# LMI Approximations for the Radius of the Intersection of Ellipsoids: a Survey


Didier Henrion*
Sophie Tarbouriech
Denis Arzelier

*LAAS-CNRS*
*7 Avenue du Colonel Roche*
*31 077 Toulouse, Cedex 4*
*France*



**Abstract**

This paper surveys various linear matrix inequality relaxation techniques for evaluating the maximum norm vector within the intersection of several ellipsoids. This difficult non-convex optimization problem frequently arises in robust control synthesis. Two randomized algorithms and several ellipsoidal approximations are described. Guaranteed approximation bounds are derived in order to evaluate the quality of these relaxations.


# 1 Introduction

## 1.1 Problem Statement

In this paper we consider the optimization problem

$$\begin{array}{rcll} p_{\text{opt}} & = & \max_x & x'x \\ & & \text{s.t.} & x \in \mathcal{F} \end{array} \qquad (1)$$

where $x$ is a vector in $\mathbb{R}^n$ and the set $\mathcal{F}$ is the intersection of $m$ ellipsoids

$$\mathcal{F} = \mathcal{E}_1 \cap \mathcal{E}_2 \cap \cdots \cap \mathcal{E}_m \qquad (2)$$


*Corresponding Author. E-mail: henrion@laas.fr

defined as

$$\mathcal{E}_i = \{x : x' P_i x \leq 1\} \tag{3}$$

for $P_i$ a given symmetric positive definite matrix in $\mathbb{R}^{n \times n}$. Feasible set $\mathcal{F}$ is the intersection of $m$ centered ellipsoids in $\mathbb{R}^n$, hence $\mathcal{F}$ is convex and centered about the origin. It is assumed that $\mathcal{F}$ is not reduced to the origin, otherwise the solution to problem (1) is trivially the zero vector. Problem (1) then merely amounts to evaluating the radius of $\mathcal{F}$, or equivalently, to finding the vector of greatest Euclidean norm that lies within the intersection of given ellipsoids.

## 1.2 Problem Relevance

Our motivation for studying problem (1) essentially stems from system and control theory. In [25], it has been shown that checking whether the radius of the intersection of ellipsoids is strictly greater than one provides a new and simple geometric interpretation of the existence of a solution to general Bilinear Matrix Inequalities (BMIs). As pointed out in [15], the solution of BMIs may be regarded as a central problem in robust control. Indeed, with the aid of suitably chosen sector transforms, a very broad class of robust control synthesis problems are representable in terms of BMIs. Such difficult key problems include fixed-order $\mathcal{H}_\infty$ synthesis, $\mu/k_m$-synthesis, decentralized control, robust gain-scheduling or simultaneous stabilization, to mention just a few. If a method can be devised to tackle general BMIs, then a substantial progress would be made in the resolution of the aforementioned control problems.

Independently, in a recent paper [16] we used the Youla-Kučera parametrization [19] and quadratic forms over Hermite-Fujiwara matrices [12] to show that the problem of robust controller design for a linear system subject to real parametric uncertainty is actually equivalent to that of solving (1). Later on [17], we showed how problem (1) occurs when trying to stabilize simultaneously a set of given plants with a unique controller, a problem widely recognized as one of the fundamental open problems in control theory [5]. Incidentally, we believe that most robust control synthesis results achieved through the polynomial approach should also be amenable to the elegant and unifying ellipsoid intersection representation proposed in [25].

Problems involving the intersection of ellipsoids also occur in estimation and filtering. In some parameter and state estimation problems, it is natural to describe the unknown parameter or state vector by an ellipsoid containing all its possible values. In this context, two problems are at the core of ellipsoidal state bounding, namely optimal outer approximation of the sum and of the intersection of several ellipsoids. For a recent reference on ellipsoidal state bounding in estimation problems, the interested reader is referred to [20]. A more mathematical treatment of the problem can be found in [9].

Finally, in the interesting papers [3, 4], Ben-Tal, El Ghaoui and Nemirovski lay the foundation of Robust Convex Optimization: they study convex optimization problems (such as Linear Programming, Quadratically Constrained Programming or Semidefinite Programming) for which the data is not specified exactly and is only known to belong to a given uncertainty set. It is argued that restricting the uncertainty to lie within the intersection of given ellipsoids leads to a good tradeoff between genericity of the uncertainty model and computational tractability. In this regard, ellipsoidal approximation of the intersection of ellipsoids proves particularly useful when measuring the level of conservativeness of convex approximations to robust counterpart of Semidefinite Programming, as illustrated for example in [3, §3.4] or [4, §2.1].

## 1.3 Problem Hardness

Besides the simplicity of its formulation, the ellipsoid intersection problem is unfortunately highly non-convex since a convex function must be maximized over a convex set. This makes problem (1) a particular case of a concave minimization problem, see [18, Chapter 1, §2]. Concave minimization problems are known to be tough, multi-extremal problems, but their properties make them easier to handle than general global optimization problems. However, from the viewpoint of computational complexity, concave minimization problems are generally NP-hard.

The ellipsoid intersection problem can also be viewed as a special instance of a general quadratic problem where a possibly indefinite quadratic function must be maximized over a set of possibly indefinite quadratic inequalities. This kind of problems is fairly general. It covers every mathematical programming problem with polynomial objective and polynomial or boolean constraints [21, §6.4.4]. Quadratic problems are known to be NP-hard when the matrix defining the quadratic criterion to be maximized has at least one positive eigenvalue [11], which is clearly the case here.

Furthermore, our problem can also be related to the ellipsoidal approximation of a subset of $\mathbb{R}^n$. Indeed, evaluating the greatest norm vector in $\mathcal{F}$ is reminiscent to finding the minimum radius sphere containing $\mathcal{F}$. As stated in [7, §3.7], the problem of describing the ellipsoid (or similarly, the sphere) of smallest volume that contains the intersection of given ellipsoids is NP-hard. Even worse, simply verifying that a given ellipsoid contains the intersection of given ellipsoids is known to be NP-complete. As a result, we do not expect to solve problem (1) in polynomial time, or a fortiori, to cast it as a convex optimization problem.

## 1.4 Approximate Solutions

Because of the importance of the ellipsoid intersection problem and in the absence of efficient numerical procedures to solve it exactly, it is highly relevant to study different methods that may provide approximate solutions. Moreover, if one is really interested in solving (1) globally, lower and upper approximation bounds on the global optimal value are required when implementing a branch-and-bound scheme [1, 18]. It is then necessary to have at hand well-worked, widely spread approximation algorithms that feature polynomial complexity. In control engineering, the tool that best fits this description is undoubtedly semidefinite programming (SDP) – see for instance [26] for a good overview. In this scope, efficient interior-point methods have been designed [21] to solve several control problems formulated in a unifying way through Linear Matrix Inequalities (LMIs), an extension of linear programming over the cone of positive semidefinite matrices [7]. Several SDP packages are now available for the scientific community [13, 28]. In a recent and stimulating survey [8], Boyd and Vandenberghe show how semidefinite programs may arise as convex relaxations of many NP-hard control problems. In the Soviet literature on control theory, these relaxation techniques are called the $\mathcal{S}$-procedure. Actually, they can similarly be derived from Lagrange duality.

In an entirely different context, it has been realized only recently that SDP relaxations may also be used to approximate various NP-hard problems of combinatorial optimization. A fundamental breakthrough was achieved in [14], where Goemans and Williamson showed that semidefinite relaxations of the maximum cut problem in a graph (a special NP-hard instance of quadratic programming with integer variables) always deliver solutions at least 87.9% the global optimal value. Their bound, derived from a randomized approximation algorithm, constitute the first substantial progress in this domain in nearly twenty years. In combinatorial optimization, the relevance of SDP is in that it usually provides tighter approximation bounds than classical linear programming relaxation.

In [8], the authors point out that the close connections between the $\mathcal{S}$-procedure used in control theory, and the various semidefinite relaxations used in combinatorial optimization, do not appear to be well known. They also wonder whether results similar to those in [14] could be obtained in control theory, and what may be the practical performance of semidefinite relaxation in non-convex quadratic problems. The work presented in the sequel aims at addressing these points while focusing on our particular ellipsoid intersection problem.

## 1.5 Contribution

Our contribution is threefold.

- First, we propose a fairly comprehensive overview of LMI relaxation techniques to date for the ellipsoid intersection problem. Up to the authors' knowledge, this is the first time that the results obtained in different settings are specialized for approximating this key control problem. Moreover, most of the results collected in this paper have only been derived very recently.
- Second, we enlighten the links existing between the semidefinite relaxations used in control theory (LMIs, $\mathcal{S}$-procedure) and those used in combinatorial optimization. Following the same lines as Goemans and Williamson [14], we derive an LMI randomized algorithm for approximating the global optimum of our ellipsoid intersection problem.
- Third, we theoretically assess the quality of these LMI relaxations and enumerate a collection of guaranteed approximation bounds on the global optimum of problem (1). We show that standard LMI relaxation schemes may result in poor approximations of the global optimum, especially when the dimension of the ambient space and/or the number of intersecting ellipsoids are high.

