Symmetric Matrix Polynomial Equation: Interpolation Results ⋆

An interpolation approach is pursued to solve a bilateral matrix polynomial equation frequently arising in control and signal processing. It results in efficient and numerically reliable resolution methods.

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Abstract

New numerical procedures are proposed to solve the symmetric matrix polynomial equation $A^T(-s)X(s) + X^T(-s)A(s) = 2B(s)$ that is frequently encountered in control and signal processing. An interpolation approach is presented that takes fully advantage of symmetry properties and leads to an equivalent reduced-size linear system of equations. It results in a simple and general characterization of all solutions of expected column degrees. Several new theoretical results concerning stability theory and reduced Sylvester resultant matrices are also developed and used to conclude a priori on the existence of a solution. By means of numerical experiments, it is shown that our algorithms are more efficient than older methods and, namely, appear to be numerically reliable.

Key words: Matrix polynomial equations, Numerical methods, Interpolation algorithms, Efficient algorithms, Continuous-time systems, Computer-aided control system design.

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1 Introduction

Symmetric matrix polynomial equation of the form

$$A^T(-s)X(s) + X^T(-s)A(s) = 2B(s) \tag{1}$$

where $A(s)$ is a given square polynomial matrix, $B(s)$ is a para-Hermitian matrix verifying the symmetry property

$$B(s) = B^T(-s) \tag{2}$$

and $X(s)$ is a polynomial matrix to be found, arises in various areas of system control theory.

So polynomial matrix spectral factorization, a standard technique applied in many H_2 control and filtering problems [3], is traditionally handled by Newton-Raphson iterative scheme consisting of solution to equation (1) in every step. Its overall performance naturally hinges on the way equation (1) is processed.

In a famous paper published in Automatica [Ježek and Kučera, 1985], a technique was presented that became standard [4]: after the calculation of a polynomial adjoint matrix, the resolution of (1) can be reduced to the application of an Euclidean division algorithm (or Schur reduction) to a set of scalar polynomial equations. The availability of efficient numerical methods to compute the adjoint of a polynomial matrix [Stefanidis *et al.*, 1992] nevertheless contrasts with the poor stability properties of the reduction process.

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[2] The most part of this work was done while this author was with the Institute of Information Theory and Automation, Prague, Czech Republic.

[3] Such as LQ optimal regulation, Kalman filtering, minimum variance LQG control, etc.

[4] A decade ago, this solver made it possible for the overall spectral factorization to perform numerically at least as well as standard ARE solution methods of that time. Since then, a substantial progress has been achieved in the solution of ARE. However, the method for (1) proposed here is believed to help the Newton-Raphson scheme to make spectral factorization again competitive.

Another algorithm based on triangularization was presented in [Ježek, 1986] but only for the discrete-time case and also with relatively bad numerical properties since still limited to elementary polynomial operations.

In a more general context the interpolation approach, comprehensively exposed in [Antsaklis and Gao, 1993], appears as a numerically reliable, flexible and easy-to-implement technique to solve unilateral matrix polynomial equations (including for instance the Diophantine equation). Their generalization of the scalar polynomial interpolation to the matrix case is shown to be particularly well suited to studying and solving a wide range of control problems. In a unified way, they provide new interpretations and demonstrations of well-known control theorems.

Finally, since equations involving just polynomials can also be solved by equating coefficients of equal powers of the indeterminate s, alternate methods of this type are extensively described in [Stefanidis *et al.*, 1992]. They are based on the Sylvester resultant formulation of the product of two polynomial matrices [Chen, 1984,Kailath, 1980]. The need of inversion of relatively high order constant matrices is not a drawback anymore, in view of the availability of powerful linear equation solvers.

This paper aims to prove that the interpolation approach developed in [Antsaklis and Gao, 1993] can be extended to handle the bilateral symmetric equation (1). In Section 2, it is first shown that, by taking advantage of symmetry property (2), para-Hermitian matrix $B(s)$ can be represented by a restricted number of interpolation points. Using this fact, an interpolation formulation of equation (1) is proposed in §3 that results in a characterization of all solutions to (1) of expected column degrees. Symmetry properties are handled to significantly reduce the original problem dimension. In Section 4 some matrix manipulations are needed to obtain a reduced Sylvester resultant matrix formulation of the bilateral symmetric equation. In addition to provide an alternative method to solve (1), it throws light on relations with polynomial stability theory and with Hurwitz matrices [Barnett, 1983,Lancaster and Tismenetsky, 1985]. Moreover, the use of interpolation theory mixed with rank properties of the reduced resultant matrix will lead under some assumptions on $A(s)$ to conditions for existence of solutions to (1). In §5, an algorithm based on the reduced resultant formulation is first derived to solve (1). It is then followed by two simple interpolation algorithms. Some basic examples underline the simplicity of the methods. Finally, numerical experiments are reported that aim at demonstrating that the proposed algorithms are more efficient and reliable than already existing methods.

As far as the authors are aware, this appears to be the first numerically reliable solution technique for (1) ever published [5]. As another advantage, no a priori assumptions on $A(s)$ and $B(s)$ are necessary for the algorithm. Hence critical cases such as for instance the presence of purely imaginary values as roots of $A(s)$ are handled without taking special precautions. If, all the same, matrix $A(s)$ is known to be stable (this is always the case when applied for spectral factorization), the method gets even simpler as now a solution is guaranteed to exist.

The algorithms presented here have been tested experimentally by running programs written in MATLAB [6] code. They are available as parts of the Polynomial Toolbox [7] being prepared [Kwakernaak and Sebek, 1997].

2 Description using interpolation of para-Hermitian polynomial matrices

In this section two methods to represent a para-Hermitian polynomial matrix using interpolation points are discussed. They parallel Theorem 2.1 in [Antsaklis and Gao, 1993] and take fully advantage of symmetry properties to reduce the original data dimensions.

For any n-by-n polynomial matrix $X(s)$, $\mathrm{vec}X(s)$ is the polynomial column vector obtained by stacking together columns of $X(s)$.

Given an n-by-n para-Hermitian polynomial matrix $B(s) = B_0 + B_1 s + \cdots + B_{d_B} s^{d_B}$, the following operators are defined :

- $\mathrm{vec}_{\mathrm{t}} B$ is a column vector obtained by stacking for $k = 0, 1, \ldots, d_B$
  - · the $n-i+1$ last elements of the i-th column of B_k for $i = 1, 2, \ldots, n$ when k is even (that is, the lower left triangular part of B_k)
  - · the $n - i$ last elements of the i-th column of B_{k+1} for $i = 1, 2, \ldots, n-1$ when k is odd (that is, the strictly lower left triangular part of B_{k+1})
- $\widehat{\mathrm{vec}}_{\mathrm{t}} B$ is a reduced column vector, derived from $\mathrm{vec}_{\mathrm{t}} B$ by suppressing all zero leading coefficients of polynomial entries of $B(s)$.

[5] Next to its control applications, equation (1) is often encountered in the field of signal processing. There it is namely used in the polynomial approach to computation of an ARMA covariance function [Södeström *et al.*, 1998].

[6] MATLAB is a trademark of The MathsWorks, Inc.

[7] A pre-commercial version of the Polynomial Toolbox has been released on July 1, 1997. It is now available as freeware through the Toolbox WWW home page at the URL `http://www.math.utwente.nl/polbox`.

Example 1 *For instance, if*

$$B(s) = \begin{bmatrix} 1 & -4s+2 \\ 4s+2 & 5s^2+3 \end{bmatrix}$$

then $\mathrm{vec_t}\,B = [1\ 2\ 3\ 4\ 0\ 0\ 5]^T$ *and* $\widehat{\mathrm{vec}}_\mathrm{t}\,B = [1\ 2\ 3\ 4\ 5]^T$.

