ROBUST STATE FEEDBACK $\mathcal{D}$ STABILIZATION VIA A CONE COMPLEMENTARY ALGORITHM

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Abstract

The problem of stabilization of a polytope of matrices in a subregion $\mathcal{D}_R$ of the complex plane is revisited. A new sufficient condition of robust $\mathcal{D}_R$ stabilization is given. It implies the solution of an $\mathcal{LMI}$ involving matrix variables constrained by a nonlinear algebraic relation. A cone complementarity formulation of this condition allows to associate an efficient iterative numerical procedure which leads to a low computational burden. This algorithm is tested on different numerical examples for which existing approaches in control literature fail.

1 Introduction

In the synthesis of feedback control systems, it is necessary to guarantee that the stability and some performance properties of the closed-loop system are robust with respect to plant perturbations. If a state-space approach is considered for modeling, plant uncertain parameters may be viewed as perturbations affecting the coefficients of system matrices and defining therefore families of system. This paper focuses on families of linear systems in state-space form where the domain of admissible system matrices is a real convex polytope. The problem of finding stability conditions for a polytope of matrices has received a considerable attention in the literature, [5],[6], [9] and some attempts have been made to solve the synthesis problem [18], [16], [3], [21], [22], [23]. In both cases the problem is known to be $\mathcal{NP}$-hard, [11] and therefore a critical tradeoff has to be faced: find testable precise conditions while keeping a weak computational complexity. Robust stability problems have been attacked via methods which rely heavily on the convexity assumption, (the results based on quadratic stability concept) or on more complex approaches for which branching operations may be required. In the first case, it is well known that we get too much conservative results while for the second case, computational complexity is a major difficulty.
The situation is more awkward in synthesis problems since a constructive method is needed to get a robust controller. The quadratic stability framework, [16], [3], has proven to be a successfull design methodology but still suffers from its conservatism when dealing with structured uncertainty. Recently, a new robust stability analysis condition simultaneously appeared in [20] and [22]. This $\mathcal{LMI}$-based condition involves parameter-dependent Lyapunov functions and extra matrix variables leading to a drastic reduction of the conservatism, (see [22] for comparison results). This analysis result has been used in [19] and [21] to tackle the problems of robust state-feedback synthesis and multiobjective synthesis for discrete-time systems. Unfortunately, such an extension is surprisingly impossible for continuous-time systems. In fact, a linearizing change of variables is proposed in [22] to get sufficient $\mathcal{LMI}$-based conditions of stabilization of a polytope of matrices in particular subregions $\mathcal{D}_R$ of the complex plane. This change of variables is no more valid for $\mathcal{D}_R$ regions such as half-planes, sectors which do not verify some basic technical assumption. For such regions, a Bilinear Matrix Inequality, $\mathcal{BMI}$, formulation may be deduced.
In this paper, we concentrate on this $\mathcal{BMI}$ formulation. Our objective, for the most part, is to propose an iterative algorithm based on the cone complementarity approach, [12], and to show that it may be efficiently used in almost all cases. A sufficient condition of robust $\mathcal{D}_R$ stabilization of a polytope of matrices is characterized by an $\mathcal{LMI}$ involving matrix variables subject to an additional non linear algebraic constraint. A cone complementarity formulation of the problem and its related numerical procedure are then proposed.
The contribution of this paper is threefold. First, it extends in a very natural way the robust analysis conditions of [22] leading to an effective synthesis procedure for any $\mathcal{LMI}$ regions. Second, it generalizes and improves the results of [19] for discrete-time systems. Although the approach is not purely $\mathcal{LMI}$-based, the computational complexity remains reasonable since each iteration consists in an $\mathcal{LMI}$ optimization and that, in general, the algorithm requires few iterations, (less than 10). Finally, the proposed method allows to deal with intersection of $\mathcal{LMI}$ regions using different Lyapunov functions for each elementary region unlike the method proposed in [8].
Notation is standard. The transpose of a matrix A is denoted A' and A^* reads for complex conjugate transpose. $\mathbf{1}$ stands for the identity matrix and $\mathbf{0}$ for the zero matrix with the appropriate dimensions. $\mathcal{S}_n$ denotes the set of symmetric matrices of $\mathbb{R}^{n\times n}$ and $\mathcal{S}_n^+$, $(\mathcal{S}_n^{+*})$, the cone of positive semi-definite, (definite) matrices in $\mathcal{S}_n$. $\mathbb{C}$ is the set of complex numbers. $\mathbb{R}^+$, $(\mathbb{R}^{+*})$ is the set of positive, (strictly positive), real numbers. $\otimes$ is the Kronecker product of two matrices. We remind that $(A\otimes B)(C\otimes D) = (AC)\otimes(BD)$. The symetric part of a square matrix A is denoted $sym[A]$, i.e.

