LMI Relaxations for Robust Stability of Linear Systems with Saturating Controls

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Abstract

A method is proposed for studying robust stability of uncertain linear systems subject to actuator saturation. It hinges upon successive LMI relaxations and therefore can readily be implemented using widely available numerical tools. The technique is illustrated on two examples and is shown to significantly improve previously published results.

Introduction

It is now widely recognized that both uncertainty and actuator saturation are ubiquitous features that must explicitly be taken into account when designing control laws for linear systems. On the one hand, theoretical achievements in the study of uncertain systems led to the so-called robust controllers, see (Zhou et al., 1996) for a good introduction. On the other hand, several new results recently emerged in the realm of systems with constrained or saturated inputs, see (Tarbouriech and Garcia, 1997) for an overview. Quite surprisingly, a very few works have been devoted so far to combining both features, in spite of their relevance in practical control problems.

Motivated by this observation, our note aims at analyzing stability of linear systems that are simultaneously uncertain and subject to actuator saturation. We assume that we are given an open-loop uncertain linear system for which a robust stabilizing state-feedback control has been computed by some design method. In addition to that, we suppose that system inputs may saturate. Our objective is then to determine the greatest set of initial conditions for which closed-loop system stability is preserved. In the system and control literature, this problem is referred to as the local stabilization problem. Without any assumption on open-loop system stability, this is a highly non-trivial issue that remains unsolved in the general case.

In this note, we approach the local stabilization problem through standard tools of modern control theory. We show that actuator saturations can be represented locally as

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polytopic system uncertainties, which may later on be combined with open-loop system uncertainties. Next, we use quadratic stabilization to show that the set of safe initial conditions may be approximated by an ellipsoid shaped by a closed-loop Lyapunov matrix. Maximization of this set hinges upon the resolution of Bilinear Matrix Inequalities (BMIs), a fundamental problem frequently arising in robust control (Goh et al., 1996). Because no efficient algorithm has been designed so far for solving general BMI problems, we follow an approached proposed in (Henrion et al., 1997; Tarbouriech and Garcia, 1997; Gomes da Silva and Tarbouriech, 1998) so as to describe an iterative relaxation scheme involving Linear Matrix Inequalities (LMIs), for which efficient polynomial-time algorithm have been designed (Boyd et al., 1994).

Problem Statement

We consider the continuous-time system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the control input. In order to represent uncertainties that may affect system (1), we restrict time-varying system matrices to

$$\begin{array}{rcl} A(t) & \in & \mathcal{A} \\ B(t) & \in & \mathcal{B} \end{array}$$

where $\mathcal{A}$ and $\mathcal{B}$ are compact sets. In the sequel we will consider two different types of uncertainty, namely

- Norm Bounded Uncertainty, where

$$\begin{array}{rcl} \mathcal{A} & = & \{A_0 + DF(t)E_1, \quad \|F(t)\|_2 \leq 1\} \\ \mathcal{B} & = & \{B_0 + DF(t)E_2, \quad \|F(t)\|_2 \leq 1\} \end{array}$$

  for given nominal matrices A_0, B_0 and constant matrices D, E_1, E_2, or

- Polytopic Uncertainty, where system matrices belong to matrix polytopes

$$\begin{array}{rcl} \mathcal{A} & = & \text{convex hull}\{A_1, \ldots, A_{n_A}\} \\ \mathcal{B} & = & \text{convex hull}\{B_1, \ldots, B_{n_B}\}. \end{array}$$

Moreover, we assume that a robust stabilizing state-feedback control matrix K was built for system (1) by some design method.

Suppose now that symmetric saturation non-linearities affect system inputs. More specifically, suppose that components u_i of input vector u verify

$$u_i = \text{sat}[K_i x] = \left\{ \begin{array}{ll} w_i & \text{if } K_i x > w_i \\ K_i x & \text{if } |K_i x| \leq w_i \\ -w_i & \text{if } K_i x < -w_i \end{array} \right. \tag{2}$$

where $w_1, \ldots, w_m$ are given positive scalars and K_i is the i-th row in matrix K. Because no assumption was made regarding stability of open-loop system matrix $A(t)$, it is well-known that, due to possible actuator saturation, some trajectories of saturated closed-loop system (1)-(2) may be unstable. Actually, the problem of finding the set of all initial conditions that are steered to the origin is a very challenging issue which is still unsolved in the general case. The following definition will be instrumental for the derivation of a method for approximating this set.

