H_2 OPTIMAL CONTROL VIA POLE PLACEMENT [1]

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Abstract: The standard H_2 optimal control problem is interpreted as a pole placement design. As a consequence, the optimal controller transfer matrix can be obtained by solving a single one-sided polynomial matrix equation, two spectral factorizations, and three matrix fraction conversions. This makes an attractive alternative to the existing solutions. Examples are included to illustrate the design procedure using Matlab Polynomial Toolbox.

Keywords: Linear Systems, H_2 Control, Pole Placement, Matrix Polynomial Equation.

1. INTRODUCTION

The standard H_2 optimal control problem is a cornerstone of linear control theory, see (Kučera, V., 1996). It consists of stabilizing a linear system in such a way that its transfer matrix attains a minimum norm in the Hardy space H_2.

Consider the feedback configuration shown Fig. 1, where S is the plant and R is the controller.

The plant is modeled by the input–state–output equations

$$\begin{aligned} \dot{x} &= Fx + G_1 v + G_2 u \\ z &= H_1 x + J_{11} v + J_{12} u \\ y &= H_2 x + J_{21} v + J_{22} u \end{aligned}$$

giving rise to the transfer matrix

$$\begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}(s) = \begin{bmatrix} H_1 \\ H_2 \end{bmatrix}(sI-F)^{-1}\begin{bmatrix} G_1 & G_2 \end{bmatrix} + \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}.$$

Denoting $R(s)$ the transfer matrix of the controller, the control system transfer matrix between v and z becomes

$$T(s) = S_{11}(s) + S_{12}(s)\,[I - R(s)\,S_{22}(s)]^{-1} R(s)\,S_{21}(s).$$

The control system is said to be stable if the states of S and R go to zero from all initial values when $v = 0$.

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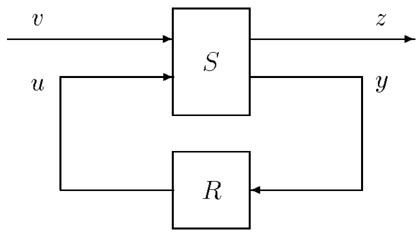

Fig. 1. Control system

The standard H_2 optimal control problem is: Given a plant S, find a controller R that stabilizes the control system and minimizes the H_2 norm of T, defined as

$$||T|| = (\frac{1}{2\pi j} \int \mathrm{tr}\, T'(-s)T(s)\, ds)^{\frac{1}{2}}$$

where tr stands for the trace, T' denotes the transpose of T, and the integral is a contour integral up the imaginary axis and then around an infinite semicircle in the left half–plane.

2. STATE SPACE SOLUTION

The following assumptions on the plant are standard, though not necessary.

$(A.1)$ (F, G_1) stabilizable, (F, H_1) detectable, and $J_{11} = 0$
$(A.2)$ (F, G_2) stabilizable, (F, H_2) detectable, and $J_{22} = 0$
$(A.3)$ $J'_{12}H_1 = 0, \quad J'_{12}J_{12} = I$
$(A.4)$ $G_1 J'_{21} = 0, \quad J_{21}J'_{21} = I$

Under these assumptions, (Doyle, J. C., K. Glover, P. P. Khargonekar, and B. A. Francis, 1989) showed that the unique optimal controller for the H_2 control problem is

$$R(s) = -L(sI - F + KH_2 + G_2L)^{-1}K$$

where

$$K = QH'_2, \quad L = G'_2P$$

and P, Q are the unique, symmetric and non–negative definite solutions of the algebraic Riccati equations

$$F'P + PF - PG_2G'_2P + H'_1H_1 = 0$$
$$FQ + QF' - QH'_2H_2Q + G_1G'_1 = 0.$$

Moreover, the matrices $F - G_2L$ and $F - KH_2$ are stable.