Polynomial equation (2) can be arranged by ascending powers of s to give

$$(B_0 - B_0^T) + (B_1 + B_1^T)s + (B_2 - B_2^T)s^2 + \cdots = 0.$$

Hence for k even

- B_k is symmetric
- B_{k+1} is skew-symmetric. (3)

Entries of $\mathrm{vec}B(s)$ are also entries of $B(s)$, so each component of $\mathrm{vec}B(s)$ can be written as the product of a polynomial vector basis by a constant vector of coefficients of $B(s)$. Since $\mathrm{vec_t}\,B$ and $\widehat{\mathrm{vec}}_\mathrm{t}\,B$ consist of these coefficients, define $S(s)$ and $\widehat{S}(s)$ as the unique polynomial matrices such that for any $B(s)$,

$$\mathrm{vec}B(s) = S(s)\,\mathrm{vec_t}\,B \tag{4}$$

and

$$\mathrm{vec}B(s) = \widehat{S}(s)\,\widehat{\mathrm{vec}}_\mathrm{t}\,B. \tag{5}$$

The above relations play the same role than equation (2.4) in [Antsaklis and Gao, 1993].

Example 2 *For the matrix $B(s)$ given in Example 1, equation (4) reads*

$$\mathrm{vec}B(s) = \left[\begin{array}{ccc|c|ccc} 1 & & & & s^2 & & \\ & 1 & & s & & s^2 & \\ & 1 & & -s & & s^2 & \\ & & 1 & & & & s^2 \end{array}\right] \left[\begin{array}{c} 1 \\ 2 \\ 3 \\ \hline 4 \\ \hline 0 \\ 0 \\ 5 \end{array}\right] = S(s)\,\mathrm{vec_t}\,B$$

and equation (5) becomes

$$\mathrm{vec}B(s) = \left[\begin{array}{ccc|c|c} 1 & & & & \\ & 1 & & s & \\ & 1 & & -s & \\ & & 1 & & s^2 \end{array}\right] \left[\begin{array}{c} 1 \\ 2 \\ 3 \\ \hline 4 \\ \hline 5 \end{array}\right] = \widehat{S}(s)\, \widehat{\mathrm{vec}}_{\mathrm{t}} B.$$

Finally for a fixed j let $a_j \neq 0$ denote an n^2-complex row vector and b_j, s_j complex scalars. With these notations, the following theorem provides an interpolation description of $B(s)$.

Theorem 3 *Given an n-by-n symmetric matrix $\Delta = [d_{ijB}]$ of non-negative integers with even diagonal elements and*

$$\hat{q} = \frac{1}{2}n(n+1) + \frac{1}{2}\sum_{i,j=1}^{n} d_{ijB}$$

interpolation triplets (s_k, a_k, b_k), $k = 1, 2, \ldots, \hat{q}$ such that the $\hat{q}$-by-$\hat{q}$ matrix

$$\widehat{S}_{\hat{q}} = \begin{bmatrix} a_1 \widehat{S}(s_1) \\ a_2 \widehat{S}(s_2) \\ \vdots \\ a_{\hat{q}} \widehat{S}(s_{\hat{q}}) \end{bmatrix} \tag{6}$$

is non-singular, there exists a unique *n-by-n para-Hermitian matrix $B(s)$ with entry degrees d_{ijB} for which*

$$a_k \mathrm{vec}B(s_k) = b_k \tag{7}$$

for $k = 1, 2, \ldots, \hat{q}$.

PROOF. This follows by direct application of basic interpolation Theorem 2.1 in [Antsaklis and Gao, 1993]. Note that $\widehat{S}(s)$ only depends on Δ (not on $B(s)$) and is made of $n(n+1)/2 + \sum_{i,j=1}^{n} d_{ijB}/2$ columns and n^2 rows, hence justifying the number of interpolation points. Since Δ contains entry degrees of $B(s)$, substitution of (7) into (5) yields

$$\widehat{S}_{\hat{q}}\, \widehat{\mathrm{vec}}_{\mathrm{t}} B = [b_1\, b_2\, \cdots\, b_{\hat{q}}]^T = T_{\hat{q}}. \tag{8}$$

Since $\widehat{S}_{\hat{q}}$ is invertible, $B(s)$ is uniquely specified by $\widehat{\mathrm{vec}}_{\mathrm{t}} B$. $\square$

A less general version of Theorem 3 can be derived if only d_B, the degree of $B(s)$, is considered instead of each entry degree. Matrix $\widehat{S}(s)$ is then replaced by $S(s)$.

Corollary 4 *Given a non-negative integer d_B and*

$$\begin{array}{ll} \bullet\ q = \frac{1}{2}n^2(d_B+1) & \text{if } d_B \text{ is odd} \\ \bullet\ q = \frac{1}{2}n(n+1) + \frac{1}{2}n^2 d_B & \text{otherwise} \end{array} \tag{9}$$

interpolation triplets (s_k, a_k, b_k) for $k = 1, 2, \ldots, q$ such that the q-by-q matrix

$$S_q = \begin{bmatrix} a_1 S(s_1) \\ a_2 S(s_2) \\ \vdots \\ a_q S(s_q) \end{bmatrix}$$

is non-singular, there exists a unique *n-by-n para-Hermitian matrix $B(s)$ of degree d_B for which (7) holds for $k = 1, 2, \ldots, q$.*

PROOF. See Theorem 3. The number of interpolation points can be obtained by inspection of the number of columns of matrix $S(s)$ in (4) and clearly verifies $q \geq \hat{q}$. That is, the lack of information on entry degrees of $B(s)$ is somehow compensated by a greater number of interpolation points. $\square$

Example 5 *For the matrix $B(s)$ given in Example 1, $d_B = 2$ and*

$$\Delta = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}.$$

Theorem 3 therefore needs $\hat{q} = 5$ interpolation points, while Corollary 4 makes use of $q = 7$ points to uniquely characterize $B(s)$. These values correspond to respective column dimensions of matrices $\widehat{S}(s)$ and $S(s)$ in Example 2.

3 Interpolation Formulation of the Symmetric Matrix Polynomial Equation

Some useful facts from the theory of Kronecker product and permutation matrices are first recalled. These results are then used to derive an alternative expression for the symmetric matrix polynomial equation. Finally, an equivalence is established between solutions $X(s)$ to (1) of expected column degrees and solutions to a linear system of equations derived by interpolation.

3.1 Preliminaries: Permutation Matrix and Kronecker Product

Let $G(s)$, $H(s)$ and $X(s)$ denote n-by-n polynomial matrices. The stacking operator vec and the Kronecker product $\otimes$ verify [Lancaster and Tismenetsky, 1985]

$$\mathrm{vec}[G(s)X(s)H(s)] = [H^T(s) \otimes G(s)]\mathrm{vec}X(s).$$

The n^2-by-n^2 constant matrix P such that

$$\mathrm{vec}X^T(s) = P\mathrm{vec}X(s) \tag{10}$$

is called *permutation matrix* and can be used to commute the Kronecker product since

$$P^T[G(s) \otimes H(s)]P = H(s) \otimes G(s). \tag{11}$$

Note that P is involutory ($P^2 = I_{n^2}$), symmetric and therefore orthogonal ($P^T P = I_{n^2}$). For details the reader is referred to [Lancaster and Tismenetsky, 1985].

3.2 Alternative Formulation of the Symmetric Matrix Polynomial Equation

Let $A(s) = A_0 + sA_1 + \ldots + s^{d_A}A_{d_A}$ and $X(s) = X_0 + sX_1 + \ldots + s^{d_X}X_{d_X}$.