Definition 1 *Uncertain saturated system (1)-(2) is locally quadratically stable if there exist a positive definite symmetric Lyapunov matrix P and a set $\mathcal{E}$ such that the inequality*

$$(A(t)x(t) + B(t)sat[Kx(t)])' Px(t) + x'(t)P (A(t)x(t) + B(t)sat[Kx(t)]) < 0 \qquad (3)$$

holds for all $x(t) \in \mathcal{E}$ and $A(t) \in \mathcal{A}$, $B(t) \in \mathcal{B}$.

In view of the above definition, a natural choice of stability domain is the ellipsoidal Lyapunov level set

$$\mathcal{E} = \{x : x'Px \leq \frac{1}{r}\} \qquad (4)$$

for a suitable positive value of scalar r. It can be shown that ellipsoid $\mathcal{E}$ is positively invariant and contractive whenever (3) holds in $\mathcal{E}$. As a result, any trajectory initiating in this ellipsoid converges to the origin. Hence $\mathcal{E}$ appears as an interesting approximation of the region of attraction of the origin. Based on these considerations, the first problem we are to solve in this note is as follows.

Problem 1 *Find Lyapunov matrix P and scalar r such that closed-loop system (1)-(2) is locally quadratically stable in Lyapunov level set $\mathcal{E}$.*

Suppose that $\mathcal{E}$ is solution to the above analysis problem. Since every homothetic set $\lambda\mathcal{E}$ for $\lambda \leq 1$ is also positively invariant and contractive, Lyapunov level set $\mathcal{E}$ should be as large as possible. Our second problem originates from this remark.

Problem 2 *Find Lyapunov matrix P and scalar r such that closed-loop system (1)-(2) is locally quadratically stable in the largest Lyapunov level set $\mathcal{E}$.*

BMI Formulation

Now we show that Problems 1 and 2 can respectively be formulated as a BMI feasibility problem and a BMI optimization problem. For not getting overwhelmed by notations, we shall suppress the dependence on t of vector x whenever convenient. First, system (1)-(2) can be written

$$\dot{x}(t) = A(t)x(t) + B(t)G_{\gamma(x(t))}Kx(t)$$

where $G_{\gamma(x(t))}$ is a diagonal matrix whose components are

$$\gamma_i(x) = \begin{cases} w_i/K_i x & \text{if } K_i x > w_i \\ 1 & \text{if } |K_i x| \leq w_i \\ -w_i/K_i x & \text{if } K_i x < -w_i. \end{cases}$$

for $i = 1, \ldots, m$. Note that $\gamma_i(x)$ lies within the interval $]0, 1]$ for any vector x. When γ_i approaches 0 there is almost no feedback from input u_i, whereas $\gamma_i = 1$ merely means that u_i does not saturate.

Recalling our formulation of Problems 1 and 2, the domain of evolution of the state of system (1)-(2) has to be limited. Therefore for any $x \in \mathcal{E}$, components of vector $\gamma(x)$ admit a lower bound

$$\gamma_{0_i} = \min\{\gamma_i(x) : \forall x \in \mathcal{E}\} \qquad (5)$$

such that $\gamma_{0_i} \leq \gamma_i(x) \leq 1$. Given such a vector γ_0 we define uncertain vertex matrices $A(t)+B(t)G^k_{\gamma_0}K$ for $k = 1, \ldots, 2^m$, where matrices $G^k_{\gamma_0}$ are the 2^m vertex diagonal matrices whose elements can take the value 1 or γ_{0_i}. In the polyhedron

$$\mathcal{S}_{\gamma_0} = \{x : |K_i x| \leq w_i/\gamma_{0_i}\} \supset \mathcal{E} \qquad (6)$$

we use difference inclusions results to describe system (1)-(2) by

$$\dot{x}(t) = A_{cl}(t)x(t) \tag{7}$$

where closed-loop matrix $A_{cl}(t)$ belongs to a polytope of uncertain matrices

$$\mathcal{A}_{cl} = \text{convex hull}\{A(t) + B(t)G^1_{\gamma_0}K, \ldots, A(t) + B(t)G^{2^m}_{\gamma_0}K\}.$$

