique up to units of the ring $\mathbb{R}_\Lambda$(s), such that

$$T(s)U(s) = \Phi_\Lambda^{-1}(s) = \begin{bmatrix} \varphi_{11}(s) & & (0) \\ \vdots & \ddots & \\ \varphi_{p1}(s) & \dots & \varphi_{pp}(s) \end{bmatrix}, \tag{4.1}$$

where the rational functions $\varphi_{ij}(s) \in \mathbb{R}_\Lambda(s)$ satisfy, for $i > j$,

$$\varphi_{ij}(s) = 0, \text{ or } \deg_\Lambda \varphi_{ij}(s) < \deg_\Lambda \varphi_{ii}(s), \tag{4.2}$$

and they are of the form

$$\varphi_{ii}(s) = \frac{\epsilon_{ii}(s)}{\pi^{k_{ii}}} \tag{4.3}$$

and for $i > j$

$$\varphi_{ij}(s) = \frac{\epsilon_{ij}(s)}{\pi^{k_{ij}}}, \tag{4.4}$$

where $\epsilon_{ii}(s)$ is an antistable polynomial, or polynomial with only Λ-unstable roots (roots out of Λ), $\pi = s + \alpha$ is a stable term ($-\alpha \in \Lambda$), $\epsilon_{ij}(s) \in \mathbb{R}[s]$ is a polynomial, and k_{ii}, k_{ij} are positive integers. The matrix $U(s)$ represents elementary column operations on $T(s)$ over $\mathbb{R}_\Lambda(s)$.

The rational matrix $\Phi_\Lambda(s)$, which is the inverse of $\Phi_\Lambda^{-1}(s)$, will be called the Λ-stable interactor of the system, since this matrix can be considered as the extension of the classic interactor [12] for the set Λ. Notice that $\Phi_\Lambda(s)$ is not unique but it is unique up to units of the ring $\mathbb{R}_\Lambda(s)$. For a fixed $\pi = s + \alpha$, the matrix $\Phi_\Lambda(s)$ is unique, and having this in mind is why we called it "the" stable interactor of the system. Actually, the algebraic properties of $\Phi_\Lambda(s)$ do not depend on the choice of π, nor the results stated in the next sections based on these properties.

The matrix $\Phi_\Lambda(s)$ is in general a rational matrix having only Λ-unstable poles. This can be seen from the fact that the numerator of the determinant of $\Phi_\Lambda^{-1}(s)$ is the product of the antistable polynomials $\epsilon_{ii}(s)$, $i = 1, \ldots, p$. Observe that if (A, B, C) has no finite zeros out of Λ, then $\Phi_\Lambda(s)$ is a polynomial matrix. The Λ-stable interactor is a state feedback invariant which contains the Λ-unstable structural information of the system, i.e. zeros out of Λ of the system, including infinite zeros.

4.2.2 Feedback realization of precompensators

It was shown before that the effect of an admissible state feedback is equivalent in transfer function terms as acting a $\mathbb{R}_\Lambda(s)$-unimodular precompensator on the system transfer function $T(s)$. The converse problem for a proper compensator is the following: A given proper compensator $P(s)$ is said to be feedback realizable on the system (A, B, C) if there exists a state feedback (F, G) such that $P(s) = [I - F(sI - A)^{-1}B]^{-1}G$. The following result, used in the proof of Theorem 4.1, states the conditions for a proper compensator to be realizable.

Lemma 4.2 [13] Let the matrices $N_1(s)$ and $D(s)$ be a right coprime matrix fraction description (MFD) of the system (A, B, I_n) with $D(s)$ column reduced, and let $P(s)$ be a nonsingular proper compensator. Then $P(s)$ is state feedback realizable on (A, B, I_n) if and only if

- $P(s)$ is biproper, and
- $P^{-1}(s)D(s)$ is a polynomial matrix. ■

4.3 Main results

The following result presents the conditions for decoupling with Λ-stability.

Theorem 4.1 The controllable and Λ-stable square system (A, B, C) is decouplable with Λ-stability if and only if its associated Λ-stable interactor $\Phi_\Lambda(s)$ is a diagonal matrix.

Proof. Necessity. Suppose that (A, B, C) is decouplable with Λ-stability. Then there exists a state feedback (F, G) such that

$$\begin{aligned} T_{F,G}(s) &= C(sI - A - BF)^{-1}BG \\ &= T(s)[I - F(sI - A)^{-1}B]^{-1}G = W(s). \end{aligned}$$

Since $[I - F(sI - A)^{-1}B]^{-1}G$ is a $\mathbb{R}_\Lambda(s)$-unimodular matrix, and $W(s) = \mathrm{diag}\{w_1(s), \ldots, w_p(s)\}$, then it follows that $\Phi_\Lambda(s)$ is diagonal.