The controller equations can be written in observer form as

$$\dot{w} = Fw + G_2u + K(y - H_2w)$$
$$u = -Lw$$

giving rise to the control system input–state–output equations

$$\begin{bmatrix} \dot{x} \\ \dot{x} - \dot{w} \end{bmatrix} = \begin{bmatrix} F - G_2L & G_2L \\ 0 & F - KH_2 \end{bmatrix} \begin{bmatrix} x \\ x - w \end{bmatrix} + \begin{bmatrix} G_1 \\ G_1 - KJ_{21} \end{bmatrix} v$$

$$z = [H_1 - J_{12}L \quad J_{12}L] \begin{bmatrix} x \\ x - w \end{bmatrix}$$

This reveals the separation property of the H_2 optimal controller: the optimal dynamics is given by $F - G_2L$ and $F - KH_2$ where K and L are determined independently of each other.

3. POLYNOMIAL CONSTRUCTION OF K AND L

The state feedback gain L and the output injection gain K can be calculated by polynomial matrix operations in place of solving the algebraic Riccati equations.

Write

$$H_2(sI - F)^{-1} = A_L^{-1}(s)B_{Ls}(s)$$

where A_L, B_{Ls} are left coprime polynomial matrices such that A_L is row reduced. Further write

$$(sI - F)^{-1}G_2 = B_{Rs}(s)A_R^{-1}(s)$$

where A_R, B_{Rs} are right coprime polynomial matrices such that A_R is column reduced.

Let C_L be a square polynomial matrix with Hurwitz determinant, such that

$$C_L(s)C'_L(-s) = [B_{Ls}(s)G_1 + A_L(s)J_{21}][B_{Ls}(-s)G_1 + A_L(-s)J_{21}]'$$

and let C_R be a square polynomial matrix with Hurwitz determinant, such that

$$C'_R(-s)C_R(s) = [H_1B_{Rs}(-s) + J_{12}A_R(s)]'[H_1B_{Rs}(s) + J_{12}A_R(s)].$$

The matrices C_L and C_R are called spectral factors and are uniquely determined up to right and left orthogonal factors, respectively.

Stronger assumptions are needed to proceed: $(A.2)$ is replaced by

$(A.2)'$ (F, G_2) controllable, (F, H_2)observable and $J_{22} = 0$.

Under these stronger assumptions, (Kučera, V., 1981) showed that

$$K = \hat{Y}_R\hat{X}_R^{-1}$$

where $\hat{X}_R, \hat{Y}_R$ is the unique constant solution pair of the matrix polynomial equation

$$A_L(s)\hat{X}_R + B_{Ls}(s)\hat{Y}_R = C_L(s)$$

and

$$L = \hat{X}_L^{-1}\hat{Y}_L$$

where $\hat{X}_L, \hat{Y}_L$ is the unique constant solution pair of the matrix polynomial equation

$$\hat{X}_LA_R(s) + \hat{Y}_LB_{Rs}(s) = C_R(s).$$

We note that if $(A.2)'$ fails to hold, the polynomial matrix equations may have more than one constant solution pair. Hence K and/or L cannot be determined.

4. TRANSFER MATRIX CONSTRUCTION OF R

Consider a controller in standard observer form with some gains K and L. The transfer matrix of the controller will now be constructed directly using the transfer matrix of the plant, namely $S_{22} = H_2(sI - F)^{-1}G_2$.

Write

$$S_{22}(s) = A_L^{-1}(s)B_L(s) = B_R(s)A_R^{-1}(s)$$

where

$$B_L(s) = B_{Ls}(s)G_2, \quad B_R(s) = H_2B_{Rs}(s).$$

Note that $(A.2)'$ implies left coprimeness of A_L, B_L and right coprimeness of A_R, B_R. Define the polynomial matrices

$$C_L(s) = A_L(s) + B_{Ls}(s)K$$
$$C_R(s) = A_R(s) + LB_{Rs}(s).$$

Following (Anderson, B. D. O. and V. Kučera, 1985), let $X_R(s), Y_R(s)$ define any polynomial matrix solution of the equation