$sym[A] = A + A'$. δ is the derivation operator for continuous-time systems, ($\delta[x(t)] = \dot{x}(t)$) and the delay operator for discrete-time ones, ($\delta[x(t)] = x(t+1)$).

2 Preliminaries

2.1 Background

Let us consider the linear uncertain dynamical system,

$$\begin{array}{l} \delta[x(t)] = Ax(t) + Bu(t) = M \begin{bmatrix} x(t) \\ u(t) \end{bmatrix} \\ M = \begin{bmatrix} A & B \end{bmatrix} \in \mathcal{M} \end{array} \quad (1)$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control vector. The dynamical matrix A and the input matrix B are in the convex polytope $\mathcal{M}$ defined by,

$$\mathcal{M} = \left\{ M = \sum_{i=1}^{N} \xi_i M^{[i]} \ : \ \xi_i \geq 0 \ , \ \sum_{i=1}^{N} \xi_i = 1 \right\} \quad (2)$$

Let

$$\mathcal{D}_R = \left\{ z \in \mathbb{C} \ : \ f_{\mathcal{D}_R}(z) = R_{11} + R_{12} z + R'_{12} z^* + R_{22} z z^* < 0 \right\} \quad (3)$$

be a given region of the complex plane, where $R \in \mathbb{R}^{2d \times 2d}$ is a symmetric matrix partitionned as:

$$R = \begin{bmatrix} R_{11} & R_{12} \\ R'_{12} & R_{22} \end{bmatrix} \ : \ \begin{array}{l} R_{11} = R'_{11} \in \mathbb{R}^{d \times d} \\ R_{22} = R'_{22} \in \mathbb{R}^{d \times d} \end{array} \quad (4)$$

Following the terminology of [7], the matrix-valued function $f_{\mathcal{D}_R}(z)$ is called the characteristic function of $\mathcal{D}_R$. In the sequel, such regions will be referred to as $\mathcal{EMI}$ regions. Without any assumption on the matrix R_{22}, $\mathcal{D}_R$ regions are not convex, but with the assumption of positive definiteness, $R_{22} \geq \mathbf{0}$, (3) appears to be a slight modification of the characterisation of $\mathcal{LMI}$ regions, [22].

• **Remarks 1** As for $\mathcal{LMI}$ regions the $\mathcal{D}_R$ regions are symmetric with respect to the real axis and an intersection of $\mathcal{D}_R$ regions is a $\mathcal{D}_R$ region.

Note that the class of $\mathcal{LMI}$ regions belongs to the class of $\mathcal{D}_R$ regions but our investigations will be restricted to the former, i.e. we assume that $R_{22} \geq \mathbf{0}$.
A straightforward extension of the usual stability property of dynamical systems may be easily extrapolated.

Definition 1
A matrix $A \in \mathbb{R}^{n \times n}$ is said to be $\mathcal{D}_R$-stable if and only if all its eigenvalues lie in the $\mathcal{D}_R$ region defined by (3).

Usually, two different robust stability concepts may been defined to study the robustness of pole clustering of a convex matrix polytope $\mathcal{A}$ defined as (2), in $\mathcal{D}_R$ regions.