After completing the first version of this manuscript, we became aware of the fact that the problem of quadratic form maximization over intersection of ellipsoids was independently studied by Nemirovski, Roos and Terlaky [22]. In this interesting paper, the issue of assessing the quality of LMI relaxations of problem (1) is also studied and a new approximation bound is derived, which actually proves to be the tightest to date. For the sake of completeness, these very recent results are also reported in our paper.

### 1.6 Paper Outline

The content of the paper is as follows. In Section 2, we briefly review existing LMI relaxation schemes for problem (1). All of them are shown to be equivalent. In Section 3, two randomized techniques are applied to problem (1). The first one is believed to be new. The second technique, more sophisticated, was elaborated in [22]. Both techniques allow to derive a guaranteed approximation bound on suboptimal solutions. Section 4 collects ellipsoidal approximation techniques and illustrates how they may be used for obtaining lower and upper bounds on the global optimum of problem (1). We shall see that the procedure developed in Section 3 may also be interpreted as an ellipsoidal approximation. The different LMI relaxation schemes and bounds are recapped in Section 5 by the way of a simple and illustrative two-dimensional example. Finally, several concluding remarks are drawn in Section 6 and potential directions for further research are mentioned.

### 1.7 Notations

The following notations will be used throughout the paper. $\mathbf{Tr}X$ stands for the trace of matrix $X$, the sum of its diagonal elements. $\mathbf{E}[x]$ is the expectation or mean of random variable $x$. $\mathbf{I}_n$ denote the identity matrix of dimension $n$. When $X$ and $Y$ are two matrices, the inequality $X > Y$ (resp. $X \geq Y$) holds with respect to the cone of positive definite matrices, i.e. matrix $X - Y > 0$ is positive definite (resp. semidefinite).

## 2 LMI Relaxations

Linear Matrix Inequalities are now widely spread in the system and control community [7]. In addition to provide a unified formulation of a great number of convex control problems, LMIs can efficiently be solved through various Semidefinite Programming techniques, including interior-point methods [21]. Alternatively, LMI relaxations can be used to derive cheap approximations to difficult non-convex problems. In this section, we review standard LMI relaxations for problem (1) and show that they are actually equivalent.

## 2.1 Rank-one Dropping

First note that, upon defining

$$X = xx',$$

problem (1) can also be written

$$\begin{array}{rcll} p_{\mathrm{opt}} & = & \max_X & \mathbf{Tr} X \\ & & \text{s.t.} & \mathbf{Tr} P_i X \leq 1, \quad i = 1, \ldots, m \\ & & & X \geq 0 \\ & & & \mathrm{rank} X = 1. \end{array} \tag{4}$$

The only non-convex term in the above formulation is the rank-one constraint on positive semidefinite matrix $X$. This constraint may be dropped, leading to the first relaxation for problem (1)

$$\begin{array}{rcll} p_{\mathrm{C}} & = & \max_X & \mathbf{Tr} X \\ & & \text{s.t.} & \mathbf{Tr} P_i X \leq 1, \quad i = 1, \ldots, m \\ & & & X \geq 0. \end{array} \tag{5}$$

The subscript $C$ in $p_{\mathrm{C}}$ stands for Cholesky. The reason behind this choice will become apparent in Section 4. The above problem is convex since a linear criterion must be maximized over a set of LMIs in matrix $X$. Every feasible $X$ in (4) is also feasible in (5) hence optimization problem (5) only yields an upper bound on the global optimum, i.e.

$$p_{\mathrm{C}} \geq p_{\mathrm{opt}}.$$

## 2.2 Lagrange Duality

Using standard duality results over the cone of positive semidefinite matrices [26], the Lagrangian for relaxed problem (5) reads

$$L(X, \lambda, \Lambda) = -\mathbf{Tr} X + \sum_{i=1}^{m} \lambda_i (\mathbf{Tr} P_i X - 1) - \mathbf{Tr} \Lambda X$$

where $\lambda = [\lambda_1, \ldots, \lambda_m]'$ is a vector of positive Lagrange multipliers and $\Lambda$ is a symmetric positive definite Lagrange multiplier matrix. The dual function $g(\lambda, \Lambda) = \min_X L(X, \lambda, \Lambda)$ is

$$g(\lambda, \Lambda) = \left\{ \begin{array}{ll} -\sum_{i=1}^{m} \lambda_i & \text{if } \mathbf{I}_n - \sum_{i=1}^{m} \lambda_i P_i + \Lambda = 0 \\ -\infty & \text{otherwise,} \end{array} \right.$$

hence the dual optimization problem

$$\begin{array}{rcll} p_{\mathrm{C}}^{\mathrm{dual}} & = & \max_{\lambda, \Lambda} & -\sum_{i=1}^{m} \lambda_i \\ & & \text{s.t.} & \Lambda = \sum_{i=1}^{m} \lambda_i P_i - \mathbf{I}_n > 0 \\ & & & \lambda_i \geq 0, \quad i = 1, \ldots, m. \end{array}$$

Dropping out Lagrange multiplier matrix $\Lambda$, the above problem can be simplified to

$$\begin{array}{rcll} p_{\mathrm{C}}^{\mathrm{dual}} & = & \min_{\lambda} & \sum_{i=1}^{m} \lambda_i \\ & & \text{s.t.} & \sum_{i=1}^{m} \lambda_i P_i > \mathbf{I}_n \\ & & & \lambda_i \geq 0, \quad i = 1, \ldots, m. \end{array} \tag{6}$$

Problem (6) is a dual LMI relaxation for problem (1). In both relaxations (5) and (6), the optimization must be performed over a feasible set defined by $m$ scalar inequalities and one semidefinite inequality of dimension $n$. However, the number of decision variables is $n(n+1)/2$ in (5) whereas it is $m$ in (6). Hence, depending on the dimension of the ambient space and the number of ellipsoids, one formulation may prove more interesting than the other. It is assumed that feasible set $\mathcal{F}$ in primal problem (5) does not reduce to the origin, hence by Slater's condition we have no duality gap and

$$p_{\mathrm{C}}^{\mathrm{dual}} = p_{\mathrm{C}}. \tag{7}$$

## 2.3 $\mathcal{S}$-procedure

In order to derive our last LMI relaxation of problem (1), we shall make use of the $\mathcal{S}$-procedure. The $\mathcal{S}$-procedure is also sometimes called Shor's relaxation, and has widely been used in the Soviet literature on control theory, see [29] for a survey. It is recalled here for the sake of clarity.

**Lemma 1** – **$\mathcal{S}$-procedure**
*Given arbitrary symmetric matrices $Q_0, Q_1, \ldots, Q_m$, if there exist $\lambda_1 \geq 0, \ldots, \lambda_m \geq 0$ such that $Q_0 \leq \sum_{i=1}^m \lambda_i Q_i$, then for any vector $z$ such that $z'Q_1 z \leq 0, \ldots, z'Q_m z \leq 0$ we have $z'Q_0 z \leq 0$.*

Now let us apply the above procedure to our problem. For any feasible vector $x$ in (1), we have

$$[x' \quad 1] \underbrace{\begin{bmatrix} P_i & 0 \\ 0 & -1 \end{bmatrix}}_{Q_i} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0, \quad i = 1, \ldots, m.$$

Let $-t$ denote an upper bound on optimal value $p_{\mathrm{opt}}$, so that for any feasible $x$,

$$[x' \quad 1] \underbrace{\begin{bmatrix} \mathbf{I}_n & 0 \\ 0 & t \end{bmatrix}}_{Q_0} \begin{bmatrix} x \\ 1 \end{bmatrix} \leq 0.$$

With these notations, the $\mathcal{S}$-procedure states that for any family of positive scalars $\lambda_i$ such that

$$Q_0 \leq \sum_{i=1}^{m} \lambda_i Q_i,$$

any feasible vector $x$ verifies the upper bound inequality $x'x \leq -t$. Shor's relaxation aims at minimizing this upper bound, and therefore reads

$$\begin{array}{rcl} p_{\mathrm{C}}^{\mathrm{Shor}} & = & \max_{\lambda,t} \quad t \\ & & \text{s.t.} \quad \begin{bmatrix} -\mathbf{I}_n & 0 \\ 0 & -t \end{bmatrix} + \sum_{i=1}^m \lambda_i \begin{bmatrix} P_i & 0 \\ 0 & -1 \end{bmatrix} \geq 0 \\ & & \qquad \lambda_i \geq 0, \quad i = 1, \ldots, m. \end{array} \tag{8}$$

By separating the diagonal term in the above LMI, one exactly obtains dual problem (6) hence

$$p_{\mathrm{C}}^{\mathrm{Shor}} = p_{\mathrm{C}}^{\mathrm{dual}}. \tag{9}$$

In view of equalities (7) and (9), LMI relaxation schemes (5), (6) and (8) provide the same upper bound on the optimal solution to problem (1).