Sufficiency. Supposing that $\Phi_\Lambda(s)$ is diagonal, we will prove this part by showing that the $\mathbb{R}_\Lambda(s)$-unimodular matrix $U(s)$ appearing in (4.1) is feedback realizable. Let $N_1(s)$ and $D(s)$ be a right coprime MFD of the system (A, B, I_n) with $D(s)$ column reduced. According to Lemma 4.2, $U(s)$ will be proved to be feedback realizable if $U^{-1}(s)D(s)$ is polynomial.

From (4.1) we have

$$T(s)U(s) = CN_1(s)D^{-1}(s)U(s) = \Phi_\Lambda^{-1}(s),$$

and from this, we get

$$\Phi_\Lambda(s) C N_1(s) = U^{-1}(s) D(s). \tag{4.5}$$

Since the left hand side of the last equation has only poles out of Λ, and the right hand side has only poles in Λ, then clearly $U^{-1}(s)D(s)$ is a polynomial matrix, which proves the result.

Given that $U^{-1}(s)D(s)$ is polynomial, then there exists a constant solution X, Y with X no singular to the polynomial matrix equation [14]

$$X D(s) + Y N_1(s) = U^{-1}(s) D(s). \tag{4.6}$$

Then, a state feedback (F, G) such that $U(s) = [I - F(sI - A)^{-1}B]^{-1}G$ is given by

$$F = -X^{-1}Y, \quad G = X^{-1}. \tag{4.7}$$

This state feedback decouples the system (A, B, C), preserves internal Λ-stability and produces $\Phi_\Lambda^{-1}(s)$ as the transfer function of the closed-loop system. ■

Remark 4.1 From the fact that the $\mathbb{R}_\Lambda$(s)-unimodular matrix $U(s)$ in (4.1) is feedback realizable, then $\Phi_\Lambda^{-1}(s)$ (diagonal or not) can be considered as the transfer function of the closed-loop system $(A + BF, BG, C)$, whose Λ-stable interactor is $\Phi_\Lambda(s)$. Thus, the matrix $U(s)$ can be regarded as the input-output result of a static state feedback which assigns poles of the system at the positions of Λ-stable zeros producing cancellation, and the remaining poles being located at $s = -\alpha \in \Lambda$.

The necessary and sufficient conditions for decoupling with Λ-stability, stated in Theorem 4.1 in terms of the Λ-stable interactor of the system, can also be expressed in terms of the row and global zero structure of the system out of Λ, which will be defined as follows.

Let n_i be the number of zeros out of Λ (Λ-unstable zeros), finite or infinite, multiplicities included, of the ith row of $T(s)$. The integers $\{n_1, n_2, \ldots, n_p\}$ will be called the row zero structure of the system out of Λ.

Remark 4.2 Let $\Phi_\Lambda^{-1}(s)$ be the column Hermite form of $T(s)$ as given by (4.1). Because of property (4.2), and since $U(s)$ in (4.1) is $\mathbb{R}_\Lambda$(s)-unimodular, it can be seen that if $\varphi_{ii}(s)$ is the only entry different from zero of the ith row of $\Phi_\Lambda^{-1}(s)$, then

$$n_i = \deg_\Lambda \varphi_{ii}(s) = k_{ii},$$

otherwise

$$n_i < \deg_\Lambda \varphi_{ii}(s) = k_{ii}.$$

The global zero structure of the system out of Λ will be defined via the Smith form of the system transfer function (see Theorem 3.2).

Let $\Psi(s)$ be the Smith form of the system transfer function $T(s)$ over the ring $\mathbb{R}_\Lambda$(s) for a given Λ, i.e. there exist $\mathbb{R}_\Lambda$(s)-unimodular matrices $V_1(s)$ and $V_2(s)$ such that

$$V_1(s) T(s) V_2(s) = \Psi(s) = \mathrm{diag}\{\psi_i(s)\}_{i=1}^p \tag{4.8}$$

where the rational functions $\psi_i(s) \in \mathbb{R}_\Lambda$(s) satisfy the divisibility conditions over $\mathbb{R}_\Lambda$(s),

$$\psi_i(s) \mid \psi_{i+1}(s), \qquad i = 1, \ldots, p-1,$$

and they are of the form

$$\psi_i(s) = \frac{\delta_i(s)}{\pi^{n'_i}}, \qquad i = 1, \ldots, p,$$

where $\delta_i(s)$ is a polynomial with only Λ-unstable roots, $\pi = s + \alpha$ is a stable term, and n'_i is a positive integer.