$$A_L(s)X_R(s) + B_L(s)Y_R(s) = C_L(s)$$

and

$$W(s) = \Pi(C_RA_R^{-1}Y_R)(s)$$

where $\Pi(\cdot)$ denotes taking the polynomial part. Then the controller transfer matrix is provided by

$$L(sI - F + KH_2 + G_2L)^{-1}K = \bar{Y}_R^{-1}(s)\bar{X}_R(s)$$

where

$$\bar{X}_R(s) = X_R(s)\bar{C}_R(s) + B_R(s)\bar{W}_R(s)$$
$$\bar{Y}_R(s) = Y_R(s)\bar{C}_R(s) - A_R(s)\bar{W}_R(s)$$

and $\bar{C}_R, \bar{W}_R$ is a right coprime polynomial matrix pair for which

$$C_R^{-1}(s)W_R(s) = \bar{W}_R(s)\bar{C}_R^{-1}(s).$$

A dual construction is possible. Let $X_L(s), Y_L(s)$ be any polynomial matrix solution of

$$X_L(s)A_R(s) + Y_L(s)B_R(s) = C_R(s)$$

and

$$W_L(s) = \Pi(Y_LA_L^{-1}C_L)(s).$$

Then the controller transfer matrix is also given by

$$L(sI - F + KH_2 + G_2L)^{-1}K = \bar{X}_L^{-1}(s)\bar{Y}_L(s)$$

where

$$\bar{X}_L(s) = \bar{C}_L(s)X_L(s) + \bar{W}_L(s)B_L(s)$$
$$\bar{Y}_L(s) = \bar{C}_L(s)Y_L(s) - \bar{W}_L(s)A_L(s)$$

and $\bar{C}_L, \bar{W}_L$ is a left coprime polynomial matrix pair for which

$$W_L(s)C_L^{-1}(s) = \bar{C}_L^{-1}(s)\bar{W}_L(s).$$

Observe that

$$A_L(s)\bar{X}_R(s) + B_L(s)\bar{Y}_R(s) = C_L(s)\bar{C}_R(s)$$

and

$$\bar{X}_L(s)A_R(s) + \bar{Y}_L(s)B_R(s) = \bar{C}_L(s)C_R(s).$$

These pole placement equations exhibit the closed loop system dynamics in terms of the zeros of $\det C_L (=$ zeros of $\det \bar{C}_L)$, which are the eigenvalues of $F - KH_2$, and the zeros of $\det C_R (=$ zeros of $\det \bar{C}_R)$, which are the eigenvalues of $F - G_2L$. The role of W_R is to modify C_R for use in the primal equation, whereas W_L matches C_L for use in the dual equation.

5. H_2 CONTROL AS A POLE PLACEMENT

It is now straightforward to combine the results of the previous sections and interpret the H_2 optimal control as a pole placement problem in which the dynamics induced by optimization is provided by the two spectral factors C_L and C_R.

The design algorithms can be summarized as follows.

Algorithm 1

Data: Plant S, either $F, G_1, G_2, H_1, H_2, J_{12}, J_{21}$ or $S_{11}, S_{12}, S_{21}, S_{22}$.
Assumptions: $(A.1), (A.2)', (A.3)$, and $(A.4)$.
Result: H_2 optimal controller transfer matrix R.

Step 1: Determine left coprime polynomial matrices A_L, B_L with A_L row reduced, such that

$$S_{22}(s) = H_2(sI - F)^{-1}G_2 = A_L^{-1}(s)B_L(s)$$

and put

$$\begin{aligned} N_L(s) &= A_L(s)S_{21}(s) \\ &= A_L(s)[H_2(sI - F)^{-1}G_1 + J_{21}]. \end{aligned}$$