# 3 Randomized LMI Approximations

## 3.1 Cholesky Approximation

The randomized technique of Goemans and Williamson [14] will now be applied for deriving approximation bounds on the LMI relaxations of problem (1). It relies upon the Cholesky decomposition of positive definite matrices. Our development is based on recent results of Nesterov and Ye, who extended Goemans and Williamson's algorithm to boolean quadratic programming [23] and to quadratic programming with bound constraints [30].

Our randomized Cholesky algorithm proceeds as follows.

**Algorithm 1 – Randomized Cholesky Approximation**

1. *Solve LMI relaxation (5) for an optimal solution* $X_{\mathrm{C}}$.
2. *Let* $V_{\mathrm{C}}$ *be the Cholesky square root of* $X_{\mathrm{C}}$, *i.e.* $X_{\mathrm{C}} = V_{\mathrm{C}}' V_{\mathrm{C}}$.
3. *Choose a random vector* $u$ *uniformly distributed on the* $n$*-dimensional unit sphere.*
4. *Let*

$$x_{\mathrm{rand}} = V_{\mathrm{C}}' u. \tag{10}$$

The motivation for the randomization in Step 3 comes from the fact that relaxation (5) is independent of the coordinate system. Indeed, with $v_i$ standing for the $i$-th column of $V_{\mathrm{C}}$, the objective value

$$\mathbf{Tr} X_{\mathrm{C}} = \sum_{i=1}^{n} v_i' v_i$$

remains unchanged for any orthogonal transformation affecting the set of vectors $v_i$. On the other hand, with the notation $V_{\mathrm{C}}' = [w_1 \ \cdots \ w_n]$ and $u = [u_1 \ \cdots \ u_n]'$, matrix $X_{\mathrm{C}}$ can be decomposed into a sum of rank-one matrices, namely $X_{\mathrm{C}} = w_1 w_1' + \cdots + w_n w_n'$. Vector $x_{\mathrm{rand}}$ is then nothing but a normalized random linear combination of the vertices $w_i$, viz. $x_{\mathrm{rand}} = u_1 w_1 + \cdots + u_n w_n$ where $u_1^2 + \cdots + u_n^2 = 1$.

In order to assess the quality of relaxation Algorithm 1, we need the following instrumental lemma. It is a stochastic version of problem (1) and proceeds along the same line as in [23] and [30].

**Lemma 2** *Let $u$ be uniformly distributed on the $n$-dimensional unit sphere. Then*

$$p_{\text{opt}} = \begin{array}{ll} \max_V & \mathbf{E}[u'VV'u] \\ s.t. & \mathbf{Tr}P_iV'V \leq 1. \end{array} \tag{11}$$

**Proof:** First, we show that the above criterion is a lower bound on $p_{\text{opt}}$. Then we find a particular feasible $V$ such that $p_{\text{opt}}$ is attained. So, on the one hand suppose that $V$ is feasible for the above problem. Since $u'u = 1$, we have $uu' \leq \mathbf{I}_n$ hence $V'uu'V \leq V'V$. From the properties of the trace operator and the definite positiveness of $P_i$, it holds $\mathbf{Tr}P_i(V'V - V'uu'V) \geq 0$, therefore $\mathbf{Tr}P_iV'uu'V = u'VP_iV'u \leq \mathbf{Tr}P_iV'V \leq 1$, for $i = 1, \ldots, m$. As a result, vector $V'u$ is feasible for (1) and

$$p_{\text{opt}} \geq \mathbf{E}[u'VV'u].$$

On the other hand, let $x_{\text{opt}}$ be the global optimum in (1) and let $x_{\text{opt}_i}$ stand for its $i$-th component. Given any unitary vector $u$, let

$$V = V_{\text{opt}} = [x_{\text{opt}_1}u \ \cdots \ x_{\text{opt}_n}u]. \tag{12}$$

One can check that $V'_{\text{opt}}u = x_{\text{opt}}$ and $V'_{\text{opt}}V_{\text{opt}} = x_{\text{opt}}x'_{\text{opt}}$ thus $\mathbf{Tr}P_iV'_{\text{opt}}V_{\text{opt}} = x'_{\text{opt}}P_ix_{\text{opt}} \leq 1$. Therefore matrix $V_{\text{opt}}$ is feasible for (11). Thus, for particular feasible $u$, $V$ we have

$$\mathbf{E}[u'V_{\text{opt}}V'_{\text{opt}}u] = \mathbf{E}[u'VV'u] = \mathbf{E}[x'_{\text{opt}}x_{\text{opt}}] = x'_{\text{opt}}x_{\text{opt}} = p_{\text{opt}}.$$

□

When we fix $V = V_{\text{C}}$ in the above problem, the expected value of the randomized criterion is characterized by the following lemma, which establishes the link between stochastic problem (11) and randomized Algorithm 1.

**Lemma 3** *Let $u$ be uniformly distributed on the $n$-dimensional unit sphere. Then*

$$\mathbf{E}[x'_{\text{rand}}x_{\text{rand}}] = \frac{1}{n}p_{\text{C}}.$$

**Proof:** First note that

$$\mathbf{E}[x'_{\text{rand}}x_{\text{rand}}] = \mathbf{E}[u'V_{\text{C}}V'_{\text{C}}u] = \mathbf{E}[\mathbf{Tr}(V_{\text{C}}V'_{\text{C}}uu')] = \mathbf{Tr}(V_{\text{C}}V'_{\text{C}}\mathbf{E}[uu']). \tag{13}$$

Vector $u$ can be anywhere on the unit sphere, thus $\mathbf{E}[u_1^2] = \cdots = \mathbf{E}[u_n^2]$. Recall that $\mathbf{E}[u_1^2 + \cdots + u_n^2] = \mathbf{E}[1] = 1$, so $\mathbf{E}[u_i^2] = \frac{1}{n}$. Moreover, by symmetry $\mathbf{E}[u_iu_j] = 0$ for $i \neq j$. The covariance matrix of $u$ is then

$$\mathbf{E}[uu'] = \frac{1}{n}\mathbf{I}_n$$

and (13) becomes

$$\mathbf{E}[x'_{\text{rand}}x_{\text{rand}}] = \frac{1}{n}\mathbf{Tr}V_{\text{C}}V'_{\text{C}} = \frac{1}{n}\mathbf{Tr}V'_{\text{C}}V_{\text{C}} = \frac{1}{n}\mathbf{Tr}X_{\text{C}}.$$

The desired equality then follows from LMI relaxation (5). □

We are now ready to state our main approximation result.

**Theorem 1** *Algorithm 1 provides the following lower and upper bounds on $p_{\mathrm{opt}}$, the global optimum of problem (1):*

$$\mathbf{E}[x'_{\mathrm{rand}} x_{\mathrm{rand}}] \leq p_{\mathrm{opt}} \leq n \mathbf{E}[x'_{\mathrm{rand}} x_{\mathrm{rand}}].$$

**Proof:** In order to prove the above theorem, we must show that

$$p_{\mathrm{opt}} \geq \mathbf{E}[x'_{\mathrm{rand}} x_{\mathrm{rand}}] \geq \frac{1}{n} p_{\mathrm{opt}}.$$

For the first part of the inequality, note that $V = V_{\mathrm{C}}$ is feasible for problem (11) and see the proof of Lemma 2. For the second part, observe that LMI relaxation (5) ensures that $p_{\mathrm{C}} \geq p_{\mathrm{opt}}$ and apply Lemma 3. □

The above theorem provides a measure of the tightness of the combination of randomized approximation and semidefinite relaxation of problem (1). Note that this bound only depends on the dimension of the space where the optimization is carried out. In particular, it does not depend on the number of constraints.

One may notice that the bound can be very loose when $n$ increases. This is in contrast with the tight semidefinite approximation bounds recently obtained in combinatorial optimization, using the same randomized technique [14, 23, 30].

## 3.2 Nemirovski Approximation

Nemirovski and co-workers recently developed another randomized technique for the ellipsoid intersection problem. The approach pursued by these authors is more involved that the one exposed in the previous section. Their algorithm is briefly reproduced here for the sake of completeness. For further details, the reader may consult [22].