The global zero structure of the system out of Λ will be defined as the set of integers $\{n'_1, n'_2, \ldots, n'_p\}$, which correspond to the degree of the rational functions $\psi_i(s)$ in the Smith form of the system transfer function $T(s)$, i.e.

$$n'_i = \deg_\Lambda\ \psi_i(s), \qquad i = 1, \ldots, p.$$

Lemma 4.3 Let $\Phi_\Lambda^{-1}(s)$ and $\Psi(s)$ be, respectively, the column Hermite form and the Smith form of $T(s)$ over $\mathbb{R}_\Lambda(s)$ as given by (4.1) and (4.8). Then

$$\sum_{i=1}^{p} k_{ii} = \sum_{i=1}^{p} n'_i \tag{4.9}$$

Proof. From (4.1) and (4.8), and due to the fact that the degree of the determinant of a $\mathbb{R}_\Lambda(s)$-unimodular matrix is zero (see Lemma 3.1), it can be seen that

$$\deg_\Lambda\ [\det \Phi_\Lambda^{-1}(s)] = \deg_\Lambda\ [\det \Psi(s)].$$

Then (4.9) results from the previous equation. ■

Theorem 4.2 The Λ-stable interactor of the system (A, B, C) is diagonal (equivalently: the system is decouplable with Λ-stability) if and only if the row and global zero structure of the system out of Λ is the same, i.e. if and only if

$$\sum_{i=1}^{p} n_i = \sum_{i=1}^{p} n'_i. \tag{4.10}$$

Proof. Necessity. Suppose that $\Phi_\Lambda(s)$ is diagonal. Then, from Remark 4.2 we have that

$$n_i = \deg_\Lambda\ \varphi_{ii}(s) = k_{ii}, \qquad i = 1, \ldots, p.$$

Thus, from Lemma 4.3 we have (4.10).

Sufficiency. Suppose that (4.10) holds, but that $\Phi_\Lambda(s)$ is not diagonal. From Remark 4.2 we have that $n_i < k_{ii}$ for some $i = 1, \ldots, p$, implying also that

$$\sum_{i=1}^{p} n_i < \sum_{i=1}^{p} k_{ii}.$$

Following Lemma 4.3, this means that

$$\sum_{i=1}^{p} n_i < \sum_{i=1}^{p} n'_i$$

contradicting our assumption. ■

The next two examples illustrate the main results.

Example 4.1 Let the system (A, B, C) be given by

$$A = \begin{bmatrix} -1 & 1 & 1 & 4 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ -1 & -2 & -1 & -4 & -3 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -4 \\ 0 & 0 \\ 0 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 1 & 0 & 2 \\ 0 & 0 & 0.5 & 0 & 0.5 \end{bmatrix}$$

whose transfer function is

$$T(s) = C(sI - A)^{-1}B = \begin{bmatrix} \frac{1}{(s+1)^2} & 0 \\ \frac{1}{(s+1)^4} & \frac{s-1}{(s+1)^3} \end{bmatrix}.$$

Let us suppose that we want to decouple this system by static state feedback, such that the closed-loop system is internally stable, i.e. such that the eigenvalues of the matrix $(A+BF)$ are in the open left half complex plane $\mathcal{C}_-$. It can be seen that the Λ-stable interactor of this system for $\Lambda = \mathcal{C}_-$ ($\mathbb{R}_\Lambda(s)=\mathbb{R}_{ps}(s)$) is not a diagonal matrix, because the entry $(2,1)$ of $T(s)$ can not be cancelled out by an elementary column operation over $\mathbb{R}_{ps}(s)$, thus the system is not decouplable with stability. In terms of the row and global zero structure of the system out of $\Lambda = \mathcal{C}_-$ (see Theorem 4.2), we have that

$$n_1 = 2, \;\; n_2 = 2,$$
$$n_1' = 2, \;\; n_2' = 3,$$

and (4.10) does not hold. The reason for this is that the row infinite zeros and the global infinite zeros coincide, but not the row and global unstable zeros, since $s=1$ is an unstable global zero but it is not a row zero of $T(s)$.

