Block Toeplitz Methods in Polynomial Matrix Computations

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Abstract

Some block Toeplitz methods applied to polynomial matrices are reviewed. We focus on the computation of the structure (rank, null-space, infinite and finite structures) of an arbitrary rectangular polynomial matrix. We also introduce some applications of this structural information in control theory. All the methods outlined here are based on the computation of the null-spaces of suitable block Toeplitz matrices.

Keywords : Polynomial matrices, numerical linear algebra, computer-aided control system design.

1 Introduction

By analogy with scalar rational transfer functions, matrix transfer functions of multivariable linear systems can be written in polynomial matrix fraction form as $N(s)D^{-1}(s)$ where $D(s)$ is a non-singular polynomial matrix [2, 12, 17]. So, in the polynomial matrices $N(s)$ or $D(s)$ we can find the structural information required to solve several control problems. Consider for example the problem of decoupling of linear systems, where the infinite structural indices of a suitable polynomial matrix are needed to determine if the system is decouplable, and also to determine the structure that could have the decoupled closed loop system [3]. The finite structure of a polynomial matrix is instrumental to its spectral factorization, which has applications in several optimal and robust control problems [18]. Obtaining the null-space of a polynomial matrix allows to solve polynomial matrix equations, such as polynomial Diophantine equations arising in the solution of several control problems [17].

It is therefore relevant to develop reliable numerical algorithms for polynomial matrix computations. In this paper we survey numerical algorithms to obtain the structure of polynomial matrices, namely the rank, the null-space, the infinite and the finite structure. The current algorithms used for these tasks follow two major approaches: the pencil approach and the Toeplitz approach.

In the classical and well known pencil approach pioneered in [20] we can obtain the structural indices of an arbitrary polynomial matrix $A(s)$ by processing a related pencil [21]. The pencil approach finds its background in the state-space methods in control. Indeed, given a linear system with strictly proper transfer function $T(s) = N(s)D^{-1}(s)$, the structural information of the system that we can obtain from polynomial matrices $N(s)$ and $D(s)$ can also by obtained from the pencil

$$P(s) = \begin{bmatrix} sI - A & B \\ C & 0 \end{bmatrix}$$

where (A, B, C) is a state-space representation of $T(s)$.

When we solve a control problem with polynomial methods we need to process directly polynomial matrices without a previous reformulation in terms of pencils. With the Toeplitz approach, obtaining the eigenstructure of $A(s)$ is naturally equivalent to obtaining null-spaces of some block Toeplitz matrices built with the coefficients of $A(s)$.

In our previous work [23] we showed how the Toeplitz approach can be an alternative to the classical pencil approach [21] to obtain infinite structural indices of polynomial matrices. In this work we describe further applications of the block Toeplitz methods. We show how all the eigenstructure of $A(s)$ can be obtained. We also present current algorithms and we outline some improvements and possible extensions. The numerical methods proposed in section 6 are analyzed in a deepest way in our oncoming works [24, 25] where we compare four different implementations of the Toeplitz algorithm to compute the null-space of a polynomial matrix.

2 Infinite structure

The *infinite structure* or structure at infinity of an $m \times n$ polynomial matrix $A(s)$ of degree d is equivalent to the finite structure in $\alpha = 0$ of the dual matrix

$$A_{\text{dual}}(s) = A_d + A_{d-1}s + \cdots + A_0 s^d,$$

From Theorem A.1 in [1], if $\alpha = 0$ has algebraic multiplicity m_A and geometric multiplicity m_G, then there exists a series of integers $k_i > 0$ for $i = 1, 2, \ldots, m_G$ such that $m_A = k_1 + k_2 + \cdots + k_{m_G}$ and a series of eigenvectors at infinity $v_{i1}, v_{i2}, \ldots, v_{ik_i}$ for $i = 1, 2, \ldots, m_G$ such that

$$\begin{bmatrix} A_d & & & 0 \\ A_{d-1} & A_d & & \\ \vdots & & \ddots & \\ A_{d-k_i+1} & \cdots & A_{d-1} & A_d \end{bmatrix} \begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{ik_i} \end{bmatrix} = 0 \tag{1}$$

with $v_{11}, v_{21}, \ldots, v_{M_G 1}$ linearly independent.