Definition 2

- *$\mathcal{A}$ is robustly $\mathcal{D}_R$-stable, if and only if, for all $A \in \mathcal{A}$, A is $\mathcal{D}_R$-stable.*
- *$\mathcal{A}$ is quadraticaly $\mathcal{D}_R$-stable, if and only if, there exists a matrix $P \in \mathcal{S}_n^{+*}$ such that for all $A \in \mathcal{A}$:*

$$R_{11} \otimes P + R_{12} \otimes (PA) + R'_{12} \otimes (A'P) + R_{22} \otimes (A'PA) < \mathbf{0} \quad (5)$$

These two notions are not in general equivalent except in special cases, e.g., complex or real unstructured uncertainty for the open left half-plane. Quadratic $\mathcal{D}_R$ stability is well-known to be more conservative though it proves useful for analysis and synthesis purpose.

2.2 The robust pole clustering analysis problem

The problem addressed in this paper is to find a robust state-feedback matrix $K \in \mathbb{R}^{m \times n}$ such that every closed-loop matrix $A_{cl} = A + BK$ belonging to the convex polytope $\mathcal{A}$ defined as,

$$\mathcal{A} = \left\{ A_{cl} = \sum_{i=1}^{N} \alpha_i (A^{[i]} + B^{[i]} K) \ : \ \alpha_i \geq \mathbf{0}, \ \sum_{i=1}^{N} \alpha_i = 1 \right\} \quad (6)$$

has all its eigenvalues in the $\mathcal{D}_R$ region. Suppose that the matrices A and B are fixed and given matrices, then the location of the eigenvalues of $A_{cl} \in \mathbb{R}^{n \times n}$ is captured by the following result,

Theorem 1 *[22]*
$A_{cl} \in \mathbb{R}^{n \times n}$ is $\mathcal{D}_R$-stable if and only if there exists a matrix $P \in \mathcal{S}_n^{+*}$ such that:

$$R_{11} \otimes P + R_{12} \otimes (PA_{cl}) + R'_{12} \otimes (A'_{cl} P) + R_{22} \otimes (A'_{cl} P A_{cl}) < \mathbf{0} \quad (7)$$

Thus $\mathcal{A}$ is robustly $\mathcal{D}_R$-stable if and only if, for each $A \in \mathcal{A}$, there exists a symmetric positive definite matrix P such that (7) holds. It is well-known that deciding whether or not every member of the polytope maintains eigenvalue locations in the specified $\mathcal{D}_R$ region is equivalent to solve an $\mathcal{NP}$-hard problem, [11]. The related problem of robust stabilization via state-feedback of the convex polytope $\mathcal{M}$ is therefore equivalent to an $\mathcal{NP}$-hard problem. Most of the approaches dealing with the synthesis problem are based on the quadratic stability notion which leads to inherently conservative stability tests.

Theorem 2 *[22]*
If, there exists two matrices $H_1 \in \mathbb{R}^{dn \times dn}$, $H_2 \in \mathbb{R}^{dn \times n}$ and N matrices $P_i \in \mathcal{S}_n^{+*}$ such that, $\forall\, i = 1, \cdots, N$:

$$R \otimes P^{[i]} + \begin{bmatrix} sym\left[(\mathbf{1}_d \otimes A_{cl}^{[i]}) H_1 \right] & (\mathbf{1}_d \otimes A_{cl}^{[i]}) H_2 - H'_1 \\ \star & -H_2 - H'_2 \end{bmatrix} < \mathbf{0} \quad (8)$$

then $\mathcal{A}$ is robustly $\mathcal{D}_R$-stable.

• **Remarks 2** *:*

If the $\mathcal{D}_R$ region of pole clustering consists in the intersection of L elementary $\mathcal{D}_{R_i}$ regions, independent parameter-dependent Lyapunov functions involving L sets of N positive definite matrices P_i may be used. The interest of this new condition is that this feature may be not only used for analysis purpose but also when synthesizing a controller as will be seen in the next section. This is a major difference with the approach proposed in [8].