Since the row infinite zeros $\{2,2\}$ and the global infinite zeros $\{2,2\}$ coincide, this system is decouplable, but not decouplable with stability. Indeed, the decoupling compensator

$$P(s) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{(s+1)(s-1)} & \frac{s+1}{s-1} \end{bmatrix}$$

is feedback realizable, and the state feedback realizing $P(s)$ can be obtained from a constant solution X, Y with X nonsingular to the polynomial matrix equation $XD(s)+YN_1(s) = P^{-1}(s)D(s)$, where $N_1(s)$, $D(s)$ is a right coprime MFD of (A, B, I_n) with $D(s)$ column reduced. Then the state feedback

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}, \;\; G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

produces the decoupled system $(A+BF, BG, C)$ whose transfer function is

$$T_{F,G}(s) = C(sI - A - BF)^{-1}BG = \begin{bmatrix} \frac{1}{(s+1)^2} & 0 \\ 0 & \frac{1}{(s+1)^2} \end{bmatrix},$$

but the closed-loop system is not internally stable, since one of the eigenvalues of the matrix $(A+BF)$ is equal to 1.

Example 4.2 Let the system (A, B, C) be given by

$$A = \begin{bmatrix} -5 & -6 & 0 & -1 & -2 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & -2 & -5 & -9 & -6 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}, \;\; B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \\ -1 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$C = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 \end{bmatrix}$$

whose transfer function is

$$T(s) = C(sI - A)^{-1}B = \begin{bmatrix} \frac{1}{(s+2)^2} & 0 \\ \frac{s-1}{(s+2)^4} & \frac{(s-1)(s+1)}{(s+2)^3} \end{bmatrix}.$$

Again, let us suppose that we want to decouple this system by static state feedback, such that the closed-loop system is internally stable. The Λ-stable interactor of this system for $\Lambda = \mathcal{C}_-$ ($\mathbb{R}_\Lambda(s)=\mathbb{R}_{ps}(s)$) is a diagonal matrix

$$\Phi_\Lambda(s) = \begin{bmatrix} \frac{1}{\pi^2} & 0 \\ 0 & \frac{s-1}{\pi^2} \end{bmatrix}^{-1} = \begin{bmatrix} \pi^2 & 0 \\ 0 & \frac{\pi^2}{s-1} \end{bmatrix},$$

or equivalently, the row and global infinite and unstable zeros of the system coincide, meaning that the system is decouplable with stability.

From (4.1), and with $\pi = s+2$, we have that

$$U(s) = \begin{bmatrix} 1 & 0 \\ -\frac{1}{(s+1)(s+2)} & \frac{s+2}{s+1} \end{bmatrix}.$$

Then, from a constant solution X, Y, with X nonsingular, to the polynomial matrix equation $XD(s) + YN_1(s) = U^{-1}(s)D(s)$, where $N_1(s)$, $D(s)$ is a right coprime MFD of (A, B, I_n) with $D(s)$ column reduced, we obtain the state feedback

$$F = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 2 & 0 & 1 & 2 \end{bmatrix}, \quad G = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which produces the decoupled and stable system $(A+BF, BG, C)$ whose transfer function is

$$T_{F,G}(s) = C(sI - A - BF)^{-1}BG = \begin{bmatrix} \frac{1}{(s+2)^2} & 0 \\ 0 & \frac{s-1}{(s+2)^2} \end{bmatrix}.$$

Actually, this system can be decoupled with stability with closed-loop transfer function

$$T_{F,G}(s) = \begin{bmatrix} \frac{1}{(s+\alpha_1)(s+\alpha_2)} & 0 \\ 0 & \frac{s-1}{(s+\alpha_3)(s+\alpha_4)} \end{bmatrix}$$

where $\alpha_{1,2,3,4}$ are arbitrary positive real numbers.

Let us suppose now that we want to decouple this system with Λ-stability, where Λ is the region shown in Fig. 4.1, starting at $s = -3$, and forming an angle of $\pm 45°$ with the negative real axis.

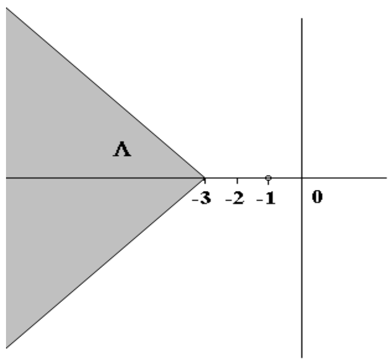

Fig. 4.1. Region Λ

The row zero structure of the system out of Λ is

$$n_1 = 2, \quad n_2 = 2,$$

and the global zero structure out of Λ is

$$n'_1 = 2, \quad n'_2 = 3.$$

Since these structures do not coincide due to the fact that the global zero at $s = -1$ is not a row zero of the system, then decoupling with Λ-stability by static state feedback can not be achieved. This can also be seen from the fact that the Λ-stable interactor of this system is not a diagonal matrix. Notice that the angle of the region Λ with respect to the real axis plays a role only if the system has complex finite zeros with imaginary part different from zero, which is not the case in this example.