**Algorithm 2 – Randomized Nemirovski Approximation**

1. *Solve LMI relaxation (5) for an optimal solution $X_{\mathrm{C}}$.*

2. *Decompose matrix $X_{\mathrm{C}}$ as a sum of rank-one matrices, i.e.*

$$X_{\mathrm{C}} = \sum_{j=1}^{n} v_j v_j'.$$

3. *Let $a_{ij} = v_j' P_i v_j$ and $c_j = v_j' v_j$ for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Let $A$ denote the matrix with entries $a_{ij}$ and let $c$ denote the vector with entries $c_j$. Let $b$ denote a vector whose $m$ entries are set to one. Solve linear program*

$$\begin{array}{ll} \max_z & c'z \\ s.t. & Az \leq b \end{array}$$

*for an optimal solution $\hat{z}$. The number of non-zero entries in vector $\hat{z}$ is less than or equal to $m$.*

4. *Build matrix*

$$X = \sum_{j=1}^{n} \hat{z}_j v_j v_j'.$$

*By construction, matrix $X$ is feasible for problem (5) and* $\mathrm{rank} X \leq \min\{m, n\}$.

5. *Let $V$ be the Cholesky square root of matrix $X$, i.e. $X = V'V$.*

6. *Decompose matrix $VV'$ as*

$$VV' = UDU'$$

*where Schur matrix $D$ is diagonal and matrix $U$ is orthonormal.*

7. *Generate a random vector $u$ with independent entries taking with equal probabilities the values $+1$ or $-1$ and let*

$$x_{\mathrm{NRT}} = \frac{1}{\sqrt{\max_j u'U'P_jUu}} V'Uu.$$

**Theorem 2** *Algorithm 2 provides the following lower and upper bounds on $p_{\mathrm{opt}}$, the global optimum of problem (1):*

$$\frac{1}{2\log(2m\min\{m,n\})}\mathbf{E}[x_{\mathrm{NRT}}' x_{\mathrm{NRT}}] \leq p_{\mathrm{opt}} \leq \mathbf{E}[x_{\mathrm{NRT}}' x_{\mathrm{NRT}}].$$

**Proof:** See [22]. □

# 4 Ellipsoidal Approximations

Various approaches may be pursued for approximating a subset of $\mathbb{R}^n$ with an ellipsoid. One usually distinguishes between inner or outer approximations. Moreover, the quality of the approximations may be measured for instance by the volume or the length of semi-axes of the ellipsoid, depending on the problem being treated. In this section, applications to the ellipsoid intersection problem are studied into detail and two approximation algorithms are described. Bounds on the global optimum of problem (1) are derived. Then, we show that randomized Algorithm 1 can also be interpreted as an ellipsoidal approximation algorithm. Finally, we propose a derandomized version of Algorithm 1 and characterize it through approximation bounds.

## 4.1 Dikin ellipsoids

In semidefinite programming, inner ellipsoidal approximations naturally follow from the notion of the analytic center of a convex set [6]. To see that, define feasible set $\mathcal{F}$ through an LMI, i.e. for symmetric matrices $F_i$ let

$$\mathcal{F} = \{x : F(x) = F_0 + \sum_{i=1}^{n} x_i F_i \geq 0\}.$$

By a simple Schur's complement argument, one can check that for the feasible set defined in (2) we have

$$F(x) = \mathrm{diag}\left\{ \begin{bmatrix} 1 & x' \\ x & P_1^{-1} \end{bmatrix}, \ldots, \begin{bmatrix} 1 & x' \\ x & P_m^{-1} \end{bmatrix} \right\}.$$

Associated to $F(x)$ is the function

$$L(x) = \begin{cases} \log\det F(x)^{-1} & x \in \mathrm{int}\mathcal{F} \\ \infty & \text{elsewhere.} \end{cases}$$

Function $L(x)$ is finite if and only if $x \in \mathcal{F}$ and becomes infinite as $x$ approaches the boundary of $\mathcal{F}$. It is called a barrier function for $\mathcal{F}$. It can be shown that when $x \in \mathcal{F}$, $L(x)$ is analytic and strictly convex. The unique minimizer of $L(x)$, denoted by $x^\star$, is referred to as the analytic center of the LMI $F(x) > 0$. Quite naturally, the level curves of $L(x)$ give a smooth approximation of the shape of the boundary of $\mathcal{F}$. Near $x^\star$ the shape of these level curves is determined by matrix $H(x^\star)$, the Hessian of $L(x)$ evaluated at $x^\star$. So the ellipsoids centered at $x^\star$ and with shape determined by $H(x^\star)$ are expected to give good quadratic approximations of the shape of $\mathcal{F}$. Such ellipsoids are called Dikin ellipsoids and can be derived for a general class of barrier functions (called self-concordant) which includes the function described above, see [21, Theorem 2.1.1]. Since

$$L(x) = \log\det F(x)^{-1} = -\log \prod_{i=1}^{m} (1 - x' P_i x) = -\sum_{i=1}^{m} \log(1 - x' P_i x)$$

the Hessian can be found upon two derivations of the above expression and reads

$$H(x) = \nabla^2 L(x) = \sum_{i=1}^{m} \left( \frac{4 P_i x x' P_i}{(1 - x' P_i x)^2} + \frac{2 P_i}{1 - x' P_i x} \right).$$

The analytic center in our case is clearly $x^\star = 0$ so that Dikin ellipsoids of $\mathcal{F}$ simplify to

$$\mathcal{E}_{\mathrm{D}}^{r} = \{x : x' P_D x \leq r\} \tag{14}$$

where

$$P_D = H(x^\star) = H(0) = 2 \sum_{i=1}^{m} P_i. \tag{15}$$

**Lemma 4** *Given the $m$-ellipsoid intersection set $\mathcal{F}$ defined in (2), we have*

$$\mathcal{E}_{\mathrm{D}}^{1} \subseteq \mathcal{F} \subseteq \mathcal{E}_{\mathrm{D}}^{m^2},$$

*where inner and outer Dikin ellipsoids $\mathcal{E}_{\mathrm{D}}^{1}$ and $\mathcal{E}_{\mathrm{D}}^{m^2}$ are defined in (14).*

**Proof:** The inclusion of $\mathcal{E}_{\mathrm{D}}^1$ into $\mathcal{F}$ is rigorously proved in [21, Theorem 2.1.1]. For a less technical and more intuitive proof, see [6, §3.3]. The inclusion of $\mathcal{F}$ into $\mathcal{E}_{\mathrm{D}}^{m^2}$ is proved in [11, Theorem 3], using the Taylor expansion of the gradient of $L(x)$. $\square$

Now we show that the inclusion relations in Lemma 4 can actually be tightened using very simple arguments.

**Lemma 5** *Given the $m$-ellipsoid intersection set $\mathcal{F}$ defined in (2), we have*

$$\mathcal{E}_{\mathrm{D}}^2 \subseteq \mathcal{F} \subseteq \mathcal{E}_{\mathrm{D}}^{2m},$$

*where inner and outer Dikin ellipsoids $\mathcal{E}_{\mathrm{D}}^2$ and $\mathcal{E}_{\mathrm{D}}^{2m}$ are defined in (14).*

**Proof:** Vector $x$ belongs to $\mathcal{E}_{\mathrm{D}}^2$ if and only if $\frac{1}{2}x'P_Dx = \sum_{i=1}^m x'P_ix \leq 1$. This implies that $x'P_ix \leq 1$ for any $i = 1, \ldots, m$, hence $x$ also belongs to $\mathcal{F}$ and the first inclusion is proved. Conversely, suppose that $x$ belongs to $\mathcal{F}$ so that $x'P_ix \leq 1$ holds for $i = 1, \ldots, m$. Upon adding these inequalities, one obtains $\sum_{i=1}^m x'P_ix \leq m$ thus $x'P_Dx \leq 2m$ and the second inclusion. $\square$

**Algorithm 3 – Dikin Ellipsoid**

1. *Compute $v_D^{\min}$, the unitary eigenvector corresponding to $\lambda_D^{\min}$, the minimal eigenvalue of matrix $\sum_{i=1}^m P_i$.*
2. *Let*
$$x_{\mathrm{D}} = \frac{1}{\sqrt{\lambda_D^{\min}}} v_D^{\min}.$$

**Theorem 3** *Algorithm 3 provides the following lower and upper bounds on $p_{\mathrm{opt}}$, the global optimum of problem (1):*

$$x_{\mathrm{D}}'x_{\mathrm{D}} \leq p_{\mathrm{opt}} \leq m x_{\mathrm{D}}'x_{\mathrm{D}}.$$

**Proof:** The theorem follows from simple eigenvalue and eigenvectors considerations in the inner and outer ellipsoids defined in Lemma 5. One can check that the vector $x_{\mathrm{D}}$ built in Algorithm 3 belongs to the boundary of $\mathcal{E}_{\mathrm{D}}^2$. In view of Lemma 5, $x_{\mathrm{D}}'x_{\mathrm{D}} = 1/\lambda_D^{\min} \leq p_{\mathrm{opt}}$. The same reasoning applies to outer ellipsoid $\mathcal{E}_{\mathrm{D}}^{2m}$, where we see that $p_{\mathrm{opt}} \leq m/\lambda_D^{\min}$. $\square$