The closed-loop matrix $A_{cl} = A + BK$ is affine in the controller parameter K. For the synthesis problem where we are looking for the gain matrix K, the inequality (8) is therefore a bilinear matrix inequality with respect to the unknown matrices due to the products between H_1, H_2 and the gain matrix K. The next section proposes a sufficient condition of robust $\mathcal{D}_R$ stabilization implying the solution of an $\mathcal{LMI}$ involving matrix variables constrained by a nonlinear algebraic condition.

3 State feedback $\mathcal{D}_R$ stabilization

3.1 Pseudo $\mathcal{LMI}$-based formulation

In [19], for stability of discrete-time systems, it is proposed to choose $H_1 = \mathbf{0}$ and to generalize a well-known linearizing change of variables, [16], by letting $S = KH_2$. An $\mathcal{LMI}$ formulation of the robust state-feedback stabilization problem is then possible. This change of variables is generalized in [22] where it is shown that it is possible to perform such a linearization only for $\mathcal{D}_R$ regions satisfying the additional assumption $R_{22} > 0$. This assumption is not satisfied by important $\mathcal{D}_R$ regions such as half-planes, conic sectors, or their intersections. The next result recasts the original bilinear problem as an $\mathcal{LMI}$ feasibility problem subject to a nonlinear algebraic constraint.

Theorem 3 *:*
If there exist N matrices $P^{[i]} \in \mathcal{S}_n^{+*}$ and four matrices $H_1 \in \mathbb{R}^{dn\times dn}$, $G_1 \in \mathbb{R}^{dm\times dn}$, $h_2 \in \mathbb{R}^{n\times n}$, $g_2 \in \mathbb{R}^{m\times n}$ solutions of the following linear inequalities, $\forall\ i = 1, \cdots, N$

$$\begin{bmatrix} R\otimes P^{[i]} + sym\left[(\mathbf{1}_d \otimes A^{[i]})H_1 + (\mathbf{1}_d \otimes B^{[i]})G_1\right] & -H_1' + \mathbf{1}_d \otimes \left(A^{[i]}h_2 + B^{[i]}g_2\right) \\ \star & -\mathbf{1}_d \otimes (h_2 + h_2') \end{bmatrix} < \mathbf{0} \tag{9}$$

under the nonlinear equation,

$$G_1 - (\mathbf{1}_d \otimes (g_2 h_2^{-1}))H_1 = \mathbf{0} \tag{10}$$

then the polytope $\mathcal{M}$ is $\mathcal{D}_R$ stabilizable and a robust state-feedback matrix is given by:

$$K_{\mathcal{D}_R} = g_2 h_2^{-1} \tag{11}$$

Let us note inequality (9),

$$\mathcal{L}^{[i]}(P^{[i]}, G_1, H_1, g_2, h_2) < \mathbf{0} \tag{12}$$

The unknown matrix H_2 must have a bloc structure given by the constraint $H_2 = \mathbf{1}_d \otimes h_2$ which induces extra conservatism in the condition for regions which have an order greater than one, i.e. $d > 1$. Similarly, a conservative change of variables is considered in [19] and [22], $H_1 = R_{12} \otimes H$, $H_2 = R_{22} \otimes H$ and $S = KH$. This leads to the following sufficient condition of robust $\mathcal{D}_R$ stabilization for regions such that $R_{22} > \mathbf{0}$:

Theorem 4 *[22]*
If there exist two matrices $H \in \mathbb{R}^{n\times n}$, $S \in \mathbb{R}^{q\times n}$ and N matrices $P^{[i]} \in \mathcal{S}_n^{+*}$ such that $\forall i \in \{i, \cdots, N\}$:

$$\begin{bmatrix} R\otimes P^{[i]} + sym\left[R_{12} \otimes (A^{[i]}H + B^{[i]}S)\right] & -R_{12}' \otimes H' + R_{22} \otimes (A^{[i]}H + B^{[i]}S) \\ \star & -R_{22} \otimes (H + H') \end{bmatrix} < \mathbf{0} \tag{13}$$

then the polytope $\mathcal{M}$ is robustly $\mathcal{D}_R$-stabilisable by state feedback and an admissible gain is:

$$K = SH^{-1} \tag{14}$$

A necessary and sufficient condition of quadratic $\mathcal{D}_R$ stabilizability is easily recovered from (13) by choosing $H = P$ and $P^{[i]} = P,\ \forall\ i = 1, \cdots, n$. In this regard, the relationships between these different conditions are investigated in the following corollary.