Determine right coprime polynomial matrices A_R, B_R with A_R column reduced, such that

$$S_{22}(s) = H_2(sI - F)^{-1}G_2 = B_R(s)A_R^{-1}(s)$$

and put

$$\begin{aligned} N_R(s) &= S_{12}(s)A_R(s) \\ &= [H_1(sI - F)^{-1}G_2 + J_{12}]A_L(s). \end{aligned}$$

Step 2: Compute square polynomial matrices C_L, C_R with Hurwitz determinants, such that

$$C_L(s)C_L'(-s) = N_L(s)N_L'(-s)$$

and

$$C'_R(-s)C_R(s) = N'_R(-s)N_R(s).$$

Step 3: Solve the equation

$$A_L(s)X_R(s) + B_L(s)Y_R(s) = C_L(s)$$

for a polynomial matrix pair X_R, Y_R.
Step 4: Calculate W_R, the polynomial part of $C_R A_R^{-1} Y_R$, and determine right coprime polynomial matrices $\bar{C}_R, \bar{W}_R$ for which

$$C_R^{-1}(s)W_R(s) = \bar{W}_R(s)\bar{C}_R^{-1}(s).$$

Step 5: Obtain the matrices

$$\begin{aligned}\bar{X}_R(s) &= X_R(s)\bar{C}_R(s) + B_R(s)\bar{W}_R(s)\\ \bar{Y}_R(s) &= Y_R(s)\bar{C}_R(s) - A_R(s)\bar{W}_R(s).\end{aligned}$$

Then

$$R(s) = -\bar{Y}_R(s)\bar{X}_R^{-1}(s).$$

Algorithm 2

Data, assumptions, result: same as above.

Step 1: same.
Step 2: same.
Step 3: Solve the equation

$$X_L(s)A_R(s) + Y_L(s)B_R(s) = C_R(s)$$

for a polynomial matrix pair X_L, Y_L.
Step 4: Calculate W_L, the polynomial part of $Y_L A_L^{-1} C_L$, and determine right coprime polynomial matrices $\bar{C}_L, \bar{W}_L$ for which

$$W_L(s)C_L^{-1}(s) = \bar{C}_L^{-1}(s)\bar{W}_L(s).$$

Step 5: Obtain the matrices

$$\begin{aligned}\bar{X}_L(s) &= \bar{C}_L(s)X_L(s) + \bar{W}_L(s)B_L(s)\\ \bar{Y}_L(s) &= \bar{C}_L(s)Y_L(s) - \bar{W}_L(s)A_L(s).\end{aligned}$$

Then

$$R(s) = -\bar{X}_L^{-1}(s)\bar{Y}_L(s).$$

The two dual algorithms merge into one simple algorithm for plants having a scalar control u and a scalar measurement y.

Algorithm 3

Data: Plant S, same as above.
Assumptions: same as above, plus u and y scalar.
Result: H_2 optimal controller transfer function R.

Step 1: Determine coprime polynomials A, B such that

$$S_{22}(s) = H_2(sI - F)^{-1}G_2 = \frac{B(s)}{A(s)}.$$

Put

$$\begin{aligned}N_L(s) &= A(s)S_{21}(s)\\ N_R(s) &= S_{12}(s)A(s).\end{aligned}$$

Step 2: Compute polynomials C_L, C_R with Hurwitz determinants, such that

$$\begin{aligned}C_L(s)C_L(-s) &= N_L(s)N'_L(-s)\\ C_R(-s)C_R(s) &= N'_R(-s)N_R(s).\end{aligned}$$

Step 3: Solve the equation

$$A(s)\bar{X}(s) + B(s)\bar{Y}(s) = C_L(s)C_R(s)$$

for a polynomial pair $\bar{X}_L, \bar{Y}_L$ such that $\deg \bar{Y} < \deg A$. Then

$$R(s) = -\frac{\bar{Y}(s)}{\bar{X}(s)}.$$

Comparing with Algorithms 1 and 2 one finds that

$$\begin{aligned}\frac{\bar{Y}(s)}{A(s)} &= \frac{Y_L(s)C_L(s)}{A(s)} - W_L(s)\\ &= \frac{C_R(s)Y_R(s)}{A(s)} - W_R(s)\end{aligned}$$

is strictly proper by the definitions of W_L and W_R. This justifies Step 3 of Algorithm 3.