## 4.2 Löwner-John ellipsoids

John and Löwner showed that the smallest volume ellipsoid that contains a given convex set is unique. This optimal ellipsoid is referred to as the outer Löwner-John ellipsoid and can be used as another means of approximating convex sets [7, §3.7]. As stated in the introduction, the problem of finding the outer Löwner-John ellipsoid of set $\mathcal{F}$ is NP-hard, so we will only seek a suboptimal but still tight approximation of this ellipsoid. The approximation ellipsoid will be denoted by

$$\mathcal{E}_{\mathrm{LJ}}^{\mathrm{out}} = \{x : x' P_0 x \leq 1\}. \tag{16}$$

Since $\mathcal{E}_{\mathrm{LJ}}^{\mathrm{out}}$ is an outer ellipsoidal approximation of $\mathcal{F}$, each vector $x$ such that $x'P_1 x \leq 1, \cdots, x'P_m x \leq 1$ must verify $x'P_0 x \leq 1$. Using the $\mathcal{S}$-procedure introduced in Lemma 1, the condition for $\mathcal{E}_{\mathrm{LJ}}^{\mathrm{out}}$ to be an outer approximation ellipsoid is that there exist positive scalars $\lambda_1, \cdots, \lambda_m$ such that

$$\left[\begin{array}{cc} P_0 & 0 \\ 0 & -1 \end{array}\right] \leq \sum_{i=1}^{m} \lambda_i \left[\begin{array}{cc} P_i & 0 \\ 0 & -1 \end{array}\right].$$

This can equivalently be written as $P_0 \leq \sum_{i=1}^m \lambda_i P_i$, $\sum_{i=1}^m \lambda_i \leq 1$ and $\lambda_i \geq 0, \quad i = 1, \ldots, m$.

As pointed out in [7, §3.7.2], this LMI in variables $P_0$, $\lambda_i$ is sufficient but not necessary for ellipsoid $\mathcal{E}_{\mathrm{LJ}}^{\mathrm{out}}$ to cover $\mathcal{F}$. However, amongst all such outer approximations, we can find the one of smallest volume by solving the following LMI optimization problem

$$\begin{array}{rl} \min_{P_0,\lambda} & \log\det {P_0}^{-1} \\ \text{s.t.} & 0 < P_0 \leq \sum_{i=1}^m \lambda_i P_i \\ & \sum_{i=1}^m \lambda_i \leq 1 \\ & \lambda_i \geq 0, \quad i = 1, \ldots, m. \end{array} \tag{17}$$

The logarithmic criterion to be minimized is proportional to the volume of $\mathcal{E}_{\mathrm{LJ}}^{\mathrm{out}}$, the product of the length of its axes. It can be shown that the above problem is convex [27] and efficient tools have been designed to solve it [28].

Another criterion, the trace of ${P_0}^{-1}$, which is proportional to the sum of the square of the lengths of semi-axes, may also be minimized in problem (17). This is discussed in further detail in Section 4.3. Recent comparisons of the two criteria and their applications to state estimation can be found in [9]. In this reference, problem (17) is also derived through more involved arguments.

Similar developments can be carried out to obtain an inner Löwner-John ellipsoid, that is to say, the greatest volume ellipsoid that is contained in $\mathcal{F}$. In contrast to its outer counterpart, the optimal inner Löwner-John ellipsoid can be computed through a convex program. Let us denote it by

$$\mathcal{E}_{\mathrm{LJ}}^{\mathrm{in}} = \{x : x' W^{-1} x \leq 1\}. \tag{18}$$

Using the $\mathcal{S}$-procedure, Boyd *et al.* [7, §3.7.3] show that matrix $W$ is the optimal value of the LMI optimization problem

$$\begin{array}{rl} \min_{W,\lambda} & \log\det W^{-1} \\ \text{s.t.} & 0 < W < \lambda_i P_i^{-1}, \quad i = 1,\ldots,m \\ & 0 \leq \lambda_i < 1, \quad i = 1,\ldots,m. \end{array} \tag{19}$$

The following lemma and theorem can be derived.

**Lemma 6** *Given the $m$-ellipsoid intersection set $\mathcal{F}$ defined in (2), we have*

$$\mathcal{E}_{\mathrm{LJ}}^{\mathrm{in}} \subseteq \mathcal{F} \subseteq \mathcal{E}_{\mathrm{LJ}}^{\mathrm{out}},$$

*where inner and outer Löwner-John ellipsoids are defined in (16) and (18).*

**Theorem 4** *Let $P_0$ and $W$ be the solutions to LMI optimization problems (17) and (19), respectively. Then the global optimum of problem (1) lies within the bounds*

$$\lambda_W^{\max} \leq p_{\mathrm{opt}} \leq \frac{1}{\lambda_{P_0}^{\min}}$$

*where $\lambda_W^{\max}$ and $\lambda_{P_0}^{\min}$ stand for the maximal eigenvalue of $W$ and minimal eigenvalue of $P_0$, respectively.*

**Proof:** Similar to that of Theorem 3. □

Finally, it has been proved that the optimal inner Löwner-John ellipsoid of a given symmetric convex set, once expanded about its center by a factor $\sqrt{n}$, is guaranteed to contain the given set [7, §3.7]. The following algorithm directly follows from this observation.

**Algorithm 4 – Löwner-John Ellipsoid**

1. *Solve LMI optimization problem (19) for an optimal solution $W$.*
2. *Compute $v_W^{\max}$, the unitary eigenvector corresponding to $\lambda_W^{\max}$, the maximal eigenvalue of $W$.*
3. *Let*

$$x_{\mathrm{LJ}} = \sqrt{\lambda_W^{\max}}\, v_W^{\max}.$$

**Theorem 5** *Algorithm 4 provides the following lower and upper bounds on $p_{\mathrm{opt}}$, the global optimum of problem (1):*

$$x_{\mathrm{LJ}}' x_{\mathrm{LJ}} \leq p_{\mathrm{opt}} \leq n x_{\mathrm{LJ}}' x_{\mathrm{LJ}}.$$

**Proof:** Similar to that of Theorem 3. □

Finally, it must be noticed that computing the Löwner-John ellipsoid in Algorithm 4 is generally more difficult than deriving the Dikin ellipsoid in Algorithm 3. However, one obtains an approximation factor depending only on the dimension of the space and not on the type or the number of constraints.

## 4.3 Derandomized Cholesky Approximation

The randomized Cholesky algorithm described in Section 3.1 can also be interpreted as an LMI optimization problem aimed at maximizing the size of an ellipsoid. To see that, recall from the proof of Lemma 2 that a feasible suboptimal solution for (1) is

$$x = V_{\mathrm{C}}' u,$$

where $u$ is an arbitrary unitary vector and $V_{\mathrm{C}}$ is the Cholesky square root matrix of the solution to LMI relaxation problem (5), i.e. $X_{\mathrm{C}} = V_{\mathrm{C}}' V_{\mathrm{C}}$. When vector $u$ describes the unit sphere, vector $x$ lies on the boundary of an ellipsoid shaped by $X_{\mathrm{C}}$. More precisely, for all $u$ such that $u'u = 1$, we have $x' X_{\mathrm{C}}^{-1} x = 1$ hence

$$x \in \mathcal{E}_{\mathrm{C}} = \{x : x' X_{\mathrm{C}}^{-1} x \leq 1\}.$$

In problem (5), maximization of the trace of $X$ amounts at maximizing the sum of the square of the lengths of the semi-axes of $\mathcal{E}_{\mathrm{C}}$. Since vector $x$ is a suboptimal solution for (1), LMI relaxation (5) provides another maximum size ellipsoid contained in $\mathcal{F}$, still distinct from the Dikin ellipsoid or the inner Löwner-John ellipsoid described above.

This interpretation allows us to derive a derandomized version of Algorithm 1.