The Kronecker product formulation of (1) leads to

$$[I_n \otimes A^T(-s)]\,\mathrm{vec}X(s) + [A^T(s) \otimes I_n]\,\mathrm{vec}X^T(-s) = 2\,vecB(s). \tag{12}$$

According to (10) and (11) and with

$$Q(s) = I_n \otimes A^T(s) = Q_0 + sQ_1 + \ldots + s^{d_A}Q_{d_A}$$

equation (12) becomes

$$Q(-s)\,\mathrm{vec}X(s) + PQ(s)\,\mathrm{vec}X(-s) = 2\,\mathrm{vec}B(s). \tag{13}$$

In addition to that, suppose that matrices $A(s)$ and $X(s)$ have respective column degrees d_{iA} and d_{iX} for $i = 1, 2, \ldots, n$. With the notation

$$X_j = [x_1^{(j)}\; x_2^{(j)}\; \cdots\; x_n^{(j)}]$$

to distinguish between columns of matrix coefficient X_j for $j = 0, 1, \ldots, d_X$, it is useful to note that for $i = 1, 2, \ldots, n$ and $j > d_{iX}$

$$x_i^{(j)} = \mathbf{0} \tag{14}$$

and then that corresponding entries of $\mathrm{vec}X_j$ are zero.

Finally, define $\mathrm{vec}X$ as the constant column vector obtained by stacking together vectors $\mathrm{vec}X_j$ for $j = 0, 1, \ldots, d_X$.

By regrouping ascending powers of s of the right hand-side, polynomial equation (13) can be written

$$L(s)\,\mathrm{vec}X = 2\,\mathrm{vec}B(s) \tag{15}$$

$$L(s) = [L_{+1}(s) \quad sL_{-1}(s) \quad s^2L_{+1}(s)\ \ldots] \tag{16}$$

and

$$\begin{aligned} L_{+1}(s) &= Q(-s) + PQ(s) \\ L_{-1}(s) &= Q(-s) - PQ(s). \end{aligned}$$

Since components of $\mathrm{vec}X$ corresponding to columns (14) are zero, a more compact formulation of (15) can be written. For that, define $\widehat{\mathrm{vec}}X$ as the reduced column vector obtained by stacking for $j = 0, 1, \ldots, d_X$ columns $x_i^{(j)}$ of X_j for which $j \leq d_{iX}$. That is to say, $\widehat{\mathrm{vec}}X$ is obtained from $\mathrm{vec}X$ by suppressing zero components corresponding to columns (14).

Rewriting of (15) follows from appropriate reduction of $L(s)$ into a matrix $\widehat{L}(s)$ of compatible column dimension :

$$L(s)\,\mathrm{vec}X = \widehat{L}(s)\,\widehat{\mathrm{vec}}X = 2\,\mathrm{vec}B(s). \tag{17}$$

Bilateral equation (1) is thus equivalent to unilateral equation (17). The polynomial unknown $X(s)$ is replaced by the constant vector unknown vec$X$ or $\widehat{\text{vec}}X$ containing coefficients of $X(s)$.

3.3 Interpolation Results

If now $\widetilde{B}(s)$ is a para-Hermitian matrix such that

$$2\,\text{vec}\widetilde{B}(s) = \widehat{L}(s)\,\widehat{\text{vec}}X \tag{18}$$

then by inspection, degree entries of $\widetilde{B}(s)$ verify for $i, j = 1, 2, \ldots, n$

$$\begin{aligned}
&\bullet\ d_{ii\tilde{B}} = d_{iA} + d_{iX} \ \text{ if } d_{iA} + d_{iX} \text{ is even} \\
&\bullet\ d_{ii\tilde{B}} = d_{iA} + d_{iX} - 1 \ \text{ if } d_{iA} + d_{iX} \text{ is odd} \\
&\bullet\ d_{ij\tilde{B}} = \max(d_{iA} + d_{jX}, d_{jA} + d_{iX}) \text{ when } i \neq j.
\end{aligned} \tag{19}$$

Now according to Theorem 3, $\widetilde{B}(s)$ is uniquely specified by

$$\hat{q} = \frac{1}{2}n(n+1) + \frac{1}{2}\sum_{i,j=1}^{n} d_{ij\tilde{B}} \tag{20}$$

interpolation points. If triplets (s_k, a_k, b_k) for $k = 1, 2, \ldots, \hat{q}$ are chosen as in Theorem 3 and such that

$$a_k \text{vec}\widetilde{B}(s_k) = a_k \text{vec}B(s_k) = b_k \tag{21}$$

then to set $s = s_k$ and to pre-multiply (15) by a_k leads to

$$\widehat{L}_{\hat{q}}\,\widehat{\text{vec}}X = 2\,T_{\hat{q}} \tag{22}$$

where

$$\widehat{L}_{\hat{q}} = \begin{bmatrix} a_1\widehat{L}(s_1) \\ a_2\widehat{L}(s_2) \\ \vdots \\ a_{\hat{q}}\widehat{L}(s_{\hat{q}}) \end{bmatrix} \tag{23}$$

has dimensions $\hat{q}$-by-$n(n + \sum_{i=1}^{n} d_{iX})$ and $T_{\hat{q}}$ is defined in (8).

We are now in position to formulate our main result.

Theorem 6 *Given $A(s)$ and $B(s)$ in (1), select non-negative integers d_{iX} to satisfy for $i, j = 1, 2, \ldots, n$*

$$d_{ijB} \leq \max(d_{iA} + d_{jX}, d_{jA} + d_{iX}) \tag{24}$$

where d_{iA} are column degrees of $A(s)$ and d_{ijB} entry degrees of $B(s)$. Moreover, suppose that linear system (22) is built from interpolation triplets verifying the assumptions of Theorem 3. Then a solution $X(s)$ to (1) of (highest) column degrees d_{iX} exists if and only if *a solution $\widehat{\mathrm{vec}}X$ to linear system (22) exists.*

PROOF. If a solution to (1) of column degrees d_{iX} exists, then clearly a solution to (22) exists. In view of (19), inequality (24) is also necessary.

Conversely, if a solution to (22) exists then according to (8) and (21),

$$\widehat{L}_{\hat{q}}\ \widehat{\mathrm{vec}}X = 2\ \widehat{S}_{\hat{q}}\ \widehat{\mathrm{vec}}_{\mathrm{t}}\widetilde{B} = 2\ \widehat{S}_{\hat{q}}\ \widehat{\mathrm{vec}}_{\mathrm{t}}B$$

which implies $\widetilde{B}(s) = B(s)$ since $\widehat{S}_{\hat{q}}$ is non-singular and $\widehat{\mathrm{vec}}_{\mathrm{t}}B$ uniquely characterizes $B(s)$ in Theorem 3. In view of (18), polynomial equation (13) admits a solution $X(s)$ with column degrees d_{iX}. $\square$

Theorem 6 shows that there exists a one-to-one correspondence between matrix polynomial solutions to (1) with given column degrees and constant vector solutions to linear system of equations (22). Therefore to apply numerical methods to (22) provides a way to characterize all solutions of expected column degrees to the symmetric matrix polynomial equation.

Note however that Theorem 6 does not specify how to find the interpolation triplets required to built linear system (22) or even guarantee the existence of a solution to (1). These points will be considered later on in Sections 4 and 5.

Furthermore, it is important to recall that up to now, *no assumptions* were made concerning polynomial matrices $A(s)$ and $B(s)$.

A simplified result can be derived if now $d_{iX} = d_X$ and $d_{iA} = d_A$ for $i = 1, 2, \ldots, n$. Inequality (24) then reads

$$d_{ijB} \leq d_B \leq d_A + d_X \tag{25}$$

which is clearly necessary for a polynomial solution $X(s)$ to (1) to exist. Now consider Theorem 6 with $\widehat{L}(s) = L(s)$ and $\widehat{\mathrm{vec}}X = \mathrm{vec}X$. According to Corollary 4, para-Hermitian matrix $B(s)$ is interpolated using q points and one obtains the linear system

$$L_q \operatorname{vec}X = 2\, T_q \tag{26}$$

where L_q is q-by-$n^2(d_X+1)$. Since now only the knowledge of d_B is required to select d_X, the following corollary to Theorem 6 has been shown.