The orders of the zeros or poles at infinity of $A(s)$ are equal to $k_i - d$. If $k_i - d < 0$ we say that $A(s)$ has one pole at infinity of degree $d - k_i$, if $k_i - d \geq 0$ we say that $A(s)$ has one zero at infinity of degree $k_i - d$.

From equation (1) we can see the direct relation between the struture at infinity of $A(s)$ and the constant null-spaces of suitable block Toeplitz matrices. An algorithm to obtain all the infinite indices and vectors process in an iterative way a block Toeplitz matrix of increasing size.

Consider the $mi \times ni$ Toeplitz matrix

$$T_i = \begin{bmatrix} A_d & 0 & \cdots & 0 \\ A_{d-1} & A_d & & \\ \vdots & & \ddots & \\ A_{d-(i-1)} & & & A_d \end{bmatrix}$$

where $A_{d-j} = 0$ if $j > d$. At step 1, integer $\beta_1 = \text{rank}\, A(s) - \text{rank}\, T_1$ is equal to the number of chains of eigenvectors at infinity. At step i we define

$$\beta_i = i\,\text{rank}\, A(s) - \text{rank} T_i - \sum_{j=1}^{i-1} \beta_j. \tag{2}$$

Notice that $\beta_{i+1} \leq \beta_i$.

At the end, when $\beta_k = 0$ for a sufficently large index k, it follows that $A(s)$ has rank A_d poles at infinity of degree d, and $t_i = \beta_i - \beta_{i+1}$ infinite elementary divisors of the form s^{i-d} for $i = 1, 2, \ldots, k-1$. If $i - d < 0$ we say that $A(s)$ has t_i poles at infinity of degree $d - i$, if $i - d \geq 0$ we say that $A(s)$ has t_i zeros at infinity of degree $i - d$.

A first version of this algorithm restricted to non-singular matrices was presented in [7]. In this section we have showed that if we know the rank of $A(s)$ we can extend the results to an arbitrary polynomial matrix. See section 4 for algorithms of rank evaluation of polynomial matrices.

3 Null-space structure

A basis of the null-space of $A(s)$ contains the $n - \text{rank}\, A(s)$ non-zero polynomial vectors $v(s)$ such that

$$A(s)v(s) = 0. \tag{3}$$

Let δ_i be the degree of each vector in the basis of the null-space, if the sum of all the degrees δ_i is minimal then we have a minimal basis in the sense of Forney [4].

The set of degrees δ_i and the corresponding polynomial vectors are called the *right null-space structure* of $A(s)$.

It is easy to show that solving equation (3) is equivalent to solving the constant equation

$$\begin{bmatrix} A_0 & & & 0 \\ \vdots & A_0 & & \\ A_d & \vdots & \ddots & \\ & A_d & & A_0 \\ & & \ddots & \vdots \\ 0 & & & A_d \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ v_\delta \end{bmatrix} = 0. \tag{4}$$

The problem remaining when solving (4) is the estimation of the degree δ. To avoid this problem we process iteratively a block Toeplitz matrix of increasing size.

Consider the block Toeplitz matrix

$$R_i = \begin{bmatrix} A_0 & & & 0 \\ \vdots & A_0 & & \\ A_d & \vdots & \ddots & \\ & A_d & & A_0 \\ & & \ddots & \vdots \\ 0 & & & A_d \end{bmatrix}$$

with dimensions $m(d+i) \times ni$ at step index $i = 1, 2, \ldots$.
Let us take $\gamma_1 = n - \operatorname{rank} R_1$, so that matrix $A(s)$ has γ_1 vectors of degree 0 in the basis of its null-space. The number of vectors of degree 1 is equal to $\gamma_2 = 2n - \operatorname{rank} R_2 - \gamma_1$ and so on. At step i we define

$$\gamma_i = ni - \operatorname{rank} R_i - \sum_{j=1}^{i-1} \gamma_j \tag{5}$$

and then the number of vectors of degree $i-1$ in the basis of the null-space of $A(s)$ is equal to $\gamma_i - \gamma_{i-1}$. Notice that $\gamma_i \leq \gamma_{i+1}$.