**Algorithm 5 – Derandomized Cholesky Approximation**

1. *Solve LMI relaxation (5) for an optimal solution $X_{\mathrm{C}}$.*
2. *Compute $v_{\mathrm{C}}^{\max}$, the unitary eigenvector corresponding to $\lambda_{\mathrm{C}}^{\max}$, the maximal eigenvalue of $X_{\mathrm{C}}$.*
3. *Let*
$$x_{\mathrm{C}} = \sqrt{\lambda_{\mathrm{C}}^{\max}}\, v_{\mathrm{C}}^{\max}.$$

**Theorem 6** *Algorithm 5 provides the following lower and upper bounds on $p_{\mathrm{opt}}$, the global optimum of problem (1):*

$$x_{\mathrm{C}}' x_{\mathrm{C}} \leq p_{\mathrm{opt}} \leq n x_{\mathrm{C}}' x_{\mathrm{C}}.$$

**Proof:** We must equivalently show that

$$p_{\rm opt} \geq x_{\rm C}' x_{\rm C} \geq \frac{1}{n} p_{\rm opt}.$$

The first part of the inequality holds since vector $x_{\rm C}$ belongs to $\mathcal{E}_{\rm C}$, hence to $\mathcal{F}$. As for the second part, note that matrix $X$ being given, the maximal value of $x_{\rm rand}' x_{\rm rand}$ that can be achieved in Algorithm 1 is ${x_{\rm C}}' x_{\rm C}$ by construction. Combined with Lemma 3, this yields ${x_{\rm C}}' x_{\rm C} \geq \mathbf{E}[x_{\rm rand}' x_{\rm rand}] = \frac{1}{n} p_{\rm C} \geq \frac{1}{n} p_{\rm opt}$. □

### 4.4 Derandomized Nemirovski Approximation

Nemirovski, Roos and Terlaky also proposed a derandomization technique for their Algorithm 2. Their approach, not reproduced here, relies upon standard derandomization techniques and is rather notationally involved. The interested reader may refer to [22].

## 5 Illustration

The approximation algorithms are now applied on a very simple and illustrative example. Throughout this section, LMI relaxations were solved using the software described in [28].

We consider a two-dimensional instance of problem (1) with $m = 3$ intersecting ellipsoids (3) shaped by positive definite matrices

$$P_1 = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 3 & -2 \\ -2 & 2 \end{bmatrix}, \quad P_3 = \begin{bmatrix} 1 & -2 \\ -2 & 6 \end{bmatrix}.$$

Ellipsoid boundaries and the boundary of their intersection $\mathcal{F}$ defined in (2) are drawn on Figure 1. We have also represented the sphere of smallest radius inscribing $\mathcal{F}$, i.e. the sphere of radius $\sqrt{p_{\rm opt}}$. This sphere intersects with $\mathcal{F}$ at two extreme points $\pm x_{\rm opt}$. Graphically, we obtained approximately

$$p_{\rm opt} \approx 0.427, \quad x_{\rm opt} \approx [0.645 \quad 0.105]'.$$

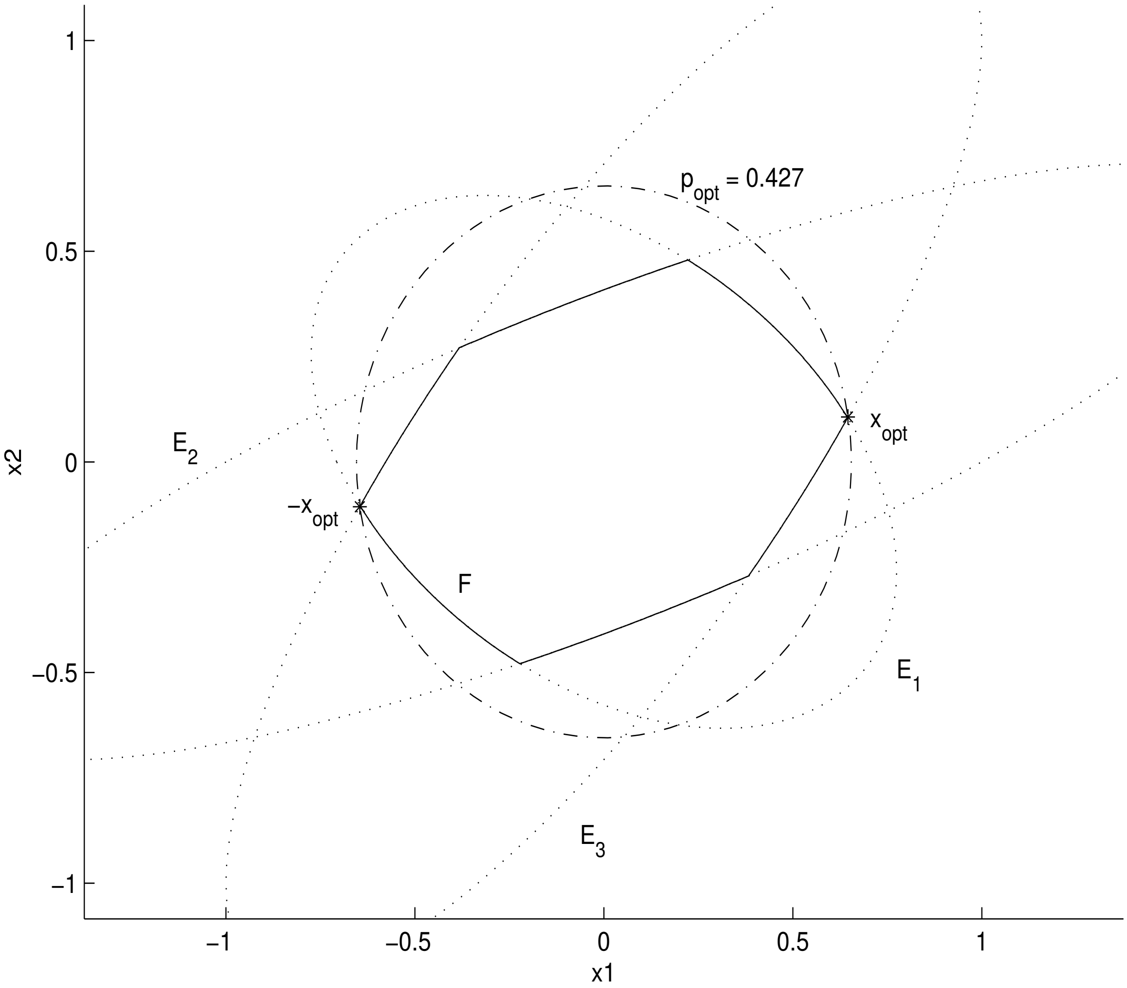


Figure 1: Boundary of $\mathcal{F}$, the intersection of ellipsoids $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$. The norm of optimal vector $x_{\text{opt}}$ is the radius of $\mathcal{F}$.

- **Dikin Ellipsoid (Algorithm 3)** First, we got

$$\sum_{i=1}^{m} P_i = \begin{bmatrix} 6 & -3 \\ -3 & 11 \end{bmatrix}$$

and the corresponding vector

$$x_{\text{D}} = \frac{1}{\sqrt{4.60}}\,[0.906 \quad 0.424]' = [0.423 \quad 0.198]'.$$

The inner ellipsoid $\mathcal{E}_{\text{D}}^{2}$ and outer ellipsoid $\mathcal{E}_{\text{D}}^{2m}$ are represented in Figure 2 in dotted line and dashed line, respectively.

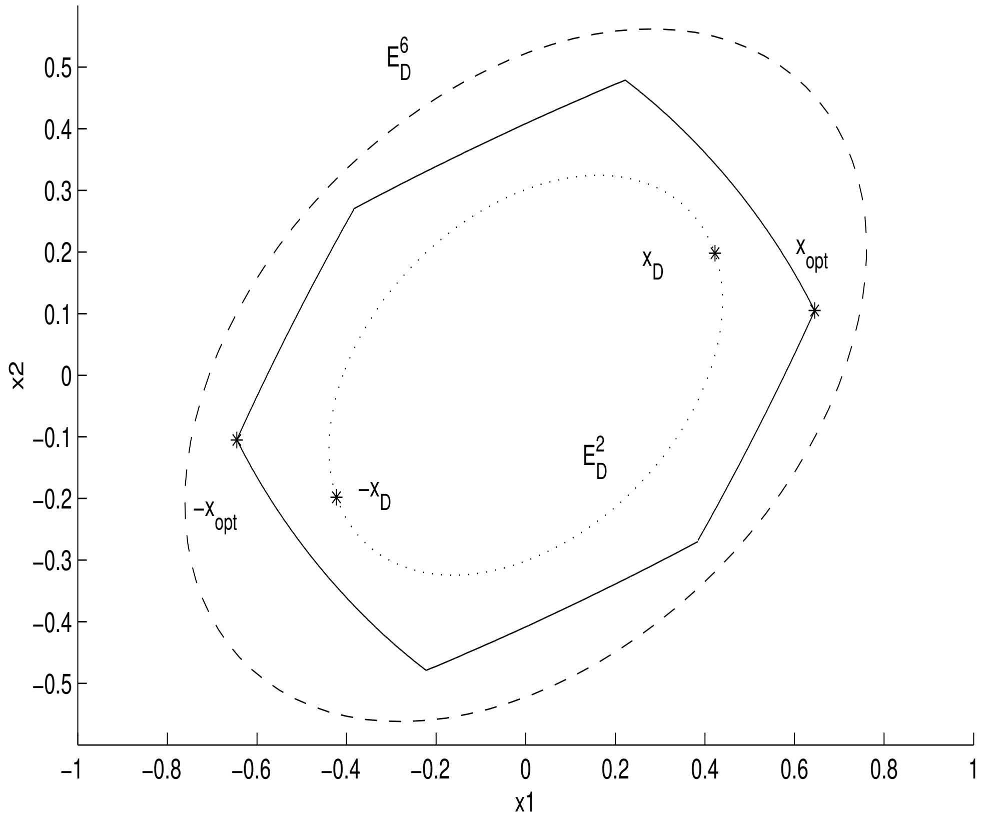


Figure 2: Dikin Ellipsoids $\mathcal{E}_{\rm D}^{2}$ and $\mathcal{E}_{\rm D}^{2m}$ with suboptimal vector $x_{\rm D}$.