Corollary 7 *Given $A(s)$ and $B(s)$ in (1), select non-negative integer d_X to satisfy (25). Suppose moreover that linear system (26) is built from interpolation triplets verifying the assumptions of Corollary 4. Then a solution $X(s)$ of (highest) degree d_X exists* if and only if *a solution* vec$X$ *to linear system (26) exists.*

Note that Corollary 7 needs more interpolation points than Theorem 6 and thus that linear system (26) is of greater size than linear system (22).

Although this is out of the scope of this paper, it is worthwhile to keep in mind that interpolation also provides useful flexibility properties. As pointed out in [Antsaklis and Gao, 1993], a crucial advantage of interpolation over other approaches is that input matrices need not be fixed. In other words, using interpolation one could solve problems where $B(s)$ is not necessarily fixed in equation (1) but is known, say, through the zeros of its determinant. This remarkable feature is specific to interpolation.

4 Reduced Resultant Formulation of the Symmetric Matrix Polynomial Equation

Algorithms in [Stefanidis *et al.*, 1992] are based on the fact that equations involving polynomials can be solved by equating coefficients of equal powers of the indeterminate s, or equivalently that the product of two polynomial matrices can be formulated via multiplication of two constant matrices. One of them is called the Sylvester resultant matrix [Chen, 1984,Kailath, 1980]. Results concerning unilateral polynomial equations involving matrix product can therefore be derived from a corresponding constant matrix formulation.

The present section aims to show that such a technique can also be applied to bilateral equations. After some preliminaries, an equivalence between polynomial solutions to (1) and constant solutions to a linear system based on a resultant matrix is stated. In particular it is shown that the dimensions of the original problem can be significantly reduced. In addition to provide an alternative way to solve (1), links with stability theory and Hurwitz matrices are enlightened. Relations with the interpolation method are drawn and both techniques are shown to be closely related. Moreover the resultant approach turns out to be useful when combined with interpolation results. In particular it is possible, under some assumptions on $A(s)$, to give *a priori* conditions of existence of solutions.

4.1 Preliminaries: Projectors and Related Matrices

Stacking operators $\mathrm{vec}B$, $\mathrm{vec_t}B$ and the n^2-by-n^2 permutation matrix P were defined respectively in Sections 2 and 3.1.

The matrix

$$P_{+1} = (I_{n^2} + P)/2$$

is called *projector* and

$$P_{-1} = I_{n^2} - P_{+1} = (I_{n^2} - P)/2$$

is its *complementary projector*. By simple inspection one can check that

$$\begin{aligned} \text{rank } P_{+1} &= n(n+1)/2 \\ \text{rank } P_{-1} &= n(n-1)/2. \end{aligned}$$

Hence let us define

- $\tilde{\Pi}_{+1}$ as a matrix with $n(n+1)/2$ rows corresponding to the linearly independent rows of P_{+1}
- $\tilde{\Pi}_{-1}$ as a matrix with $n(n-1)/2$ rows corresponding to the linearly independent rows of P_{-1}

and

$$\begin{aligned} \Pi_{+1} &= \begin{bmatrix} \tilde{\Pi}_{+1} \\ \mathbf{0} \end{bmatrix} \\ \Pi_{-1} &= \begin{bmatrix} \mathbf{0} \\ \tilde{\Pi}_{-1} \end{bmatrix} \end{aligned}$$

as square matrices of size n^2. According to properties (3), for k even

$$\begin{aligned} \tilde{\Pi}_{+1}\mathrm{vec}B_k &= \mathrm{vec}_{\mathrm{t}} B_k \\ \tilde{\Pi}_{-1}\mathrm{vec}B_{k+1} &= \mathrm{vec}_{\mathrm{t}} B_{k+1} \\ [\Pi_{+1} \quad \Pi_{-1}] \begin{bmatrix} \mathrm{vec}B_k \\ \mathrm{vec}B_{k+1} \end{bmatrix} &= \mathrm{vec}_{\mathrm{t}}(B_k + sB_{k+1}) \end{aligned} \tag{27}$$

and

$$\begin{aligned} \tilde{\Pi}_{-1}\mathrm{vec}B_k &= \mathbf{0} \\ \tilde{\Pi}_{+1}\mathrm{vec}B_{k+1} &= \mathbf{0} \\ [\Pi_{-1} \quad \Pi_{+1}] \begin{bmatrix} \mathrm{vec}B_k \\ \mathrm{vec}B_{k+1} \end{bmatrix} &= \mathbf{0}. \end{aligned} \tag{28}$$

Note finally that

$$\mathrm{rank} \begin{bmatrix} \tilde{\Pi}_{+1} \\ \tilde{\Pi}_{-1} \end{bmatrix} = n^2 \tag{29}$$

and whereas Π_{+1} and Π_{-1} are both singular, that

$$\mathrm{rank}\,[\Pi_{+1} \quad \Pi_{-1}] = \mathrm{rank} \begin{bmatrix} \Pi_{+1} \\ \Pi_{-1} \end{bmatrix} = n^2. \tag{30}$$

4.2 Reduced Resultant Formulation

Suppose that inequality (25) holds, i.e. $d_X \geq d_B - d_A \geq 0$. This is a natural hypothesis since only polynomial solutions to (1) are expected. Finally without loss of generality, assume that $d_B = d_A + d_X$ [8].

[8] If $d_B < d_A + d_X$ simply note $B(s) = B(s) + B_{d_B+1}s^{d_B+1} + \cdots + B_{d_B+\delta}s^{d_B+\delta}$ where $\delta = d_A + d_X - d_B$ and B_{d_B+i} for $i = 1, 2, \ldots, \delta$ are zero matrices. Then let $d_B = d_B + \delta$.

In Section 3.2 the Kronecker product formulation of (1) has been shown to lead to equation (13) :

$$Q(-s)\,\mathrm{vec}X(s) + PQ(s)\,\mathrm{vec}X(-s) = 2\,\mathrm{vec}B(s).$$

The Sylvester resultant formulation is derived by equating coefficients of the powers of s in the above matrix polynomial equation :

$$T\begin{bmatrix}\mathrm{vec}X\\ \mathrm{vec}X\end{bmatrix} = 2\mathrm{vec}B \tag{31}$$

where

$$T = \left[\begin{array}{cccc|cccc}
Q_0 & & & \mathbf{0} & PQ_0 & & & \mathbf{0}\\
-Q_1 & Q_0 & & & PQ_1 & -PQ_0 & & \\
\vdots & -Q_1 & \ddots & Q_0 & \vdots & -PQ_1 & \ddots & (-1)^{d_X}PQ_0\\
(-1)^{d_A}Q_{d_A} & \vdots & & -Q_1 & PQ_{d_A} & \vdots & & (-1)^{d_X}PQ_1\\
 & (-1)^{d_A}Q_{d_A} & \ddots & \vdots & & -PQ_{d_A} & \ddots & \vdots\\
\mathbf{0} & & & (-1)^{d_A}Q_{d_A} & \mathbf{0} & & & (-1)^{d_X}PQ_{d_A}
\end{array}\right] \tag{32}$$

is an $n^2(d_A+d_X+1)$-by-$2n^2(d_X+1)$ constant matrix. Pre- and post-multiplication by the following matrices transform T into block-diagonal form :

$$\frac{1}{2}\begin{bmatrix}M_1\\ M_2\end{bmatrix} T \begin{bmatrix}I & I\\ I & -I\end{bmatrix} = \begin{bmatrix}R_{d_X} & \mathbf{0}\\ \mathbf{0} & R^*_{d_X}\end{bmatrix} \tag{33}$$

with if $m = d_A + d_X + 1$ is even

$$\begin{bmatrix}M_1\\ M_2\end{bmatrix} = \begin{bmatrix}I_{\frac{m}{2}} \otimes [\Pi_{+1} \quad \Pi_{-1}]\\ I_{\frac{m}{2}} \otimes [\Pi_{-1} \quad \Pi_{+1}]\end{bmatrix} \tag{34}$$

and otherwise

$$\begin{bmatrix}M_1\\ M_2\end{bmatrix} = \left[\begin{array}{cc}
I_{\frac{m-1}{2}} \otimes [\Pi_{+1} \quad \Pi_{-1}] & \mathbf{0}\\
\mathbf{0} & \widetilde{\Pi}_{+1}\\ \hline
I_{\frac{m-1}{2}} \otimes [\Pi_{-1} \quad \Pi_{+1}] & \mathbf{0}\\
\mathbf{0} & \widetilde{\Pi}_{-1}
\end{array}\right]. \tag{35}$$