When $\gamma_f = n - \operatorname{rank} A(s)$ for some index f sufficently large, we have obtained all the vectors of a minimal basis of the null-space of $A(s)$.

The *left null-space structure* of $A(s)$, i.e. the set of vectors $w(s)$ such that $w(s)A(s) = 0$, can be obtained from the transposed relation $A^T(s)w^T(s) = 0$. So, in a similar way, processing at each step the block Toeplitz matrix

$$\bar{R}_i = \begin{bmatrix} A_0^T & & & 0 \\ \vdots & A_0^T & & \\ A_d^T & \vdots & \ddots & \\ & A_d^T & & A_0^T \\ & & \ddots & \vdots \\ 0 & & & A_d^T \end{bmatrix}$$

with dimensions $n(d+i) \times mi$ at step index $i = 1, 2, \ldots$, we can obtain integers

$$\zeta_i = mi - \operatorname{rank} \bar{R}_i - \sum_{j=1}^{i-1} \zeta_j \tag{6}$$

A version of this algorithm is currently implemented in the function `null` of the Polynomial Toolbox for Matlab [19]. Function `null` uses the Column Echelon Form (CEF) of R_i to reveal its rank at each step [7, 10]. Nevertheless, we have showed that others rank revealing numerical methods can be more efficient than the CEF [24]. See section 6 for a brief description of these numerical methods.

4 Rank evaluation

From the algorithms outlined in previous sections, we can see that it is important to have a method to obtain the rank of an $m \times n$ polynomial matrix $A(s)$ with degree d. Several methods have been documented, see for example [8], and are currently implemented in function `rank` of the Polynomial Toolbox. These methods are based on the fact that $A(s)$ has, at most, $z = d \min(m, n)$ finite zeros, which are those values $s = \alpha_i$ with $i = 1, 2, \ldots, z$ for which $\operatorname{rank} A(\alpha_i) < \operatorname{rank} A(s)$. So, if we compute the rank of $A(s)$ evaluated at $z + 1$ arbitrary, yet distinct, values of s we can be sure that the maximum achieved rank is equal to the rank of $A(s)$. See the next section for more about the finite structure of $A(s)$. The upper bound z for the number of finite zeros of $A(s)$ is a consequence of the following lemma from [22].

Lemma 1 *Consider an $m \times n$ polynomial matrix $A(s)$ with degree d. The number of poles of $A(s)$ (we only have poles at infinity) must be equal to its number of zeros (finite or infinite) including multiplicities plus the degrees of the vectors in the null-spaces of $A(s)$.*

Now we show how the rank of $A(s)$ can also be obtained from the block Toeplitz matrices T_i and R_i used in previous sections. For notational simplicity we explain the algorithm with an example. Consider that we want to obtain the eigenstructure of polynomial matrix

$$A(s) = \begin{bmatrix} 1 & s^3 & 0 & 0 \\ 0 & 1 & s & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Supposing that we do not know the rank ρ of $A(s)$, we start with the guess $\rho = \min(m, n) = 3$.

First we compute

$$\operatorname{rank} T_1 = \operatorname{rank} A_3 = 1$$

and therefore we have $\beta_1 = \rho - 1 = 2$. Then we compute

$$\operatorname{rank} R_1 = \operatorname{rank} [A_0^T \quad A_1^T \quad A_2^T \quad A_3^T]^T = 3$$

and so $\gamma_1 = n - 3 = 1$. This shows that $A(s)$ has a vector of degree 0 in the basis of its right null-space.

