Improved Polynomial Matrix Determinant Computation*

Didier Henrion[†‡]
Michael Šebek[§¶]

Abstract

An early result on the Smith-MacMillan form of a rational matrix is used for evaluating the degree of the determinant of a polynomial matrix using numerically reliable techniques. This allows for accurate determinant zeroing and determinant interpolation, thus improving existing numerical methods for polynomial matrix determinant computation.

Keywords

Polynomial Matrix, Determinant Computation, Numerical Methods.

1 Motivation

Whenever transfer functions of linear multivariable systems and/or circuits are described by polynomial matrix fractions, analysis and design procedures consists of various manipulations with polynomial matrices. See [5, 6] for polynomial matrices used in control and [1, 2, 7, 8, 11, 12, 13] for their application in signal processing-oriented filter design.

A typical job of this kind may include determinant degree computation of a square polynomial matrix. The degree may reveal system order, the number of its poles or zeros and/or can further be used to investigate both finite and infinite structure of the system.

*This work was supported by the Barrande Project No. 97/005-97/026, by the Grant Agency of the Czech Republic under contract No. 102/99/1368 and by the Ministry of Education of the Czech Republic under contract No. VS97/034.

†Corresponding author. E-mail: `henrion@laas.fr`. FAX: +33 5 61 33 69 69.

‡LAAS-CNRS, 7 Avenue du Colonel Roche, 31 077 Toulouse, Cedex 4, France.

§Trnka Laboratory for Automatic Control, Faculty of Electrical Engineering, Czech Technical University, Technická 2, 166 27 Prague 6, Czech Republic.

¶Institute of Information Theory and Automation, Czech Academy of Sciences, Pod vodárenskou věží 4, 182 08 Prague 8, Czech Republic.

Next to this, the knowledge of the degree may improve reliable calculation of the determinant itself and hence becomes an important step when evaluating various characteristic polynomials, seeking for poles or zeros locations, etc.

As pointed out in [10], one of the main stumbling block when designing algorithms for numerical computation of the determinant

$$d(s) = d_0 + d_1 s + \cdots + d_\delta s^\delta$$

of a given non-singular $n \times n$ polynomial matrix

$$A(s) = A_0 + A_1 s + \cdots + A_\alpha s^\alpha$$

is the correct evaluation of the degree δ. Due to unavoidable numerical roundoff errors, it is sometimes difficult to decide which leading coefficients are relevant in the computed determinant

$$\hat{d}(s) = \hat{d}_0 + \hat{d}_1 s + \cdots + \hat{d}_{\hat{\delta}} s^{\hat{\delta}}$$

where $\hat{\delta} \geq \delta$. Typically, some threshold ϵ is selected as a function of n, α and the machine precision. Then, the smallest index i such that $|\hat{d}_i| < \epsilon$ is determined and coefficients $\hat{d}_i, \hat{d}_{i+1}, \ldots, \hat{d}_{\hat{\delta}}$ are artificially set to zero. Such an operation, referred to as zeroing, is a popular technique that proves necessary for preventing unduly degree swelling when performing successive operations on polynomial matrices.

The same kind of problem arises when deriving $\hat{d}(s)$ via interpolation [10] or fast Fourier transform [4], the later being the most powerful and reliable numerical method to date for performing polynomial matrix determinant computation, as illustrated by intensive computational experiments [4]. An estimate is always required on the number of sample points needed for performing interpolation or fast Fourier transform. The sum of column or row degrees of $A(s)$ is usually chosen as a rough upper bound on $\delta + 1$, the correct number of points.

In this note, we solve the determinant degree evaluation problem while keeping with our main impetus, which is the development of reliable numerical algorithms for dealing with polynomial matrices [3] and their implementation in a user-friendly MATLAB[1] package called the Polynomial Toolbox [9].

In this regard, the aim of this note is to point out that the value of d can be evaluated in a numerically stable way, thus allowing for proper determinant zeroing and determinant interpolation. The result that proves instrumental to this computation is not new. It was published some twenty years ago by Van Dooren and co-workers [14]. It was derived when studying the Smith-MacMillan of a rational matrix from its Laurent expansion at its poles and zeros. Our main contribution is in showing that this result can actually be used for improving polynomial matrix determinant computation.

[1] MATLAB is a trademark of The MathWorks, Inc.

2 Main Result

Theorem 1 *Let*

$$T_i = \begin{bmatrix} A_\alpha & \cdots & A_{-i+1} & A_{-i} \\ & \ddots & & \vdots \\ & & A_\alpha & A_{\alpha-1} \\ 0 & & & A_\alpha \end{bmatrix}$$

denote a Toeplitz matrix built from matrix coefficients of a non-singular $n \times n$ polynomial matrix $A(s)$, where it is assumed that $T_{i-\alpha} = 0$ and $A_i = 0$ when $i < 0$. Let

$$r_i = \mathrm{rank} T_i - \mathrm{rank} T_{i-1} \tag{1}$$

and

$$k = \min\{i : r_i = n, i \geq 0\}.$$

Then the degree δ of the determinant of $A(s)$ is given by

$$\delta = \mathrm{rank} T_k - n(k+1) \tag{2}$$

Proof: It is well-known that any non-singular rational matrix features the same number of poles and zeros, finite or infinite, counting multiplicities [15]. Moreover, degree δ is equal to the number of finite zeros of $A(s)$. Since a polynomial matrix has no finite poles [15], it follows that

$$\delta = p_\infty - z_\infty \tag{3}$$

where p_∞ and z_∞ denote algebraic multiplicities of the pole and zero of $A(s)$ at infinity, respectively. Using the Smith-MacMillan form of $A(s)$ at infinity – see Corollary 3.7 and Remark 4 in Section IV in [14] – one can show that

$$p_\infty = \sum_{i=-\alpha}^{-1} r_i \tag{4}$$

and

$$z_\infty = \sum_{i=0}^{k} (n - r_i). \tag{5}$$

Equation (2) readily follows from relations (1), (3), (4) and (5). $\square$

It must be underlined that a fast recursive algorithm is described in [14] for performing the successive rank computations and determining index k. The algorithm takes advantage of the special Toeplitz structure of matrices T_i and hinges upon the singular value decomposition, a numerically stable operation. Therefore Theorem 1 provides a numerically reliable method for evaluating the degree of the determinant of a polynomial matrix using coefficients of the matrix only.