Properties (29) and (30) show that (34) and (35) are invertible matrices and therefore that (33) is an equivalence relation. Note that matrix R_{d_X} has row dimension $n^2m/2$ when m is even and $n^2(m-1)/2 + n(n+1)/2$ otherwise. After substitution of (33) into (31), one obtains

$$\begin{bmatrix} R_{d_X} \operatorname{vec} X \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} M_1 \operatorname{vec} B \\ M_2 \operatorname{vec} B \end{bmatrix}. \tag{36}$$

According to equation (27) the upper part of the right hand-side matrix in (36) is $\operatorname{vec}_{\mathrm{t}} B$ and the lower part is readily seen to be zero by property (28). Therefore (36) is equivalent to

$$R_{d_X} \operatorname{vec} X = \operatorname{vec}_{\mathrm{t}} B, \tag{37}$$

hence leading to a significant reduction of original problem dimensions.

Now suppose that matrix $X(s)$ has column degrees d_{iX} for $i = 1, 2, \ldots, n$ and that $\widehat{\operatorname{vec}} X$ is obtained from $\operatorname{vec} X$ by suppressing zero components associated to columns (14). Moreover, suppose that in a similar fashion, corresponding columns of R_{d_X} are left out to yield a matrix $\widehat{R}_{d_X}$ such that

$$\widehat{R}_{d_X} \widehat{\operatorname{vec}} X = \operatorname{vec}_{\mathrm{t}} B. \tag{38}$$

The equivalence between polynomial matrix solutions to (1) and constant vector solutions to (37) and (38) is summarized by the following statements, which are respective alternatives to Theorem 6 and Corollary 7.

Theorem 8 *Given $A(s)$ and $B(s)$ in (1), select non negative integers $d_{iX} \leq d_X$ for $i = 1, 2, \ldots, n$ such that d_X satisfy (25). Then a solution $X(s)$ to (1) of column degrees d_{iX} exists if and only if a solution $\widehat{\operatorname{vec}} X$ to linear system (38) exists.*

Corollary 9 *Given $A(s)$ and $B(s)$ in (1), select non negative integer d_X to satisfy (25). Then a solution $X(s)$ to (1) of degree d_X exists if and only if a solution* $\operatorname{vec} X$ *to linear system (37) exists.*

4.3 Relation with Stability Theory and Hurwitz Matrices. Rank Properties of the Reduced Resultant

A particularly interesting case which deserves a mention is when

$$d_X = d_A - 1 \tag{39}$$

and

$$\begin{aligned} &\bullet\ A(s) \text{ is regular, i.e. } A_{d_A} \text{ is non-singular} \\ &\bullet\ \text{determinants of } A(s) \text{ and } A(-s) \text{ are relatively prime.} \end{aligned} \tag{40}$$

Lemma 10 *Under assumptions (39) and (40), the reduced resultant matrix* R_{d_X} *in (37) is* non-singular.

PROOF. Matrix R_{d_X} in (37) is square and built from a Sylvester resultant matrix in (32) which is made of four square Toeplitz block-matrices of striped form :

$$T = \left[\begin{array}{cccc|cccc} Q_0 & & & \mathbf{0} & PQ_0 & & & \mathbf{0} \\ -Q_1 & Q_0 & & & PQ_1 & -PQ_0 & & \\ \vdots & & \ddots & & \vdots & & \ddots & \\ (-1)^{m-1}Q_{m-1} & & \cdots & Q_0 & PQ_{m-1} & & \cdots & (-1)^{m-1}PQ_0 \\ \hline (-1)^m Q_m & & \cdots & -Q_1 & PQ_m & & \cdots & (-1)^m PQ_1 \\ & (-1)^m Q_m & & Q_2 & & -PQ_m & & (-1)^m PQ_2 \\ & & \ddots & \vdots & & & \ddots & \vdots \\ \mathbf{0} & & & (-1)^m Q_m & \mathbf{0} & & & (-1)^m PQ_m \end{array}\right].$$

Theorem 4.26 in [Barnett, 1983] states that for $A(s)$ *regular* the above matrix is non-singular if and only if the determinants of $A(s)$ and $A(-s)$ are *relatively prime*. Furthermore, in view of equivalence relation (33), T non-singular implies R_{d_X} non-singular. $\square$

In the scalar case ($n = 1$) R_{d_X} can be directly related to the Hurwitz matrix of polynomial $A(s)$ [Lancaster and Tismenetsky, 1985], which is known to be non-singular when $A(s)$ does not have symmetric roots with respect to the imaginary axis. A simple example should help to clarify that.

Example 11 *Consider the symmetric polynomial equation* ($n = 1$)

$$a(-s)x(s) + x(-s)a(s) = 2b(s)$$

where $a(s) = a_0 + a_1 s + a_2 s^2 + a_3 s^3$, $x(s) = x_0 + x_1 s + x_2 s^2$ *and* $b(s) = b_0 + b_2 s^2 + b_4 s^4$. *Note that assumption (39) is verified and therefore that* R_{d_X}

will be square. Using the Sylvester resultant formulation, (31) reads

$$\left[\begin{array}{ccc|ccc} a_0 & 0 & 0 & a_0 & 0 & 0 \\ -a_1 & a_0 & 0 & a_1 & -a_0 & 0 \\ a_2 & -a_1 & a_0 & a_2 & -a_1 & a_0 \\ \hline -a_3 & a_2 & -a_1 & a_3 & -a_2 & a_1 \\ 0 & -a_3 & a_2 & 0 & -a_3 & a_2 \\ 0 & 0 & -a_3 & 0 & 0 & a_3 \end{array}\right] \begin{bmatrix} x_0 \\ x_1 \\ x_2 \\ x_0 \\ x_1 \\ x_2 \end{bmatrix} = 2 \begin{bmatrix} b_0 \\ 0 \\ b_2 \\ 0 \\ b_4 \\ 0 \end{bmatrix}$$

which after reduction becomes (37)

$$R_2 \, \mathrm{vec} X = \begin{bmatrix} a_0 & 0 & 0 \\ a_2 & -a_1 & a_0 \\ 0 & -a_3 & a_2 \end{bmatrix} \begin{bmatrix} x_0 \\ x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_0 \\ b_2 \\ b_4 \end{bmatrix} = \mathrm{vec}_{\mathrm{t}} B.$$

The Hurwitz matrix of $a(s)$ defined in [Lancaster and Tismenetsky, 1985] is

$$H(a) = \begin{bmatrix} a_2 & a_0 & 0 \\ a_3 & a_1 & 0 \\ 0 & a_2 & a_0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{bmatrix} R_2^T \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

and therefore $a(s)$ strictly stable and regular ($a_3 \neq 0$) implies $H(a)$ and R_2 non-singular.

The following extension of Lemma 10 will be useful when combined with interpolation results.

Lemma 12 *Under assumptions (40) and if*

$$d_X \geq d_A - 1 \tag{41}$$

the reduced resultant matrix R_{d_X} in (37) has full row-rank.

PROOF. The reduction process of Theorem 8 leads to a reduced resultant of the form

$$R_{d_X} = \left[\begin{array}{c|c} R_S & R_C \\ \hline \mathbf{0} & R_R \end{array}\right]$$

where R_S is the non-singular reduced resultant corresponding to $d_X = d_A - 1$ (see Lemma 10) and R_R is a matrix whose rows are obtained after reduction of two block triangular matrices with respectively Q_{d_A} and PQ_{d_A} on their diagonals. By assumption (40), Q_{d_A} is non-singular and therefore R_R has full row-rank. $\square$

Note that Lemma 12 is a strengthened version of Lemma 10 since equality (39) is only a particular case of inequality (41).