We also compute

$$\operatorname{rank} \bar{R}_1 = \operatorname{rank} [A_0 \quad A_1 \quad A_2 \quad A_3]^T = 2$$

from which it follows that $\zeta_1 = m - 2 = 1$. This shows that $A(s)$ has a vector of degree 0 in the basis of its left null-space. At this step we know that the rank of $A(s)$ cannot be $\rho = 3$ but at most $\rho = 2$. Knowing this we update $\beta_1 = \rho - 1 = 1$.

Now we proceed with the next step and we compute $\operatorname{rank} T_2 = 2$, $\operatorname{rank} R_2 = 6$ and $\operatorname{rank} \bar{R}_2 = 4$ and therefore from (2), (5) and (6), it follows that $\beta_2 = 1$, $\gamma_2 = 1$ and $\zeta_2 = 1$.

At following step we compute the rank of T_3 and we obtain $\beta_3 = 0$. At this step, remembering that $\beta_{i+1} \leq \beta_i$ we can be sure that $\rho = \operatorname{rank} A(s) = 2$. Notice also that $\zeta_2 = m - \rho = 1$, so we have found all the degrees in the basis of the left null-space of $A(s)$. Finally, we can continue computing the ranks of R_i until we find $\gamma_f = n - \rho = 2$.

In computational terms, computing the rank of $A(s)$ evaluated at $z + 1$ points amounts to performing $nd + 1$ singular value decompositions (SVD) of $m \times n$ matrices [6]. On the other hand, computing the rank of $A(s)$ as in the last example amounts to obtaining the ranks of block Toeplitz matrices T_i, R_i and $\bar{R}_i$ for $i = 1, 2, \ldots, f$. From Lemma 1 we can show that, at most, the final number of steps f is equal to $nd + 1$. It follows that we have to compute at most the ranks of $nd + 1$ block Toeplitz matrices. In summary, when computing all the structure of matrix $A(s)$ it can be more efficient to follow the algorithm outlined above instead of computing first the rank and then the eigenstructure.

5 Finite structure

A finite zero of $A(s)$ is a number α such that there exists a non-zero vector v satisfying $A(\alpha)v = 0$. Vector v is called characteristic vector or eigenvector associated to the zero α.

From Theorem A.1 in [1], if α is a finite zero of $A(s)$ with algebraic multiplicity m_A and geometric multiplicity m_G, then there exists a series of integers $k_i > 0$ for $i = 1, 2, \ldots, m_G$ such that $m_A = k_1 + k_2 + \cdots + k_{M_G}$ and a series of characteristic vectors $v_{i1}, v_{i2}, \ldots, v_{ik_i}$ for $i = 1, 2, \ldots, m_G$ associated to α such that

$$\begin{bmatrix} \bar{A}_0 & & & 0 \\ \bar{A}_1 & \bar{A}_0 & & \\ \vdots & & \ddots & \\ \bar{A}_{k_i-1} & \cdots & \bar{A}_1 & \bar{A}_0 \end{bmatrix} \begin{bmatrix} v_{i1} \\ v_{i2} \\ \vdots \\ v_{ik_i} \end{bmatrix} = 0 \tag{7}$$

with $v_{11}, v_{21}, \ldots, v_{M_G 1}$ linearly independent and where

$$\bar{A}_j = \frac{1}{j!} \left[\frac{\mathrm{d}^j A(s)}{\mathrm{d}s^j} \right]_{s=\alpha}.$$

Integer k_i is the length of the ith chain of characteristic vectors associated to α.

Notice that system (1) is a particular case of system (7). So, if we know all the finite zeros of $A(s)$ we can use the algorithm outlined in section 2 to obtain the structure associated to each zero.

The problem of obtaining the finite zeros or roots of a polynomial matrix $A(s)$ also deserves attention. Several methods are already programmed in the function `roots` of the Polynomial Toolbox. One approach consists in obtaining the roots of $A(s)$ by computing the roots of the determinant of $A(s)$. See [9] for reference on polynomial matrix determinant computation. Another approach consists in obtaining the roots of $A(s)$ by computing the generalized eigenvalues of the associated companion matrix. For that we can use classical algebraic methods such as the QZ algorithm [6]. The discussion about the respective numerical stability of these approaches remains open. In any case, a well identified problem is to detect the presence of the infinite zeros while computing the finite zeros. This problem can also tackled by several approaches. Here we consider an application of the algorithm for polynomial matrix factor extraction proposed in [10].