3 Illustration

Let

$$A(s) = \begin{bmatrix} 1+2s^2 & 2s^2+s^3 & s^2 \\ -3+s-2s^2 & -1+s-s^2-s^3 & -1+s-s^2 \\ 2 & 2+s & 1 \end{bmatrix}.$$

Build Toeplitz matrices

$$T_{-3} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad T_{-2} = \left[\begin{array}{ccc|ccc} 0 & 1 & 0 & 2 & 2 & 1 \\ 0 & -1 & 0 & -2 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \cdots$$

Rank offsets (1) are collected in Table 1. Note that $r_k = n = 3$ for $k = 4$.

i	-3	-2	-1	0	1	2	3	**4**	5	6
r_i	1	1	2	2	2	2	2	**3**	3	3

Table 1: Rank offsets r_i.

Recalling equation (2), the degree of the determinant of $A(s)$ is given by

$$\delta = \text{rank} T_4 - 15 = 0.$$

Matrix $A(s)$ is therefore unimodular and its determinant $d(s)$ can readily be computed by interpolation at only one point, i.e.

$$d(s) = \det A(0) = \det \begin{bmatrix} 1 & 0 & 0 \\ -3 & -1 & -1 \\ 2 & 2 & 1 \end{bmatrix} = 1.$$

References

[1] A. Ahlen and M. Sternad, "Optimal Filtering Problems", Chapter 5 in K. Hunt (Ed.), "Polynomial Methods in Optimal Control and Filtering", Control Engineering Series, *Peter Peregrinus*, London, 1993.

[2] A. Ahlen and M. Sternad, "Derivation and Design of Wiener Filters using Polynomial Equations", pp. 353–418 in C. T. Leondes (Ed.), "Control and Dynamic Systems", Stochastic Techniques in Digital Signal Processing Systems, Vol. 64, *Academic Press*, New York, 1994.

[3] D. Henrion, "Reliable Algorithms for Polynomial Matrices", *Ph. D. Thesis*, Institute of Information Theory and Automation, Czech Academy of Sciences, Prague, Czech Republic, December 1998.

[4] M. Hromčík and M. Šebek, "New Algorithms for Polynomial Matrices Based on FFT", *Technical Report*, Institute of Information Theory and Automation, Czech Academy of Sciences, Prague, Czech Republic, 1998. Submitted for publication.

[5] V. Kučera, "Discrete Linear Control: The Polynomial Approach", *John Wiley and Sons*, Chichester, 1979.

[6] V. Kučera, "Analysis and Design of Discrete Linear Control Systems", *Prentice Hall*, London, 1991.

[7] L. Lindbom, "A Wiener Filtering Approach to the Design of Tracking Algorithms with Applications in Mobile Radio Communications", *Ph. D. Thesis*, Uppsala University, Sweden, ISBN 91-506-1126-7, November 1995.

[8] K. Öhrn, "Design of Multivariable Cautious Discrete-Time Wiener Filters: A Probabilistic Approach", *Ph. D. Thesis*, Uppsala University, Sweden, ISBN 91-506-1157-7, May 1996.

[9] M. Šebek, H. Kwakernaak, D. Henrion and S. Pejchová, "Recent Progress in Polynomial Methods and Polynomial Toolbox for MATLAB Version 2.0", *Proceedings of the Conference on Decision and Control*, IEEE, pp. 3661-3668, Tampa, Florida, December 1998. See the home page `http://www.polyx.cz`.

[10] M. Šebek, F. Kraus, S. Pejchová, H. Kwakernaak and D. Henrion, "Numerical Methods for Zeros and Determinant of Polynomial Matrix", *Proceedings of the Mediterranean Symposium on New Directions in Control and Automation*, IEEE, pp. 488-491, Chania, Crete, Greece, June 1996.

[11] M. Sternad and A. Ahlen, "LQ Controller Design and Self-Tuning Control", Chapter 3 in K. Hunt (Ed.), "Polynomial Methods in Optimal Control and Filtering", Control Engineering Series, *Peter Peregrinus*, London, 1993.

[12] M. Sternad and A. Ahlen, "$\mathcal{H}_2$ Design of Model-Based Nominal and Robust Discrete-Time Filters", Chapter 5 in M. Grimble and V. Kučera (Eds.), "A Polynomial Approach to $\mathcal{H}_2$ and $\mathcal{H}_\infty$ Robust Control Design", *Springer Verlag*, London, 1996.

[13] C. Tidestav, A. Ahlen and M. Sternad, "Realizable MIMO Decision Feedback Equalizers", *Proceedings of the International Conference on Acoustics, Speech and Signal Processing*, IEEE, Phoenix, Arizona, March 1999.

[14] P. M. Van Dooren, P. Dewilde and J. Vandewalle, "On the Determination of the Smith-MacMillan Form of a Rational Matrix From Its Laurent Expansion", *IEEE Transactions on Circuits and Systems*, Vol. 26, No. 3, pp. 180–189, 1979.

[15] A. I. G. Vardulakis, "Linear Multivariable Control. Algebraic Analysis and Synthesis Methods", *Wiley*, Chichester, 1991.