4.4 Comparison with Interpolation Formulation

It is stimulating to understand relations existing between linear system (26) :

$$L_q \, \mathrm{vec} X = 2\, T_q$$

obtained in the scope of interpolation, and linear system (37) :

$$R_{d_X} \, \mathrm{vec} X = \mathrm{vec}_{\mathrm{t}} B$$

derived from the Sylvester resultant theory.

By construction one can check that matrix $L(s)$ given in (17) verifies

$$L(s) = 2\, S(s)\, R_{d_X} \tag{42}$$

where $S(s)$ was defined in (4) and R_{d_X} is the reduced resultant of Corollary 9. Pre-multiplication of (42) by a_j at points s_j for $j = 1, 2, \ldots, q$ yields

$$L_q = 2\, S_q\, R_{d_X} \tag{43}$$

Since $S_q \, \mathrm{vec}_{\mathrm{t}} B = T_q$ one can conclude that linear system (26) is simply linear system (37) multiplied to the left by non-singular matrix $2\, S_q$.

4.5 Existence of Solution

It is often of interest to know beforehand whether a solution to (1) can be found or not. The following conditions for the symmetric matrix polynomial equation is established in this regard.

Lemma 13 *Given $A(s)$ and $B(s)$ in (1) where $A(s)$ verifies assumptions (40), select d_X such that (41) holds. Then a solution $X(s)$ of (highest) degree d_X to (1) exists and is given by linear system of equation (26).*

PROOF. Consider Corollary 7 where equivalence between existence of a polynomial solution to (1) and existence of a constant solution to linear system (26) was established.

Observe that a solution vec$X$ to (26) exists if and only if

$$\text{rank}\,[L_q \quad T_q] = \text{rank}\; L_q. \tag{44}$$

A sufficient condition for (44) to hold is that L_q has *full row-rank*. Otherwise, some relations between L_q and T_q (that is between $A(s)$ and $B(s)$) must exist. Since S_q is invertible, equation (43) yields

$$\text{rank}\; L_q = \text{rank}\; R_{d_X}. \tag{45}$$

According to Lemma 12 the corresponding reduced resultant R_{d_X} has full row-rank. Equation (45) shows that this is also the case for L_q and therefore that (26) always has a solution. □

5 Numerical Methods

It is now possible to expose numerical methods to solve (1). Even if our main impetus is to expose an interpolation technique to solve the symmetric bilateral matrix polynomial equation, the results developed in Section 4.2 will be first shown to supply an alternative resolution method. Two interpolation algorithms are then described. The choice of interpolation points, a crucial point, is discussed at length. By the use of simple examples these methods are shown to be easily extended to handle cases like presence of purely imaginary zeros or special requirements on the expected solution. Finally, comparative numerical results are exposed that clearly demonstrate the efficiency and reliability of our algorithms over already existing techniques.

5.1 Reduced Resultant Algorithm

According to Theorem 8, there is a correspondence between polynomial solutions to (1) and constant vector solutions to the reduced linear system (38) obtained by equating the coefficients of s. This leads to the subsequent straightforward formulation of a first algorithm to solve (1).

Algorithm 1 *(1) Given $A(s)$, $B(s)$ in (1), select d_{iX} for $i = 1, 2, \ldots, n$ to be expected column degrees of solution $X(s)$.*
(2) Using relation (33), build matrix $\widehat{R}_{d_X}$ and linear system (38).
(3) Each solution $\widehat{\mathrm{vec}} X$ to linear system (38) corresponds to a solution $X(s)$ to (1) of column degrees d_{iX}.

5.2 Interpolation Algorithm

In contrast with Algorithm 1, the following procedures are now based on interpolation and are implementations of Theorem 6 and related linear system (22).

Algorithm 2 *(1) Given $A(s)$, $B(s)$ in (1), select d_{iX} for $i = 1, 2, \ldots, n$ to be expected column degrees of solution $X(s)$. They must verify (24).*
(2) Compute $d_{ij\tilde{B}}$ using (19), and then the number of interpolation points $\hat{q}$ according to (20).
(3) Build $\widehat{L}(s)$ by selecting columns of $L(s)$ in (16) corresponding to d_{iX}.
(4) Select distinct *scalars s_i and* random *vectors a_i for $i = 1, 2, \ldots, \hat{q}$ to build $\widehat{L}_{\hat{q}}$ in (23) and $T_{\hat{q}}$ in (8).*
(5) Each solution $\widehat{\mathrm{vec}} X$ to linear system (22) corresponds to a solution $X(s)$ to (1) of column degrees d_{iX}.

Theorem 6 states that in order to build linear system (22), interpolation points must verify the assumptions of Theorem 3, that is, matrix $\widehat{S}_{\hat{q}}$ in (6) must be non-singular. Lemma 2.4 in [Antsaklis and Gao, 1993] shows that for a choice of $\hat{q}$ distinct scalars s_i, $\widehat{S}_{\hat{q}}$ is non-singular for almost any set of vectors a_i, thus justifying the use of random vectors in step (4) of the above algorithm.

However, the special structure of $\widehat{S}(s)$ suggests a more selective choice for vectors a_i and scalars s_i. In that sense, the modification to Algorithm 2 described below allows to build linear system (22) given a restricted number, say n_e, of scalars t_i verifying for $i, j = 1, 2, \ldots, n_e$ and $i \neq j$

$$|t_i| \neq |t_j| \neq 0 \tag{46}$$

and with a relatively simple set of vectors a_i such that matrix $\widehat{S}_{\hat{q}}$ is guaranteed to be non-singular. Therefore Theorem 6 can also be invoked.

Algorithm 3 *only step (4) differs from Algorithm 2:*

(4) Construct matrix $\Delta^* = [d^*_{ijB}]$ *where*

$$\begin{aligned}
&\bullet\ d^*_{ijB} = \tfrac{1}{2}d_{ij\tilde{B}} + 1 \quad \text{if } d_{ij\tilde{B}} \text{ is even and } i \geq j \\
&\bullet\ d^*_{ijB} = \tfrac{1}{2}d_{ij\tilde{B}} \qquad\quad \text{if } d_{ij\tilde{B}} \text{ is even and } i < j \\
&\bullet\ d^*_{ijB} = \tfrac{1}{2}(d_{ij\tilde{B}} + 1) \ \text{if } d_{ij\tilde{B}} \text{ is odd.}
\end{aligned} \tag{47}$$

Choose $n_e = \max\{d^*_{ijB} \,:\, i,j = 1,2,\ldots,n\}$ *distinct scalars* t_i *verifying (46). Then for* $i = 1,2,\ldots,n_e$
(a) set $s = t_i$ *in (17) to obtain the linear system*

$$\widehat{L}(t_i)\widehat{\mathrm{vec}}X = 2\,\mathrm{vec}B(t_i) \tag{48}$$

(b) for $k = 1,2,\ldots,n^2$, *if the k-th component of* $\mathrm{vec}\Delta^*$ *is greater than or equal to i, the k-th row of linear system (48) is selected to build linear system (22).*

Row selection step (4b) corresponds, for a fixed $s = t_i$, to choose a set a_j of *unitary* vectors. Since polynomial equations (1) and (17) are equivalent, the above procedure amounts

- to evaluate the symmetric matrix polynomial equation at n_e interpolation points t_i verifying assumption (46)
- to interpolate a solution $X(s)$ by selecting a sufficient amount of data in terms of rows of equation (48) when i varies from 1 to n_e.

Note that the total number of rows selected is $\sum_{i,j=1}^{n} d^*_{ijB} = \hat{q}$.