6. EXAMPLES

[Example 1]

Consider the two-mass/spring system described in (Wie, B., and D. S. Bernstein, 1990). The system matrices in standard state-space form are given by

$$\left[\begin{array}{c|c|c} F & G_1 & G_2 \\ \hline H_1 & J_{11} & J_{12} \\ \hline H_2 & J_{21} & J_{22} \end{array}\right] = \left[\begin{array}{cccc|cc|c} 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\alpha & \alpha & 0 & 0 & 0 & 0 & \alpha \\ \beta & -\beta & 0 & 0 & \beta & 0 & 0 \\ \hline 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 1 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

where state components are the position of the first body, the position of the second body, the velocity of the first body, and the velocity of the second body. A control force u acts on the first body, and the position y of the second body is measured. The performance variable z is also the position of the second body. Some disturbance v corrupts measurement y and the fourth component of the state vector. Let $\alpha = 0.625$, $\beta = 0.8333333$.

We illustrate the polynomial method and the state-space method to design the controller transfer function that minimizes the H_2 norm of the transfer function between v and z. The corresponding MATLAB scripts based on macros of the POLYNOMIAL TOOLBOX 2.0 – see (Polyx, Ltd., 1998) – are given below.

```
% ----------- Polynomial solution (Algorithm 3) ----------
% -> Step 1 (A = AL = AR, B = BL = BR)
>> [BLs,AL] = ss2lmf(F,eye(4),H2); BL = BLs*G2; NL = BLs*G1+AL*J21;
>> [BRs,AR] = ss2rmf(F,G2,eye(4)); BR = H2*BRs; NR = H1*BRs+J12*AR;
% -> Step 2
% First spectral factorization
>> CL = spcof(NL*NL')
CL =
     0.63 + 1.4s + 2.5s^2 + 1.3s^3 + 1.2s^4
% Second spectral factorization
>> CR = spf(NR'*NR)
CR =
     1 + 2.9s + 4.1s^2 + 2.2s^3 + 1.9s^4
% -> Step 3
% Solve the Diophantine equation for a minimum degree YP
>> [XP,YP] = axbyc(A,B,CL*CR,'min');
>> YP = YP/lcoef(XP), XP = XP/lcoef(XP) % normalize YP and XP
YP =
     0.52 + 2.6s + 0.31s^2 + 1.4s^3
XP =
     2.6 + 4.1s + 4s^2 + 2.3s^3 + s^4

% ---------- State-space solution -----------
>> P = are(F, G2*G2', H1'*H1); L = G2'*P; % Solve first Riccati equation
>> Q = are(F', H2'*H2, G1*G1'); K = Q*H2'; % Solve second Riccati equation
>> [YS, XS] = ss2rmf(F-K*H2-G2*L, K, L); % Convert to transfer function
>> YS = YS/lcoef(XS), XS = XS/lcoef(XS) % normalize YS and XS
YS =
     0.52 + 2.6s + 0.31s^2 + 1.4s^3
XS =
     2.6 + 4.1s + 4s^2 + 2.3s^3 + s^4
```

[Example 2]

Consider the longitudinal aircraft model of an F-15 fighter at flight condition Mach=0.5, Altitude=1000ft and Weight=45000lbs proposed in Example 7.6.2 by (Saberi, A., P. Sannuti, and B. M.Chen, 1995). The control input u represents the usual elevator control and the vehicle states are velocity, angle of attack, pitch rate and pitch attitude. The disturbance inputs for the H_2 norm evaluation are associated with the longitudinal and transverse components of atmospheric turbulence having white noise spectra. We suppose that only the first two components of the state vector are available for feedback. The system matrices in standard state-space form are given by