- **Löwner-John Ellipsoid (Algorithm 4)** Next we computed the outer Löwner-John ellipsoid via LMI optimization problem (17). We obtained

$$P_0 = \begin{bmatrix} 1.86 & -0.310 \\ -0.310 & 3.72 \end{bmatrix}.$$

The boundary of the corresponding ellipsoid is represented in Figure 3 in dotted line. We can see that $\mathcal{E}_{\rm out}^{\rm LJ}$ is not optimal since it is still possible to find an ellipsoid of smallest volume that contains $\mathcal{F}$. Next, we computed the inner Löwner-John ellipsoid through Algorithm 4 and LMI problem (19). We obtained

$$W = \begin{bmatrix} 0.329 & 0.0530 \\ 0.0530 & 0.164 \end{bmatrix}.$$

and the corresponding vector

$$x_{\rm LJ} = \sqrt{0.344}\,[0.959 \quad 0.283]' = [0.563 \quad 0.166]'.$$

Inner ellipsoid $\mathcal{E}_{\rm LJ}^{\rm in}$ is represented in Figure 3 in dashed line. The homothetic ellipsoid $\sqrt{n}\mathcal{E}_{\rm LJ}^{\rm in}$, which is guaranteed to contain set $\mathcal{F}$, is drawn in dash-dotted line.

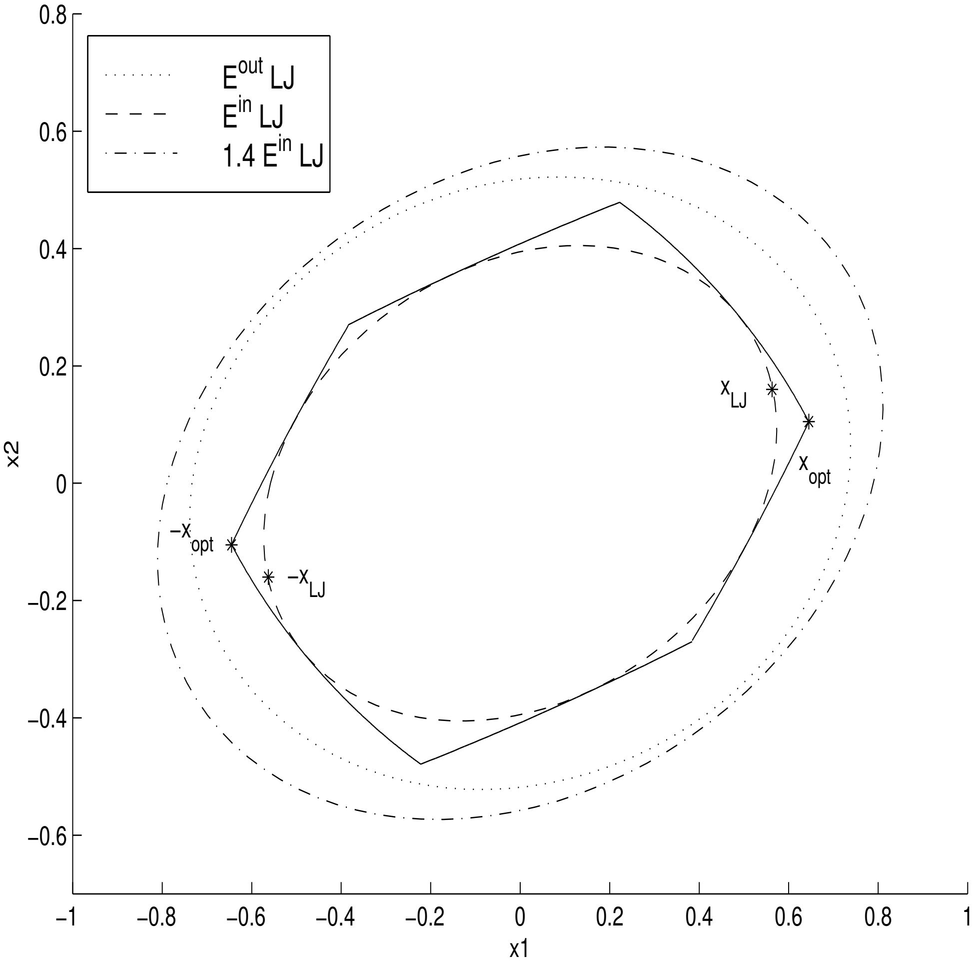


Figure 3: Löwner-John ellipsoids $\mathcal{E}_{\mathrm{LJ}}^{\mathrm{out}}$, $\mathcal{E}_{\mathrm{LJ}}^{\mathrm{in}}$ and $\sqrt{n}\mathcal{E}_{\mathrm{LJ}}^{\mathrm{out}}$ with suboptimal vector $x_{\mathrm{LJ}}$.

- **Cholesky Approximation (Algorithm 5)** LMI relaxation (5) yields

$$X_{\mathrm{C}} = \begin{bmatrix} 0.416 & 0.0675 \\ 0.0675 & 0.0110 \end{bmatrix}.$$

It can be checked that matrix $X_{\mathrm{C}}$ is almost rank-one – its singular values are $1.09 \cdot 10^{-7}$ and 0.427. Moreover, $\mathbf{Tr}X_{\mathrm{C}} = 0.427 \approx p_{\mathrm{opt}}$. According to Algorithm 5, we built vector

$$x_{\mathrm{C}} = \sqrt{0.427}\,[0.987 \quad 0.160]' = [0.645 \quad 0.105]' \approx x_{\mathrm{opt}}.$$

The ellipsoid $\mathcal{E}_{\mathrm{C}}$ shaped by $X_{\mathrm{C}}^{-1}$, is almost degenerate and points towards the two greatest norm points of $\mathcal{F}$. Note that here the global optimal is almost attained, but it is not necessarily the case in general.

- **Nemirovski Approximation (Algorithm 2)** In this case, LMI relaxation (5) already produced an almost rank-one solution, hence

$$x_{\mathrm{NRT}} = [0.645 \quad 0.105]' \approx x_{\mathrm{opt}}.$$

- **Bounds on the global optimum:** Lower and upper bounds on the global optimum $p_{\mathrm{opt}} \approx 0.427$ obtained through the different relaxation techniques described in this paper are reported in Table 1.

| Method | Approx. Gap | Lower bound | Upper bound |
|---|---|---|---|
| Dikin (Theorem 3) | $m$ | 0.218 | 0.653 |
| Löwner-John (Theorem 4) | $n$ | 0.344 | 0.553 |
| Löwner-John (Theorem 5) | $n$ | 0.344 | 0.688 |
| Cholesky (Theorem 6) | $n$ | 0.427 | 0.851 |
| Nemirovski (Theorem 2) | $2\log(n\min\{m,n\})$ | 0.154 | 0.427 |

Table 1: Comparative lower and upper bounds on the global optimum $p_{\mathrm{opt}} \approx 0.427$.

Generally speaking, the tightest approximation gap is clearly provided by Nemirovski's algorithm, since the gap is logarithmic in $m$ and $n$. As far as other approximation methods are concerned, we can readily state that, when $m$ is greater than $n$, $\frac{1}{n}$-approximation algorithms are likely to perform better than the $\frac{1}{m}$-approximation algorithm. However, no definitive conclusion can be drawn from this simple two-dimensional example. The example was only aimed at illustrating the different concepts introduced in this paper.

# 6 Conclusion

This work is the continuation of the research efforts initiated in [16, 17]. In this reference, we pursued a polynomial approach to derive necessary and sufficient conditions for robust stabilization of a linear system subject to real parametric uncertainty, and we faced the problem of checking whether the radius of the intersection of given ellipsoids is strictly greater than one. In other words, we want to know if $p_{\mathrm{opt}} > 1$ in problem (1). In [25], it was noticed that this ellipsoid intersection problem is actually at the core of robust control synthesis problems since it provides solvability conditions for general BMIs in an elegant and unified way. Because this problem was shown to be NP-hard in its most general form, in this paper we focused on polynomial-time techniques to solve it approximately. Lower and upper bounds on the optimal solution were given that readily lead to necessary or sufficient robust stabilizability conditions in [16, 17]. Alternatively, the bounds may be used in a branch-and-bound optimization algorithm if one is willing to solve problem (1) for its global optimum. These bounds were derived through the resolution of LMIs, for which efficient polynomial-time SDP algorithms are now widely available.