Furthermore, matrix $\widehat{S}_{\hat{q}}$ in (6) can be checked (after some row and column permutations) to be a block diagonal matrix compound of *Vandermonde matrices* which, for choice (46) of scalars t_i, are known to be invertible, as shown by the simple following example.

Example 14 *Consider the strictly Hurwitz polynomial matrix*

$$A(s) = \begin{bmatrix} s^2 + 3s + 1 & s + 1 \\ -1 & 2s^2 + 4s \end{bmatrix}$$

and the para-Hermitian polynomial matrix

$$B(s) = \begin{bmatrix} s^2+1 & \frac{1}{2}(s^3-2s^2-s+1) \\ \frac{1}{2}(-s^3-2s^2+s+1) & s^2+1 \end{bmatrix}.$$

First of all we study solutions $X(s)$ *to (1) with column degrees* $d_{1X}=0$ *and* $d_{2X}=1$. *Inequalities (24) are verified and polynomial matrix* $\widetilde{B}(s)$ *in (18) has entry degrees*

$$\begin{aligned} \tilde{d}_{11} &= d_{1X}+d_{1A} = 2 \\ \tilde{d}_{12} &= \max(d_{1X}+d_{2A}, d_{2X}+d_{1A}) = 3 \\ \tilde{d}_{21} &= \tilde{d}_{12} = 3 \\ \tilde{d}_{22} &= d_{2X}+d_{2A}-1 = 2. \end{aligned} \tag{49}$$

The number of points required to interpolate $\widetilde{B}(s)$ *in Theorem 3 is then* $\hat{q} = 3+(2+3+3+2)/2 = 8$. *With* $d_{1X}=0$ *and* $d_{2X}=1$, *polynomial matrix* $\widehat{L}(s)$ *in (17) is built on* $L_{+1}(s)$ *and only the two last columns of* $sL_{-1}(s)$ *to give*

$$\widehat{L}(s) = \left[\begin{array}{cccc|cc} 2s^2+2 & -2 & 0 & 0 & 0 & 0 \\ -s+1 & 2s^2-4s & s^2+3s+1 & -1 & -s^3-3s^2-s & s \\ s+1 & 2s^2+4s & s^2-3s+1 & -1 & s^3-3s^2+s & -s \\ 0 & 0 & 2 & 4s^2 & -2s^2 & -8s^2 \end{array}\right].$$

Using equalities (47), one obtains $d^*_{ijB}=2$ *for* $i,j=1,2$ *and clearly* $n_e=2$. *Since all elements in matrix* Δ^* *are the same, to set* $s=t_1=1$ *and* $s=t_2=2$ *into equation (17) provides an easy way to write (22)*

$$\begin{bmatrix} \widehat{L}(1) \\ \widehat{L}(2) \end{bmatrix} \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_2^{(1)} \end{bmatrix} = \begin{bmatrix} 2\,\mathrm{vec}B(1) \\ 2\,\mathrm{vec}B(2) \end{bmatrix}. \tag{50}$$

The matrix $\widehat{S}_{\hat{q}}$ *in (6) corresponding to entry degrees (49) reads*

$$\widehat{S}_{\hat{q}} = \begin{bmatrix} \widehat{S}(1) \\ \widehat{S}(2) \end{bmatrix}$$

and after some row and column permutations, $\widehat{S}_{\hat{q}}$ appears to be equivalent to

$$\begin{bmatrix} 1 & t_1^2 & & & & & & \\ 1 & t_2^2 & & & & & & \\ & & 1 & t_1^2 & & & & \\ & & 1 & t_2^2 & & & & \\ & & & & 1 & t_1 & t_1^2 & t_1^3 \\ & & & & 1 & -t_1 & t_1^2 & -t_1^3 \\ & & & & 1 & t_2 & t_2^2 & t_2^3 \\ & & & & 1 & -t_2 & t_2^2 & -t_2^3 \end{bmatrix}.$$

For choice (46) of scalars t_i, that is $|t_1| \neq 0$, $|t_2| \neq 0$, $|t_1| \neq |t_2|$, this block Vandermonde matrix is non-singular and Theorem 6 can be applied. Linear system (50) is composed of 8 *equations for* 6 *scalar unknowns and admits a (unique) solution $\widehat{\mathrm{vec}}X = [1\ 0\ 1\ 1\ 1\ 0]^T$, i.e.*

$$X(s) = \begin{bmatrix} 1 & s+1 \\ 0 & 1 \end{bmatrix} \tag{51}$$

is the unique solution to (1) with column degrees $d_{1X} = 0$ and $d_{2X} = 1$.

If d_{1X} is now expected to be 1*, the values of $d_{ij\tilde{B}}$ in (49) are not modified in view of (19). The characterization of all solutions $X(s)$ to (1) is then direct, since $A(s)$ verify assumptions (40), inequality (41) holds and the number of interpolation points given by Corollary 13 is still* 8*. L_q is known by advance to have full row-rank and is checked to be square. The unique solution to (1) of degree $d_X = 1$ is therefore still (51).*

5.3 Critical Case

Since Theorem 6 does not make any assumption on $A(s)$ and $B(s)$, it can also handle the case when $A(s)$ has *purely imaginary* zeros. Clearly Lemma 13 cannot be applied anymore but equation (22) still provides a way to conclude on the existence of a solution.

Example 15 *Let*

$$A(s) = \begin{bmatrix} s+1 & 1 \\ s & 0 \end{bmatrix}$$

and

$$B(s) = \begin{bmatrix} -s^2+1 & \frac{1}{2}(-s+1) \\ \frac{1}{2}(s+1) & 0 \end{bmatrix}$$

polynomial matrices for which all solutions $X(s)$ to (1) with degree $d_X = 1$ have to be found. Note that $A(s)$ has a (purely imaginary) zero at $s = 0$. Inequalities (24) are verified and (19) yields $d_{11\tilde{B}} = 2$, $d_{12\tilde{B}} = 2$ and $d_{22\tilde{B}} = 0$ which set the number of interpolation points to $\hat{q} = 3 + (2+2+2+0)/2 = 6$. Simple polynomial matrix manipulations show that

$$L_{+1}(s) = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 1 & 0 & s+1 & s \\ 1 & 0 & -s+1 & -s \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

and

$$L_{-1}(s) = \begin{bmatrix} -2s & -2s & 0 & 0 \\ 1 & 0 & -s-1 & -s \\ -1 & 0 & -s+1 & -s \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Equations (47) give

$$\Delta^* = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$$

and $n_e = 2$. To set for instance $s = t_1 = 1$ and $s = t_2 = 2$ in (15) provides 8 linear constraints on $\widehat{\text{vec}}X = \text{vec}X$ from which (according to step (4b) in

Algorithm 3) only 6 *are selected to build* $\widehat{L}_{\hat{q}} = L_{\hat{q}}$ *and* $T_{\hat{q}}$. *Equation (22) then reads*

$$\left[\begin{array}{cccc|cccc} 2 & 0 & 0 & 0 & -2 & -2 & 0 & 0 \\ 1 & 0 & 2 & 1 & 1 & 0 & -2 & -1 \\ 1 & 0 & 0 & -1 & -1 & 0 & 0 & -1 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ \hline 2 & 0 & 0 & 0 & -8 & -8 & 0 & 0 \\ 1 & 0 & 3 & 2 & 2 & 0 & -6 & -4 \end{array}\right] \begin{bmatrix} \mathrm{vec} X_0 \\ \mathrm{vec} X_1 \end{bmatrix} = \left[\begin{array}{c} 0 \\ 2 \\ 0 \\ 0 \\ \hline -6 \\ 3 \end{array}\right].$$

It can be verified that

$$\mathrm{rank}\, L_{\hat{q}} = \mathrm{rank}\, [L_{\hat{q}} \quad T_{\hat{q}}] = 5. \tag{52}$$