In [10] it is showed how a non-singular polynomial matrix $A(s)$ with a set of zeros $Z = \{z_1, \ldots, z_p, z_{p+1}, \ldots, z_q\}$ can be factorized as $A(s) = L(s)R(s)$ where polynomial matrix $L(s)$ has zeros $z_1, \ldots, z_p$ and polynomial matrix $R(s)$ has zeros $z_{p+1}, \ldots, z_q$. In section 2 we have seen how to compute the infinite zeros of an arbitrary polynomial matrix $A(s)$, so a natural extension of the algorithm of [10] to non-square matrices can be used to obtain the factorization $A(s) = L(s)R(s)$ where matrix $L(s)$ contains only the finite zeros of $A(s)$.

6 Numerical methods

In previous sections we outlined different algorithms to obtain the eigenstructure of an arbitrary rectangular polynomial matrix. These algorithms are based in successively obtaining ranks and null spaces of block Toeplitz matrices. In this section we survey the different numerical linear algebra methods that can be used for that purpose.

In [7, 10] the reduction of a Toeplitz matrix into its column echelon form (CEF) is used in order to reveal its rank. The CEF method is based on the application of successive Householder transformations onto the rows of the Toeplitz matrix. Using this method at each step of our algorithms, the complete Toeplitz matrix has to be analyzed. In other words the CEF method does not take advantage of the block Toeplitz structure.

In [23] we proposed the singular value decomposition (SVD) as a way to reveal the rank of a Toeplitz matrix. We showed how, by taking advantage of the block structure of the Toeplitz matrix, we can use some of the information computed at step $i-1$ in the computations at step i. Another difference between the CEF and the SVD methods, when they are used in our algorithms, is that when computation the SVD of a Toeplitz matrix we also obtain a basis of its null-space [5, 6], namely the solution of systems (1), (4) or (7).

Other rank revealing methods that we can propose are the QR factorization [6] or the dual LQ factorization yielding a factor in lower triangular form along with an orthogonal transformation matrix. With the QR or LQ factorization methods we can also take advantage of the block Toeplitz structure. In computational terms, the QR or LQ factorization and the CEF perform less operations than the SVD method. Choosing between one of these methods depends on the requirements in accuracy [11] and performance of each particular application.

In an oncoming work [24] we analyze in a deepest way the application of these different numerical methods. We focus only in obtaining the null-space of a polynomial matrix and we develop the algorithm outlined in section 3 using the methods CEF and LQ.

A drawback of the Toeplitz approaches presented here is that the dimension of the corresponding Toeplitz matrices can be very large. In order to overcome this drawback, in [25] we also introduce some displacement structure methods. These methods take full advantage of the Toeplitz structure and can reduce in one order of magnitude the algorithmic complexity [13, 14]. Applications of displacement structure methods in block Toeplitz matrix computations can be found in [15, 16].

7 Conclusions

In our previous work [23] we showed how the Toeplitz approach can be a competitive alternative to the pencil approach [21] to obtain infinite structural indices of polynomial matrices. In this paper we have outlined how these Toeplitz methods can be extended to obtain all the eigenstructure of polynomial matrices. The extension of existing algorithms to the non-square case was also reviewed here.

Several numerical methods are proposed to process block Toeplitz matrices. Therefore several implementations of the algorithms can be done. In our oncoming works [24, 25] we present four different implementations of an algorithm to obtain the null-space of a polynomial matrix. From the results presented in [24, 25] we anticipate that it could be possible to improve significantly some functions of the Polynomial Toolbox in terms of numerical accuracy and computational burden.

In a future work we are also planning to implement other functions to obtain the whole eigenstructure of a polynomial matrix based in the algorithms outlined here. The ultimate objective is a numerically reliable and computationally cheap algorithm to perform J-spectral factorization of polynomial matrices [18].

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