Corollary 1 *:*

1- *If the order of the $\mathcal{D}_R$ region is equal to 1 then the quadratic $\mathcal{D}_R$ stability condition is a sufficient condition for the condition of theorem 3.*

2- *If $R_{22} > \mathbf{0}$ then the quadratic $\mathcal{D}_R$ stability condition is a sufficient condition for the condition of theorem 4.*

3- *If $R_{22} > \mathbf{0}$ and the order of the $\mathcal{D}_R$ region is equal to 1 then the quadratic $\mathcal{D}_R$ stability condition is a sufficient condition for the condition of theorem 4 which implies the condition of theorem 3.*

• **Remarks 3** *Note that in the previous corollary little is said about regions of order greater than 1. In fact, the main regions of concern when imposing explicit constraints on the closed-loop dynamics are the disk, the half-plane, the conic sector which is considered as the intersection of two half-planes and their intersection which are or may be converted into $\mathcal{D}_R$ regions of order 1. In the sequel, the case of intersection of elementary $\mathcal{D}_R$ regions is carefully studied. $\mathcal{D}_R$ regions of order 2 are, for instance, the ellipse and the hyperbolic sector for which the condition of theorem 4 cannot be applied, ($R_{22} \geq \mathbf{0}$).*

3.2 Intersections of $\mathcal{EMI}$ regions

One of the main feature of the new proposed condition is that it is possible to deal with the intersection of $\mathcal{D}_R$ regions by considering a parameter-dependent Lyapunov function for each elementary $\mathcal{D}_R$ regions. This is in stark contrast with the approach proposed in [8]. In case of an intersection of regions of identical orders, this may be done by choosing single extra variables $G_1,\ g_2,\ H_1,\ h_2$. When the regions defining the intersection are of different orders, it is necessary to give a structure to the extra variables defined for each j region H_{1j}, G_{1j} as is shown in the following theorem.

Theorem 5 *:*

Let us define the region $\mathcal{D}_R$ of the complex plane as the intersection of L elementary regions $\mathcal{D}_{R_j}$ of respective order d_j and characterized by the symmetric matrix $R_j \in \mathbb{R}^{d_j \times d_j}$.

$$\mathcal{D}_R = \bigcap_{j=1}^{L} \mathcal{D}_{Rj} \tag{15}$$

1- *Suppose that $d_j = d,\ \forall\ j = 1, \cdots, L$. If there exist $L \times N$ matrices $P_j^{[i]} \in \mathcal{S}_n^{+*}$ and matrices $H_1 \in \mathbb{R}^{dn \times dn}$, $G_1 \in \mathbb{R}^{dm \times dn}$, $h_2 \in \mathbb{R}^{n \times n}$, $g_2 \in \mathbb{R}^{m \times n}$ solutions of the following linear inequalities, $\forall\ i = 1, \cdots, N$ and $\forall\ j = 1, \cdots, L$,*

$$R_j \otimes P_j^{[i]} + \begin{bmatrix} sym\left[(\mathbf{1}_d \otimes A^{[i]})H_1 + (\mathbf{1}_d \otimes B^{[i]})G_1\right] & -H_1' + \mathbf{1}_d \otimes \left(A^{[i]}h_2 + B^{[i]}g_2\right) \\ \star & -\mathbf{1}_d \otimes (h_2 + h_2') \end{bmatrix} < \mathbf{0} \tag{16}$$