Using standard numerical tools the following parameterization of all solutions $X(s)$ *of degree* $d_X = 1$ *to (1) is obtained :*

$$X(s) = \begin{bmatrix} (1-a)s + 1 & -bs \\ as + c & bs + a - b \end{bmatrix}$$

where a, b *and* c *are any scalars.*

5.4 Additional Constraints

It is worthwhile to note that when in system (22) there are more unknowns than equations, i.e

$$n(n + \sum_{i=1}^{n} d_{iX}) > \hat{q},$$

this freedom can be handled so that $X(s)$ satisfies supplementary constraints. Linear system (22) can then be augmented of $n(n + \sum_{i=1}^{n} d_{iX}) - \hat{q}$ additional linear equations on elements of $\widehat{\mathrm{vec}} X$, that is on scalar coefficients of $X(s)$:

$$\begin{bmatrix} \widehat{L}_{\hat{q}} \\ D \end{bmatrix} \widehat{\mathrm{vec}} X = \begin{bmatrix} 2T_{\hat{q}} \\ E \end{bmatrix}.$$

Example 16 *To force $X(s)$ in Example 15 to be for instance upper triangular is equivalent to add to system (22) the two following linear constraints :*

$$\left[\begin{array}{cccc|cccc} 0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \end{array}\right] \begin{bmatrix} \mathrm{vec}X_0 \\ \mathrm{vec}X_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

All upper triangular solutions of degree $d_X = 1$ are then

$$X(s) = \begin{bmatrix} s+1 & -ds \\ 0 & ds \end{bmatrix}$$

where d is any scalar. The presence of a parameter is not surprising since the original $L_{\hat{q}}$ does not have full row-rank, as shown by equation (52).

5.5 Computer Simulations

Computer simulations were conducted to evaluate the performance of our methods in comparison with that of [Ježek and Kučera, 1985].

We implemented and compared four numerical routines :

- The elementary polynomial operations algorithm based on Euclidean division, proposed in [Ježek and Kučera, 1985].
- Algorithm 1, resolution of a linear system built on a reduced Sylvester resultant matrix.
- Algorithm 2, resolution of a linear system obtained from random interpolation points.
- Algorithm 3, resolution of a linear system obtained from specific interpolation points.

Some comments regarding implementation are in order.

- Even though performance of Algorithms 1,2 and 3 mainly rely upon the linear system solver being used, we have here simply invoked the MATLAB built-in Gaussian elimination algorithm.
- Crucial steps in the Euclidean division algorithm are the computation of an adjoint polynomial matrix and the extraction of a polynomial matrix left factor (see equations (11) and (12) in [Ježek and Kučera, 1985]). Theoretically, both tasks can be achieved using triangularization via elementary polynomial operations. Practically, numerical results were so poor that we had to implement Sylvester resultant methods instead [Stefanidis *et al.*, 1992].

Elementary polynomial operations are thus only used to solve via Euclidean division scalar versions of polynomial equation (1), that is equation (14) in [Ježek and Kučera, 1985].

For our computer simulations, $A(s)$ and $X(s)$ were chosen to be random square polynomial matrices of size n. For convenience, we supposed that degrees of both matrices were the same, i.e. $d_A = d_X$. Each constant scalar coefficient of polynomial entries of $A(s)$ and $X(s)$ was a Gaussian random variable with zero mean and unitary variance, except for the strictly lower triangular entries of $X(s)$ that were set to zero. Matrix $B(s)$ is evaluated from $A(s)$ and $X(s)$ using equation (1).

Input matrices $A(s)$ and $B(s)$ being given, the performance of each equation solver was measured by the root mean square error, defined as

$$\mathrm{RMSE} = \sqrt{\frac{1}{N_r}\sum_{k=1}^{N_r}\gamma(\hat{X}_k(s) - X(s))^2}$$

where N_r is the number of independent runs (100 in our case), $\hat{X}_k(s)$ denotes the solution computed in the k-th run, $X(s)$ is the exact solution to equation (1) and finally

$$\gamma(X(s)) = \sqrt{(\mathrm{vec}X)^T \mathrm{vec}X}.$$

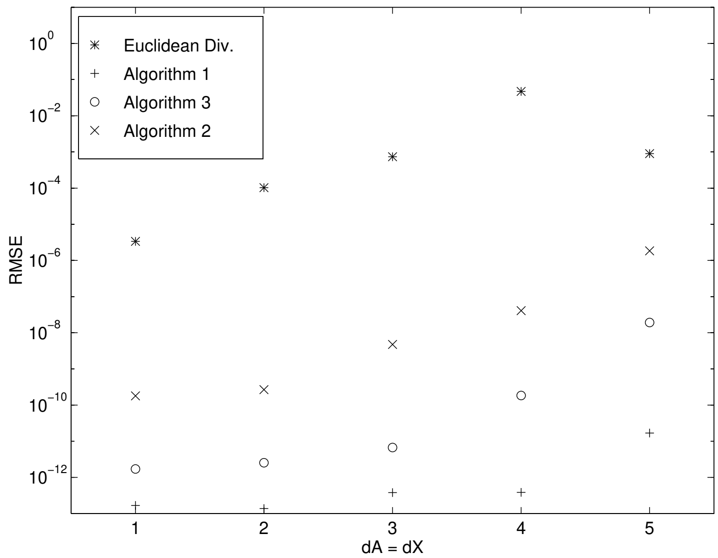

Fig. 1. Comparative performance for $n = 3$.

Figure 1 shows the RMSEs of the four methods as functions of $d_A = d_X$. The following comments naturally arise.

- One can see that our routines are more accurate and reliable than the Euclidean division algorithm. Generally speaking, there exist well-worked standard solvers to deal with linear systems of equations (even more powerful than the one used for our simulations). On the contrary, elementary polynomial operations can behave erratically.
- Interpolation Algorithms 2 and 3 are less accurate than reduced resultant Algorithm 1. We can even notice than the selective choice of Algorithm 3 seems numerically preferable to the random selection of interpolation points in Algorithm 2. Actually, the choice of interpolation points is known to affect the condition number of matrix $\widehat{L}_{\hat{q}}$ in linear system (22). A random choice typically guarantees an adequate conditioning. However, alternative (other that $[1\, s\, s^2 \ldots]$) polynomial basis such as Chebychev polynomials and corresponding interpolation points can improve the condition number many times, see Section 3 in [Antsaklis and Gao, 1993]. This possibility has not been exploited in the present paper.

6 Conclusion

Interpolation is a simple and powerful tool playing a key role in a great number of problems involving polynomials and polynomials matrices. The objective of this paper was to demonstrate that interpolation techniques can also apply to bilateral equation (1). Towards that end, two interpolation formulations of para-Hermitian matrices were proposed that fully take advantage of symmetry properties.

The first one was used together with a Kronecker product version of the symmetric matrix polynomial equation to characterize all solutions $X(s)$ to (1) of expected column degrees via a linear system of equations of reduced size. An efficient numerical method was proposed in that sense and since no assumptions on $A(s)$ or $B(s)$ were made, critical cases such as for instance the presence of purely imaginary values as roots of $A(s)$ are handled without special care. These results gave rise to two algorithms based on interpolation. The choice of interpolation triplets was shown to be the crucial step and was discussed at length.

The second one was combined with a reduced Sylvester resultant formulation of (1) to provide a way to conclude a priori on the existence of a solution, assuming now special requirements on $A(s)$. This development was also motivated for theoretical reasons since some relations with stability theory and Hurwitz matrices have been shown to exist. Moreover, the reduced resultant approach provides an alternative method to solve the symmetric matrix polynomial equation. An algorithm illustrated that.

Some simple examples were then proposed to demonstrate that, in addition to a straightforward numerical method to solve (1), the linearity of the given equivalent constant formulation can be exploited to handle the case when more than one solution exists and to select among possible choices the ones corresponding to requirements like non-singularity or triangularity of $X(s)$.

Finally, reliability and numerical advantages of these linear system algorithms over already existing elementary polynomial operation methods were enlightened by means of computer simulations.

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