We believe that the problem of evaluating the radius of the intersection of ellipsoids is worth studying in further detail because of its relevance in robust control synthesis. Potential directions of future research are mentioned next.

- The approximation bounds given in this paper are only theoretical. We aim at carrying out intensive numerical computations to assess the practical performance of respective relaxation algorithms.
- A specialized global optimization algorithm must be devised for solving problem (1) up to a given accuracy. Existing global optimization algorithms [18] do not sufficiently take into account the very strong properties of the problem at hand: convexity of the criterion to be maximized, convexity of the feasible set.

- New relaxations have just been proposed for general quadratic problems involving linear constraints [24]. These relaxations are convex and stronger that LMI relaxations, although still NP-hard. It might be interesting to know whether such stronger relaxations could be derived for problem (1).

- In [2], the authors considered the problem of maximizing a quadratic form $x'Qx$ over the intersection of finitely many ellipsoids when matrix $Q$ has rank one. Using a double minimization formulation and non-linear transformations, they showed that this apparently tough problem is actually convex, and can be solved in polynomial time. Note that it is not possible to apply these results here since matrix $Q$ in problem (1) is a full-rank identity matrix. One might however get inspired by these ideas and try to exploit structural properties of problem (1).

- Further structural properties may be derived from the sufficient Karush-Kuhn-Tucker conditions for problem (1), provided in [10].

- Finally, relations existing between the ellipsoid intersection problem and the general problem of studying the Hurwitz stability of a polynomial were unveiled in [16, 17] through the use of Hermite-Fujiwara matrices. Thus, problem (1) is closely related to the study of the domain of stability of a polynomial in the Euclidean space of its coefficients. What are the direct implications of the bounds derived in the present paper for the convex approximation of the Hurwitz stability set remains an open and challenging problem.

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# References

[1] V. Balakrishnan and S. Boyd, "Global Optimization in Control System Analysis and Synthesis", in C. T. Leondes (Editor), "Control and Dynamic Systems", Volume 53, *Academic Press*, 1992.

[2] A. Ben-Tal and M. Teboulle, "Hidden Convexity in some Nonconvex Quadratically Constrained Quadratic Programming", *Mathematical Programming*, Vol. 72, pp. 51–63, 1996.

[3] A. Ben-Tal and A. Nemirovski, "Robust Convex Optimization", *Mathematics of Operations Research*, Vol. 23, No. 4, pp. 769–805, 1998.

[4] A. Ben-Tal, L. El Ghaoui and A. Nemirovski, "Robust Semidefinite Programming", to appear in R. Saigal, L. Vandenberghe and H. Wolkowicz (Editors), "Handbook of Semidefinite Programming", *Kluwer Academic Publishers*, Waterloo, Canada, 2000.

[5] V. Blondel and J. Tsitsiklis, "NP-hardness of Some Linear Control Design Problems", *SIAM Journal on Control and Optimization*, Vol. 35, No. 6, pp. 2118–2127, 1997.

[6] S. Boyd and L. El Ghaoui, "Method of Centers for Minimizing Generalized Eigenvalues", *Linear Algebra and its Applications*, Special Issue on Systems and Control, Vol. 188, p. 63–111, 1993.

[7] S. Boyd, L. El Ghaoui, E. Féron and V. Balakrishnan, "Linear Matrix Inequalities in System and Control Theory", *SIAM Studies in Applied Mathematics*, Vol. 15, Philadelphia, Pennsylvania, 1994.

[8] S. Boyd and L. Vandenberghe, "Semidefinite Programming Relaxations of Nonconvex Problems in Control and Combinatorial Optimization", in A. Paulraj, V. Roychowdhury and C. Schaper (Editors), "Communications, Computation, Control and Signal Processing – a Tribute to Thomas Kailath", *Kluwer Academic Publishers*, 1997.

[9] C. Durieu, B. T. Polyak and E. Walter, "Trace Versus Determinant in Ellipsoidal Outer-Bounding, with Application to State Estimation", *Proceedings of the IFAC World Congress*, San Francisco, California, pp. 43–48, 1996.

[10] O. E. Flippo and B. Jansen, "Duality and Sensitivity in Quadratic Optimization over a Sphere", *Technical Report* No. 92-65, Faculty of Technical Mathematics and Informatics, Delft University of Technology, The Netherlands, 1992.

[11] M. Fu, Z. Q. Luo and Y. Ye, "Approximation Algorithms for Quadratic Programming", *Journal of Combinatorial Optimization*, Vol. 2, No. 1, pp. 29–50, 1998.

[12] P. A. Fuhrmann and B. N. Datta, "On Bézoutians, Vandermonde matrices and the Liénard-Chipart stability criterion", *Linear Algebra and its Applications*, Vol. 120, pp. 23–37, 1989.

[13] P. Gahinet, A. Nemirovski, A. J. Laub and M. Chilali, "Matlab LMI Control Toolbox", *The MathWorks Inc.*, 1995.

[14] M. X. Goemans and D. P. Williamson, "Improved Approximation Algorithms for Maximum Cut and Satisfiability Problems Using Semidefinite Programming", *Journal of the ACM*, Vol. 42, No. 6, pp. 1115–1145, 1995.

[15] K. C. Goh, M. G. Safonov and J. H. Ly, "Robust Synthesis via Bilinear Matrix Inequalities", *International Journal of Robust and Nonlinear Control*, Vol. 6, No. 9/10, pp. 1079–1095, 1996.

[16] D. Henrion, S. Tarbouriech and M. Šebek, "Algebraic Approach to Robust Controller Design: A Geometric Interpretation", *Proceedings of the American Control Conference*, pp. 2703–2707, Philadelphia, Pennsylvania, 1998.

[17] D. Henrion, S. Tarbouriech and M. Šebek, "Rank-one LMI Approach to Simultaneous Stabilization of Linear Systems", *Systems and Control Letters*, Vol. 38, No. 2, pp. 79–89, 1999.

[18] R. Horst and H. Tuy, "Global Optimization: Deterministic Approaches", Third Revised and Enlarged Edition, *Springer Verlag*, Berlin, Germany, 1996.

[19] V. Kučera, "Diophantine Equations in Control - A Survey", *Automatica*, Vol. 29, No. 6, pp. 1361–1375, 1993.

[20] D. G. Maksarov and J. P. Norton, "State Bounding with Ellipsoidal Set Description of the Uncertainty", *International Journal of Control*, Vol. 65, No. 5, pp. 847–866, 1996.

[21] Yu. Nesterov and A. Nemirovski, "Interior-Point Polynomial Methods in Convex Programming", *SIAM Studies in Applied Mathematics*, Vol. 13, Philadelphia, Pennsylvania, 1994.

[22] A. Nemirovski, C. Roos and T. Terlaky, "On Maximization of Quadratic Form over Intersection of Ellipsoids with Common Center", *Mathematical Programming*, Vol. 86, No. 3, pp. 463–473, 1999.

[23] Yu. Nesterov, "Semidefinite Relaxation and Nonconvex Quadratic Optimization", *Optimization Methods and Software*, Vol. 9, pp. 141–160, 1998.

[24] A. J. Quist, E. de Klerk, C. Roos and T. Terlaky, "Copositive Relaxation for General Quadratic Programming", *Optimization Methods and Software*, Vol. 9, pp. 185–209, 1998.

[25] M. G. Safonov and G. P. Papavassilopoulos, "The Diameter of an Intersection of Ellipsoids and BMI Robust Synthesis", *Proceedings of the IFAC Symposium on Robust Control Design*, Rio de Janeiro, Brazil, pp. 313–317, 1994.

[26] L. Vandenberghe and S. Boyd, "Semidefinite Programming", *SIAM Review*, Vol. 38, pp. 49–95, 1996.

[27] L. Vandenberghe, S. Boyd and S. P. Wu, "Determinant Maximization with Linear Matrix Inequality Constraint", *SIAM Journal on Matrix Analysis and Applications*, Vol. 19, No. 2, pp. 499–533, 1998.

[28] S. P. Wu and S. Boyd, "A Parser/Solver for Semidefinite Programming and Determinant Maximization Problems with Matrix Structure", *Technical Report*, Information Systems Laboratory, Stanford University, California, 1996.

[29] V. A. Yakubovich, "The $\mathcal{S}$-procedure in Nonlinear Control Theory", *Vestnik Leningrad University of Mathematics*, Vol. 4, pp. 73–93, 1977. In Russian, 1971.

[30] Y. Ye, "Approximating Quadratic Programming with Bound and Quadratic Constraints", *Mathematical Programming*, Vol. 84, pp. 219–226, 1999.