In view of inclusion relation (6), non-linear system (1)-(2) may be represented by polytopic system (7) within $\mathcal{S}_{\gamma_0}$. Note that this result was already derived in (Henrion et al., 1997; Tarbouriech and Garcia, 1997; Gomes da Silva and Tarbouriech, 1998). It is reformulated here for the sake of clarity.

With these notations, standard results on quadratic stability (Barmish, 1985) entail that a sufficient condition for polytopic system (7) to be robustly stable is that the Lyapunov matrix P introduced in Definition 1 is simultaneously solution to the following family of matrix inequalities, depending on the uncertainty being considered:

- Norm Bounded Uncertainty

$$\begin{bmatrix} (A_0 + B_0G^k_{\gamma_0}K)'P + P(A_0 + B_0G^k_{\gamma_0}K) & PD & E_1' + K'G^k_{\gamma_0}E_2' \\ D'P & -I & 0 \\ E_1 + E_2G^k_{\gamma_0}K & 0 & -I \end{bmatrix} < 0 \tag{8}$$

  for all $k = 1, \ldots, 2^m$, where I denotes an identity matrix of appropriate dimension.

- Polytopic Uncertainty

$$(A_i + B_jG^k_{\gamma_0}K)'P + P(A_i + B_jG^k_{\gamma_0}K) < 0 \tag{9}$$

  for all $i = 1, \ldots, n_A$, $j = 1, \ldots, n_B$ and $k = 1, \ldots, 2^m$.

In the above inequalities, decision variables are Lyapunov matrix P and saturation vector γ_0, whose components are entries of diagonal matrices $G^k_{\gamma_0}$. Recall that the equivalence between non-linear system (1)-(2) and polytopic system (7) holds within $\mathcal{S}_{\gamma_0}$, which in turn must inscribe Lyapunov level set $\mathcal{E}$. This inclusion can also be written as a matrix inequality

$$\begin{bmatrix} P & \gamma_{0i}K_i' \\ \gamma_{0i}K_i & rw_i^2 \end{bmatrix} \geq 0 \tag{10}$$

for $i = 1, \ldots, m$, where decision variables are matrix P, scalars γ_{0i} and scalar r (Henrion et al., 1997; Tarbouriech and Garcia, 1997; Gomes da Silva and Tarbouriech, 1998). The following result has been shown.

Theorem 1 *If there exist a positive definite symmetric matrix P, a vector γ_0 of components $0 < \gamma_{0i} \leq 1$ and a positive scalar r such that matrix inequalities (8) or (9) and (10) hold, then the Lyapunov level set $\mathcal{E}$ defined in (4) is solution to Problem 1.*

Depending on how the size of an ellipsoid is measured, there are various candidates to the largest ellipsoid introduced in the statement of Problem 2. Some of them are enumerated next, but this list is far from being exhaustive. The interested reader may consult (Gomes da Silva and Tarbouriech, 1999, §3) for a discussion on this topic.

Corollary 1 *In Problem 2, maximization of the initial condition ellipsoid $\mathcal{E}$ given in Theorem 1 can be performed by minimizing r,* trace P *or* $\log\det P$ *subject to matrix inequalities (8) or (9) and (10).*

Note that matrix inequalities (8) and (9) are bilinear (BMIs) in decision variables P and γ_0, whereas (10) is an LMI in P, γ_0 and r. In (Goh et al., 1996) it has been shown that many of the problems considered in the robust control literature can be formulated as BMIs. In view of Theorem 1 and Corollary 1, this is also the case for Problems 1 and 2.