Since the symmetric region Λ of the complex plane can be arbitrarily chosen, it is to be expected that previous results on decoupling can be deduced from our results. Indeed, the conditions for decoupling of square linear multivariable systems, and decoupling with stability can be easily deduced from our solution. For these two cases, the conditions of Theorem 4.2 can be interpreted as follows:

- Consider the decoupling problem with closed-loop modes at any finite position. In this case $\Lambda = \mathcal{C}$ and the region out of Λ is $\Lambda' = \infty$. Then the system (A, B, C) is decouplable if and only if the row and global infinite zero structure of the system is the same [6,7].

- In the decoupling problem with stability, it is required that all modes of the closed-loop system are in the left hand complex plane, i.e. $\Lambda = \mathcal{C}_-$ and $\Lambda' = \mathcal{C}_+ \cup \infty$. Then the system is decouplable in this case if and only if the row and global zero structure of the system out of Λ is the same, i.e. if and only if the row and global infinite zeros, plus the row and global zeros in $\mathcal{C}_+$ are the same [8,9].

5 Conclusions

In this paper we have examined algebraic properties of Λ-generalized polynomials, and applied these properties to the problem of decoupling with Λ-stability of linear square multivariable systems. The structural necessary and sufficient conditions for this problem to have a solution are stated in terms of the Λ-stable interactor of the system or in terms of the row and global zero structure of the system out of Λ. Summarizing: decoupling as such, i.e. each system output being controlled independently by one and only one input regardless of the position of the closed-loop modes, can be achieved if and only if the row infinite zero structure of the system is the same as the global infinite zero structure. If the system is decouplable, decoupling with closed-loop modes in a particular region Λ of the complex plane can be achieved if and only if the global finite zeros out of Λ of the system, multiplicities included, are also row finite zeros of the system.

Observe that if a system (A, B, C) with no finite zeros is decouplable, then it can be decoupled with closed-loop modes located at any finite position of the complex plane. If a decouplable system has finite zeros, then for the system to be decouplable with Λ-stability, the region Λ must contain every global zero that it is not a row zero of the system. For instance, in Example 4.2, the only global finite zero that it is not a row finite zero is $s = -1$; then the system can be decoupled with Λ-stability if and only if the region Λ contains this point.

Theorems 4.1 and 4.2 state the necessary and sufficient conditions for decoupling with Λ-stability, Theorem 4.1 in terms of the Λ-stable interactor of the system, and Theorem 4.2 in terms of the global and row zero structure of the system out of Λ. From the practical point of view, since elementary operations in a particular ring Λ could be very involved, in general is simpler to test the conditions of Theorem 4.2 than to obtain the Λ-stable interactor of the system. The row zero structure of the system out of Λ can be obtained for instance from a gcd of the i-th row of $T(s)$, and the global zero structure of the system out of Λ can be obtained from the Smith form of $T(s)$ using the determinantal divisors of $T(s)$, avoiding thus the use of elementary operations in $\mathbb{R}_\Lambda(\mathrm{s})$. This procedure, however, can also be very complicated. If the system (A, B, C) is decouplable with Λ-stability, a decoupling state feedback can be computed as indicated in the proof of Theorem 4.1.

Observe that the main feature of our extension to generalized polynomials as presented in this paper, namely the inclusion of the point at infinity, is not fully exploited in the study of the decoupling problem with Λ-stability. Indeed, though generally defined as a subset of the extended complex plane, for the problem of decoupling the region Λ is taken as a subset of the complex plane, since the poles of a linear system (A, B, C) are all finite. Infinite poles may appear in the study of the so-called singular systems, i.e. systems of the form

$$\begin{aligned} E\dot{x}(t) &= Ax(t) + Bu(t) \\ y(t) &= Cx(t) \end{aligned}$$

where E is not square, or is singular. But even in this case, since infinite pole behavior is undesirable from the practical point of view, control strategies are usually designed to deal only with finite poles. It is then hoped that the concepts of generalized polynomials will be applied in a near future to solve in a unified way other multivariable control problems besides decoupling.

As shown in Theorems 4.1 and 4.2, obtaining the Hermite and Smith canonical forms is a cornerstone of the design algorithm to achieve multivariable decoupling. For the same reason as Gaussian elimination without pivoting can be numerically unstable in the field of real or complex-valued matrices, it is now recognized that elementary operations on polynomial or rational objects are not numerically sound. An important future research topic is therefore the development of numerical stable, efficient algorithms to deal with Λ-generalized polynomials. Preliminary work in this direction was described in [15] in the special case of polynomial matrices.

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