under the nonlinear equation,

$$G_1 - (\mathbf{1}_d \otimes (g_2 h_2^{-1}))H_1 = \mathbf{0} \tag{17}$$

then the polytope $\mathcal{M}$ is $\mathcal{D}_R$ stabilizable and a robust state-feedback matrix is given by:

$$K_{\mathcal{D}_R} = g_2 h_2^{-1} \tag{18}$$

2- *Suppose now that the $\mathcal{D}_{R_j}$ are of different order. If there exist $L \times N$ matrices $P_j^{[i]} \in \mathcal{S}_n^{+*}$ and matrices $h_1 \in \mathbb{R}^{n \times n}$, $g_1 \in \mathbb{R}^{m \times n}$, $h_2 \in \mathbb{R}^{n \times n}$, $g_2 \in \mathbb{R}^{m \times n}$ solutions of the following linear inequalities, $\forall\ i = 1, \cdots, N$ and $\forall\ j = 1, \cdots, L$,*

$$R_j \otimes P_j^{[i]} + \begin{bmatrix} sym\left[R_{12j} \otimes (A^{[i]}h_1 + B^{[i]}g_1)\right] & -R_{12j}' \otimes h_1' + \mathbf{1}_d \otimes \left(A^{[i]}h_2 + B^{[i]}g_2\right) \\ \star & -\mathbf{1}_d \otimes (h_2 + h_2') \end{bmatrix} < \mathbf{0} \tag{19}$$

under the nonlinear equation,

$$g_1 - (g_2 h_2^{-1})h_1 = \mathbf{0} \tag{20}$$

then the polytope $\mathcal{M}$ is $\mathcal{D}_R$ stabilizable and a robust state-feedback matrix is given by:

$$K_{\mathcal{D}_R} = g_2 h_2^{-1} \tag{21}$$

• **Remarks 4** *:*
For intersection of regions of different order, a similar condition may be deduced using a different change of variables $H_{1j} = \mathbf{1}_{d_j} \otimes h_1$ which leads to a different sufficient condition. The first one is particularly interesting since it ensures that the quadratic stability as well as the result in [22] are included by this one. Another way to tackle the problem of the intersection of $\mathcal{D}_R$ regions consists in applying results of theorem 3 with the matrix R defined as the concatenation of elementary R_j matrices. Of course, a single parameter-dependent Lyapunov function is used for the different regions.

3.3 A numerical algorithm based on cone complementarity

It is important to note that in the conditions of theorem 3, the non convex feature of the original problem is entirely defined by the nonlinear algebraic equality (10) involving extra matrix variables only. In [2] and in [12], it is shown that such problems may be recast as the minimization of a suitable nonlinear functional on a convex set defined by $\mathcal{LMI}$ constraints. The nonlinear constraint holds exactly at each local optimal where this functional vanishes. Following the lines in [13], a conic complementarity problem is associated to the conditions of theorem 3.

Problem 1 *:*

$$min \quad Trace \begin{bmatrix} T_1 & T_2 \\ T_2' & T_3 \end{bmatrix} \begin{bmatrix} Z_1 & Z_2 \\ Z_2' & Z_3 \end{bmatrix}$$

$under$

$$\left[\begin{array}{cc|cc} Z_1 & Z_2 & G_1 & \mathbf{1}_d \otimes g_2 \\ Z_2' & Z_3 & H_1 & \mathbf{1}_d \otimes h_2 \\ \hline \star & \star & \mathbf{1} & \mathbf{0} \\ \star & \star & \mathbf{0} & \mathbf{1} \end{array}\right] \geq \mathbf{0}$$

$$\begin{bmatrix} T_1 & T_2 \\ T_2' & T_3 \end{bmatrix} \geq \mathbf{0}$$

$$T_1 \geq \mathbf{1}$$

$$\forall\ i = 1, \cdots, N \qquad \mathcal{L}^{[i]}(P^{[i]}, G_1, H_1, g_2, h_2) < \mathbf{0} \tag{22}$$

The relationships between theorem 3 and problem 1 are now stated more formally.

