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Positioning of Symmetric and Non-Symmetric Parts Using Radial and Constant Fields: Computation of all Equilibrium Configurations

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Abstract

Programmable force fields have been used as an abstraction to represent a whole new class of devices that have been proposed for part manipulation. The general idea behind these devices is that a force field is implemented in a plane upone which the part is placed. The forces and torques exerted on the contact surface of the part translate and rotate the part. Manipulation plans for these devices can therefore be considered as strategies for applying a sequence of force fields to bring parts to some desired configuration. Instances of these novel devices are currently implemented using MEMS technology, small airjets, vibration, and small motors. Manipulation in this case is sensorless and non-prehensile and promises to address the handling of very small or very fragile parts, such as electronics components, that can not be handled with conventional pick and place robotics techniques.

In this paper we consider the problem of bringing a part to a stable equilibrium configuration using force fields. We study the combination of a unit radial field with a small constant field. A part placed on the radial field moves toward the origin of the radial field but cannot be oriented due to symmetry. Perturbing the radial field with a constant force field breaks the symmetry and gives rise to a finite number of equilibria. Under certain conditions there is a unique stable equilibrium configuration. In the case when these conditions are not fulfilled, we provide a comprehensive and unified analysis of the problem that leads to an algorithm to compute all stable equilibrium configurations. The paper contains a detailed discussion on how to implement the algorithm for any part. In our analysis, we make extensive use of potential fields. Using the theory of potential fields, the stable equilibrium configurations of a part are equivalent to the local minima of a scalar function. Our work leads to the design of a new generation of efficient, open-loop parts feeders that can bring a part to a desired orientation from any initial orientation without the need of sensing or a clock.

1 Introduction

In automated manufacturing parts are typically stored in boxes and they have to be positioned and oriented before assembly. This task is critical and strongly affects the productivity of the assembly line. Orientation has been traditionally performed by vibratory bowl feeders [25]. Vibratory

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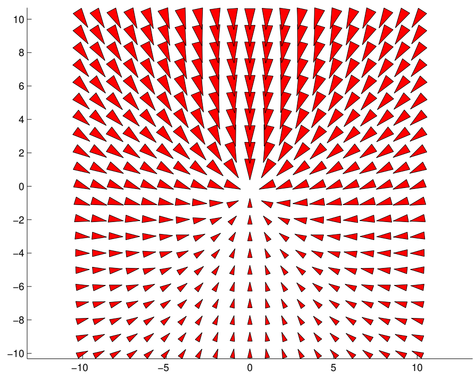

Figure 1: A Unit radial field combined with a constant field of magnitude 0.4 in the vertical direction. (Reprinted from [13].)

bowl feeders are designed to orient a single part shape, therefore they have to be re-designed and re-built to handle different shapes. Some recent research attempts to develop systematic approaches for designing and analyzing vibratory bowl feeders [3, 35, 49], while the mainstream research in manufacturing has focused in developing more flexible and more robust platforms, such as programmable part feeders. This type of part feeder can be programmed to handle different parts without the need for hardware modification [2, 6, 26, 34, 31, 30, 48, 55]. In particular, methods that do not require extensive sensors are favored [2, 26, 31, 34, 55].

Another alternative that has been proposed a few years ago is the use of a new class of non-prehensile devices for distributed manipulation. The general idea behind these devices is that a force field is implemented in a plane and the part is placed on that plane. The forces and torques exerted on the contact surface of the part translate and rotate the part. Examples of such devices can be built, in microscale, with the use of MEMS actuators arrays [21, 7, 15, 8, 14, 20, 19], and in macroscale, with the use of small mechanical motors [45, 46], vibrating plates [6, 51], or airjets actuators [5]. Programmable force fields have been used as an abstraction to represent this new class of devices. Manipulation plans for these devices can be considered as strategies for applying a sequence of fields to bring parts to some desired configurations. Manipulation in this case is sensorless and non-prehensile and promises to address the handling of very small or very fragile parts, such as electronics components, that can not be handled with conventional pick and place robotics techniques.

Several algorithmic results have been published associated with the capabilities and the limitations of these new devices. A theory for programmable force fields was first introduced in [22, 21] and developed in later work [15, 14, 16, 23, 17, 36, 13]. In the next section we survey previously published results. A central assumption behind the algorithmic study of programmable force fields is that practical implementations of programmable force fields are dramatically improving and hence it is justifiable to look at the capabilities and limitations of programmable force fields independent of their implementation. Our goal in this paper is to further understand what can be expected from programmable force fields and not to advocate a particular implementation. We also hope that the thorough study of the capabilities of programmable force fields will contribute to the design of more robust and more efficient parts feeders.