LMI Relaxations

At this point, it must be emphasized that our problem formulation entails various sources of potential conservatism such as the representation of uncertainty or the existence of a unique Lyapunov matrix shared by all matrix polytope vertices. Moreover, there exist several global optimization methods for solving the BMI problem introduced in Corollary 1, but their worst-case complexities are exponential and the computational effort that would be required may be unreasonably large. As a consequence, when tackling Problem 2 we need not necessarily to solve the BMI optimization problem for its global optimum. We will be satisfied with an approximate feasible solution, i.e. a sufficiently large size solution to Problem 1. As pointed out in (Boyd and Vandenberghe, 1997), there are several ways to approximate solutions to BMI optimization problems using efficient polynomial-time algorithms. Since bilinear matrix inequalities (8) or (9) become linear as soon as one decision variable is fixed, in this note we have opted for simple approximation schemes based on LMI Relaxations (LMIR). Following Corollary 1, the relaxations that have proved useful are listed below.

LMIR 1	Given P, solve for γ_0 and r, $\min r$	subject to LMIs (8) or (9) and (10).
LMIR 2	Given P, solve for γ_0 and r, $\min(r + \sum_{i=1}^{m} \gamma_{0_i})$	
LMIR 3	Given γ_0 and r, solve for P, $\min \operatorname{trace} P$	
LMIR 4	Given γ_0 and r, solve for P, $\min \log\det P$	

It must be pointed out that LMIR 4 is actually performed by minimizing the convex criterion $\log\det W^{-1}$ upon a suitable change of variable $W = P^{-1}$, see (Gomes da Silva and Tarbouriech, 1998).
Our algorithm alternates between the four relaxation procedures described above and provides a solution to Problem 1 and an approximate solution to Problem 2.

Algorithm

Step 1 Let $\gamma_{0_1} = \cdots = \gamma_{0_m} = r = 1$. Solve for P the LMI set (8) or (9) and (10).
Step 2 Solve LMIR 1, 2, 3 or 4.
Step 3 If some conditions on the size of $\mathcal{E}$ are fulfilled, then stop.
Otherwise, go to Step 2.

Remark 1 *In the above algorithm, the key point is the selection of relaxation procedure in Step 2. At first glance, it does not seem to be possible to know beforehand which sequence features the best rate of convergence. Our implementation of the algorithm allows the user to select the relaxation procedure of his choice at each step.*

Remark 2 *The stopping criterion in Step 3 depends on the algorithm behavior. Clearly, further iterations always increase the size of $\mathcal{E}$. As there are no convergence properties available for this type of relaxation algorithms, one can, for instance, decide to stop the process if there is no significant variation on the size of $\mathcal{E}$.*

Our method is now illustrated on two examples borrowed from the system and control literature. In both cases, it is shown that our approach significantly improves previously published results. For solving LMI optimization problems, we have used the MATLAB[1] LMI solver described in (Gahinet et al., 1995) conjointly with the MAXDET software proposed in (Wu et al., 1996).

FIRST EXAMPLE: NORM BOUNDED UNCERTAINTY

We consider the uncertain system studied in (Kim and Bien, 1994) with nominal matrices

$$A_0 = \begin{bmatrix} 0.1 & -0.1 \\ 0.1 & -3 \end{bmatrix} \quad B_0 = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}.$$

In (Kim and Bien, 1994), matrices of system (1) are affected by additive unstructured uncertainties of norm less than or equal to 0.1. Following our formalism, we have opted for a norm bounded description of this uncertainty with matrices

$$D = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad E_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \end{bmatrix} \quad E_2 = 1.$$

In addition to that, as in (2) actuators are subject to saturation levels $w_1 = 5$ and $w_2 = 2$. The goal of the authors in (Kim and Bien, 1994) was to derive the largest set of initial conditions for which the optimal control state feedback

$$u = -\begin{bmatrix} 0.7283 & 0.0338 \\ 0.0135 & 1.3583 \end{bmatrix} x$$

ensures robust stabilization even when saturation occurs. The largest initial condition set they obtained was

$$x(0) \in \mathcal{Q} = \{x : x'x \leq 2265\}.$$

In Tables 1 and 2, we report two sequences of LMI relaxations together with the volume[2] ratios between ellipsoid $\mathcal{E}$ obtained at each step of our iterative algorithm and sphere $\mathcal{Q}$ given above. After a few iterations, we were able to expand the set of safe initial conditions by a factor of 676.8 in the first case and 1877 in the second case. These sets are represented on Figure 1. This illustrates the substantial improvement brought by our method over the analysis approach pursued in (Kim and Bien, 1994).