$$\left[\begin{array}{c|c|c} F & G_1 & G_2 \\ \hline H_1 & J_{11} & J_{12} \\ \hline H_2 & J_{21} & J_{22} \end{array}\right] = \left[\begin{array}{rrrr|rrrr|r} -0.00819936 & -25.708390 & 0 & -32.170950 & 8.19936 & 46.21354 & 0 & 0 & -6.809399 \\ -0.00019451 & -1.276261 & 1 & 0 & 0.19451 & 2.29521 & 0 & 0 & -0.149683 \\ 0.00069573 & 1.021763 & -2.405226 & 0 & -0.69573 & -1.83673 & 0 & 0 & -14.061107 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0.070711 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0.316228 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

We compare the polynomial solution and the state-space solution to find the H_2 optimal controller. The corresponding MATLAB scripts based on macros of the POLYNOMIAL TOOLBOX are reproduced below.

```
% ---------- Polynomial solution (Algorithm 1) ----------
% -> Step 1
>> [BLS, AL] = ss2lmf(F,eye(4),H2); NL = BLS*G1 + AL*J21; BL = BLS*G2;
>> [BRS, AR] = ss2rmf(F,G2,eye(4)); NR = H1*BRS + J12*AR; BR = H2*BRS;
% -> Step 2
>> CL = spcof(NL*NL') % First Hurwitz factor
CL =
     2.015 + 1.426s + 0.03108s^2     1.695 + 1.859s
     2.857 + 1.935s                  3.107 + 4.151s + s^2
>> CR = spf(NR'*NR) % Second Hurwitz factor
CR =
     4.485 + 9.031s + 4.598s^2 + 1.163s^3 + 0.1089s^4
% -> Step 3
>> [XR, YR] = axbyc(AL, BL, CL); % Solve Diophantine equation
% -> Step 4
% Compute the polynomial part WR of CR*inv(AR)*YR as the quotient
% of the division of CR*YL by AL
>> [YL, AL] = lmf2rmf(YR, AR); [WR, RR] = rdiv(CR*YRL, ARL);
>> [WR2, CR2] = lmf2rmf(WR, CR);
% -> Step 5
>> XP = XR*CR2 + BR*WR2; YP = YR*CR2 - AR*WR2; % Controller right MFD..
>> C = lcoef(XR, 'col'); XP = XP/C, YP = YP/C % ..in reduced form
XP =
     1080 + 64.74s + s^2    -104.4 + 0.443s
     108.1 + 104.6s         -9.163 - 7.732s + s^2
YP =
    -0.4276 - 2.638s     0.06557 + 0.2968s

% ---------- State-space solution -----------
>> P = are(F, G2*G2', H1'*H1); L = G2'*P; % Solve first Riccati equation
>> Q = are(F', H2'*H2, G1*G1'); K = Q*H2; % Solve second Riccati equation
>> [YS, XS] = ss2rmf(F-K*H2-G2*L, K, L); % Controller right MFD..
>> C = lcoef(XS, 'col'); YS = YS/C, XS = XS/C % ..in reduced form
YS =
    -0.4276 - 2.638s     0.06557 + 0.2968s
XS =
     1080 + 64.74s + s^2    -104.4 + 0.443s
     108.1 + 104.6s         -9.163 - 7.732s + s^2
```

7. CONCLUSIONS

In this paper, an alternative algorithm has been described for computing an H_2-optimal controller via polynomial methods.

In contrast to the standard polynomial approach to H_2-optimal design, which is based on two coupled two-sided polynomial matrix equations (Kučera, V., 1980) the algorithm proposed here relies only on a single one-sided polynomial matrix equation of lower dimension. Thorough numerical experiments must however be carried out to check whether significant savings are achieved in computational time and memory requirement.

Another point that deserves further attention is an extensive numerical comparison between the polynomial approach proposed in this paper and the state-space approach (Doyle, J. C., K. Glover, P. P. Khargonekar, and B. A. Francis 1989).

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