Lemma 1 *:*

The sufficient condition of theorem 3 is verified if and only if the global minimum of problem 1 is 0.

Algorithm 1 *: CCA*

Step 1: *Let $k = 0$. Find a feasible point T_0, Z_0, Ψ_0, $P_0^{[i]}$ for $\mathcal{LMI}$ problem (22). If there is no solution, stop. Problem (12)-(10) is not feasible.*

Step 2: *Solve the $\mathcal{LMI}$ problem:*

$$\begin{array}{ll} \min & trace(TZ_k + T_k Z) \\ s.t. & (22) \end{array}$$

for T_{k+1}, Z_{k+1}.

Step 3: *Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{n+m}$ denote the singular values of matrix M. If $\sigma_{n+1} \leq \epsilon\sigma_n$ then a solution to problem (12)-(10) is found and the $\mathcal{D}$-stabilizing state-feedback matrix is $K = YG^{-1}$, (ϵ defines some accuracy level).*

Step 4: *Let $k = k + 1$. If $k > k_{max}$, then matrix K was not found. Otherwise go to step 2.*

The initialization step amounts to solve an $\mathcal{LMI}$ problem. The solvability of this first step is therefore a necessary condition for the feasibility of the original $\mathcal{BMI}$ problem. In [12], it is shown that the sequence $t_k = trace(T_k Z + Z_k T)$ is a decreasing sequence bounded below by 0 and therefore always converges.
The iterative process is stopped as soon as the ratio of two successive singular values, (testing the rank), of the matrix M is less than some accuracy level or as soon as a prescribed maximum number of iterations is exceeded or as soon as the relative variation of the criterion is less than 0.01%. With this stopping criteria, the behaviour of this algorithm has been carefully studied and extensively compared with existing methods: quadratic-based conditions and conditions of theorem 4. Thousands of random polytopes of matrices have been tested for the continuous-time and discrete-time stabilization problem. Due to the vertexization of the different conditions, the generated systems are limited to 4 states, 5 vertices and 2 inputs. For continuous-time polytopes, the new algorithm is compared to the quadratic approach while for discrete-time, it is compared to the $\mathcal{LMI}$-based method from [19]. For both cases the behavior is similar and the new proposed approach stabilizes between 15 % and 25 % of polytopes for which all the other methods fails. It is important to note that in each case, the algorithm is "plateauing" for less than 5 % of polytopes which are stabilized by the quadratic approach or by the method from [19]. This behavior has been noticed in [1] where an efficient hybrid algorithm based on a combination of conditional gradient and second-order Newton methods is proposed. The interest of the new proposed approach is now illustrated by three numerical examples corresponding to three characteristic cases.

4 Illustrative examples

These numerical examples are intended to illustrate three main features of the proposed approach. First, a continuous-time polytope of matrices example is considered and a comparison with quadratic state-feedback stabilization is done. The second example shows that the cone complementarity approach is a valuable extension of the purely $\mathcal{LMI}$-based one proposed in [19]. A discrete-time polytope of matrices is defined for which this last method fails while the one proposed in this article succeeds in few iterations. Finally, a particular region of regional pole placement is considered. It consists in the intersection of three subregion, a disk, a half-plane and a sector. $\mathcal{D}_R$ stabilization. Such a region cannot be considered by the method of [19] and has therefore to be approximated by a disk.

4.1 Example 1

The previous algorithm is now applied to the robust stabilization problem of an uncertain continuous-time system borrowed from [10], (example 7.2 p. 271) defined by the following system matrices:

$$A = \begin{bmatrix} 0 & \alpha - 1 \\ \beta & 0 \end{bmatrix} \quad B = \begin{bmatrix} \alpha \\ 1 - \beta \end{bmatrix}$$