LMIR #	2	3	2	3	2	3
Volume Ratio	0.0428	0.2123	0.2123	0.2125	21.8	**676.8**

Table 1: First Relaxation Sequence

[1]Matlab is a trademark of the MathWorks, Inc.

[2]The volume of ellipsoid $\{x : x'Px \leq 1\}$ is proportional to $(\det P)^{-\frac{1}{2}}$.

LMIR #	4	2	4	2
Volume Ratio	0.2125	0.2125	59.41	**1877**

Table 2: Second Relaxation Sequence

Second Example: Polytopic Uncertainty

The linearized equations of motion for a satellite given in (Dolphus and Schmitendorf, 1991) yield the following matrices for system (1):

$$A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3p(t)^2 & 0 & 0 & 2p(t) \\ 0 & 0 & 0 & 1 \\ 0 & -2p(t) & 0 & 0 \end{bmatrix} \quad B(t) = B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

Time-varying uncertain parameter $p(t)$ represents the period of rotation and lies within the interval $[0.5, 1.5]$. The saturation levels in (2) are $\omega_1 = \omega_2 = 15$. In order to represent uncertainty on $p(t)$, we assume that matrix $A(t)$ belongs to the polytope

$$\mathcal{A} = \text{convex hull} \left\{ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.75 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.75 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0.75 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.75 & 0 & 0 & 3 \\ 0 & 0 & 0 & 1 \\ 0 & -3 & 0 & 0 \end{bmatrix} \right\}.$$

In (Dolphus and Schmitendorf, 1991), the authors show that the linear state feedback

$$u = \begin{bmatrix} -19 & -13 & -2 & 0 \\ 0 & 2 & -7 & -8 \end{bmatrix} x \tag{11}$$

stabilizes the saturated linear system for any initial condition in the unit sphere, i.e.

$$x(0) \in \{x : x'x \leq 1\}.$$

when $p(t)$ is set to 1. However, as stated in (Dolphus and Schmitendorf, 1991), robust feedback (11) is not necessarily stabilizing for all admissible uncertainties. In other words, the unit sphere is not a set of safe initial conditions when $p(t)$ is varied. For the sake of comparison, we applied our LMI relaxation method. The volume ratios between the ellipsoids $\mathcal{E}$ obtained through our algorithm and the unit sphere are

- 7.90 for one iteration with LMIR 3
- 11.30 for one iteration with LMIR 4.

Other relaxation schemes or further iterations did not significantly improve these figures. These results clearly illustrate the relevance of our approach since our algorithm returns a set $\mathcal{E}$ of initial conditions which are guaranteed to be stabilized even when $p(t)$ varies. It must be noticed that, although the volume of $\mathcal{E}$ is larger than the volume of the unit sphere, the latter set is not entirely contained into the former when feedback (11) is applied. Nevertheless, another saturating linear control law is proposed in (Henrion et al., 1997) so that this inclusion holds.

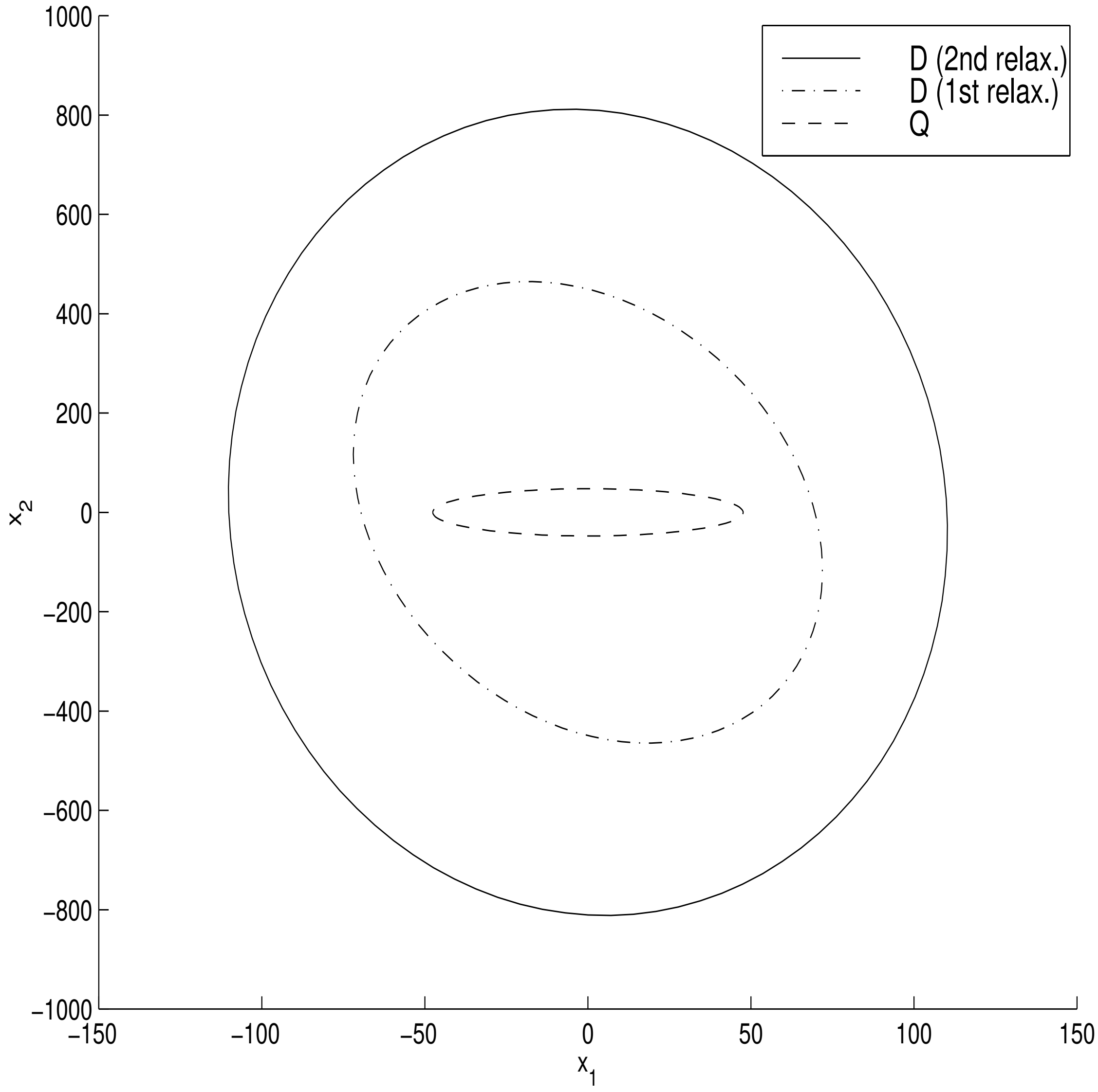

Figure 1: Comparison of initial condition sets

Concluding Remarks

Based on the local equivalence between a non-linear saturated system and a linear polytopic system, we have proposed a BMI method for analyzing robust stability of an uncertain linear system subject to actuator saturation. A tractable algorithm has been devised that relies upon LMI relaxation procedures.

In contrast with constrained robust design methods, generally aimed at avoiding actuator saturation while preserving closed-loop performance (Henrion et al., 1998), our approach explicitly handles control saturation. However, it should be interesting to study whether both approaches could be combined and what are the trade-offs between system performance and the size of the set of safe initial conditions.

Another direction of research could be of analyzing saturating systems with other uncertainty models. Indeed, positive real or bounded real uncertainties may sometimes prove less conservative than norm bounded uncertainties (Daafouz, 1997, §II.1).

In spite of the inevitable conservatism induced by the polytopic representation of saturation non-linearities, the existence of a unique and constant Lyapunov matrix and the non-convex nature of the BMI formulation, we believe that our LMI relaxation algorithm will fulfill the need for efficient, widely spread tractable numerical methods for solving this kind of difficult control problem.

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