The uncertain parameters are defined as,

$$|\alpha - 0.5| \leq \gamma \quad |\beta - 0.5| \leq \gamma$$

This defines a polytope of matrices (A, B) with four vertices. In [10], a quadratic state feedback stabilizing gain is computed. Moreover, it is shown that the set of quadratic state feedback stabilizing gain is not empty for all $\gamma \in [0\ ,\ 0.36]$. For $\gamma = 0.36$ the quadratic state-feedback stabilizing gain is given by $K_{quad.} = \begin{bmatrix} -0.26 & -4.94 \end{bmatrix}$. Applying the algorithm 1 based on cone complementarity, we are able to find robust state feedback stabilizing gains for all $\gamma \in [0\ ,\ 0.5 - \epsilon]$ where ϵ is of the order of the relative accurracy of the $\mathcal{LMI}$ solver. Note that two of the vertices of the polytope are not controllable pairs for $\gamma = 0.5$. For $\epsilon = 0.002$, after four iterations, the robust state-feedback gain is $K_{rob.} = \begin{bmatrix} -0.00038 & -0.88784 \end{bmatrix}$.

4.2 Example 2

The second example is given as a discrete-time polytope of matrices with three vertices defining the three following couples of $(A^{[i]}, B^{[i]})$ matrices:

$$A_1 = \begin{bmatrix} 0 & 2 & 0 \\ 0 & 0 & -1 \\ -1 & 2 & 0 \end{bmatrix} \quad B_1 = \begin{bmatrix} -2 \\ 0 \\ -2 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 0 & 0 \\ -1 & -1 & 0 \\ -2 & 0 & 0 \end{bmatrix}$$
$$A_3 = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad B_3 = \begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} \quad B_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \tag{23}$$

This polytope of matrices is not quadratically stabilizable and the approach proposed in [19] also fails, (the associated $\mathcal{LMI}$'s are found infeasible). After three iterations, our algorithm gives the following robust state-feedback gain $K = \begin{bmatrix} -0.0592 & 0.5758 & -0.1080 \end{bmatrix}$.

4.3 Example 3

Let the continuous-time polytope of matrices be given by,

$$A_1 = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix} \quad B_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad A_2 = \begin{bmatrix} -2 & 1 \\ -1 & 1 \end{bmatrix} \quad B_2 = \begin{bmatrix} -1 \\ 2 \end{bmatrix} \tag{24}$$

The polytope of matrices defined by the two vertices A_1, A_2 is not stable. The $\mathcal{D}_R$ region of pole placement is the intersection between the disk centered at $\alpha_1 = -0.4$ with radius $r = 1.5$, the conic sector defined by its angle with the vertical $\theta = \pi/6$ and its apex $\alpha_2 = -0.15$ and the left half-plane defined for $x = \alpha_3 = -0.4$. After 6 iterations, the algorithm gives a $\mathcal{D}_R$ stabilizing gain $K = \begin{bmatrix} -0.1178 & -0.3914 \end{bmatrix}$ and $6 = 3 \times 2$ Lyapunov matrices. The figure 2 shows the closed-loop poles of the polytope in the considered region along the convex combination of the closed-loop matrices. Note that it is impossi-

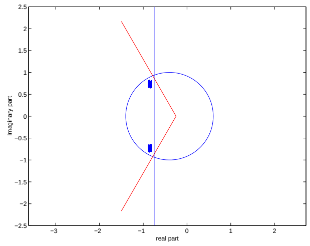

Figure 1: Uncertain system closed-loop poles

ble to find a disk included in this region for which the method of [19] or [22] succeeds. Moreover, other similar approaches using a single parameter-dependent Lyapunov function for the three different regions also fail.

5 Conclusions

In this paper, a tractable state-feedback synthesis method has been proposed for robust regional pole placement in $\mathcal{D}_R$ regions. A recently developped framework based on parameter-dependent Lyapunov functions is used to generalize existing conditions. The original nonlinear problem is linearized via a cone complementarity formulation. A first-order descent technique, the Franck and Wolfe algorithm, is used to get local solutions of this problems. The relevance of the approach is then illustrated by different numerical examples.
A current area of research is the synthesis of output-feedback controller assigning the closed-loop poles in a prescribed $\mathcal{D}_R$ region via parameter-dependent Lyapunov functions.

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