In this paper we focus on studying the stable equilibrium configurations of a part under the

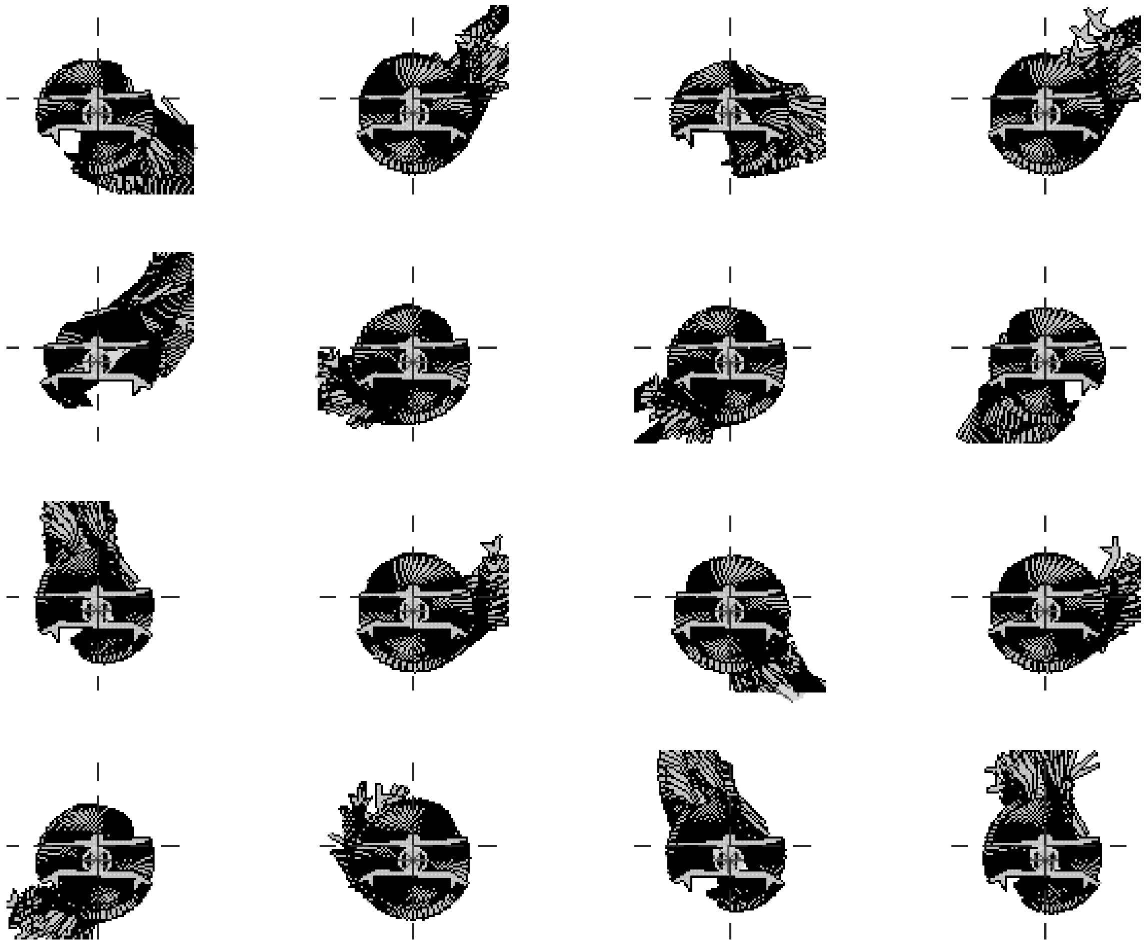

Figure 2: Simulation runs for a part that has a unique stable equilibrium in the combined unit radial and constant field. The magnitude of the constant field is 0.46 in the y-direction. In all runs the part reaches the same final pose. (Reprinted from [13].)

influence of a combined unit radial and constant field. An illustration of an instance of the field is offered in Figure 1. A part placed on a unit radial field alone, moves toward the origin of the radial field but cannot be oriented due to symmetry. Perturbing the radial field with a constant force field breaks the symmetry and gives rise to a finite number of equilibria. Hereafter, when we talk about an equilibrium of a part, we refer to both the position and orientation of the part. In our earlier related work [13], we gave the conditions for the uniqueness of a stable equilibrium configuration (see Figure 2), proving a long standing conjecture of [15]. In this paper we revisit and expand this result. In the case where the conditions are not fulfilled, we propose an algorithm to compute all stable equilibrium configurations. Our work settles some of the open questions in [13]. We make extensive use of potential fields in our analysis. Using the potential field theory, stable equilibrium configurations are equivalent to the local minima of a scalar function. To arrive to the proposed algorithm, we provide a comprehensive and unified analysis of the problem using geometric interpretations when possible.

This paper is organized as follows. Section 2 surveys related work and highlights the contributions of this paper. Section 3 introduces the notation used. Because of the extensive notation in this paper, we provide a table of all the symbols used at the end of the paper. Section 4 defines stable equilibrium configurations using the traditional approach of forces and torques. In contrast, Section 5 defines the stable equilibrium configurations through the theory of potential fields. With that approach equilibrium configurations become the minima of a scalar function. In Section 6 we analyze the properties of a combined unit radial and constant field. Section 7 defines the conditions under which a stable equilibrium is expected. An algorithm to compute all stable equilibrium configurations in the case of non unique equilibrium are given in the same section. We present some computed examples in Section 8 and conclude with a discussion and open problems in Section 9.

2 Related Work

Since 1994 a new class of devices has emerged for distributed part manipulation. Abstractly, these devices implement a force field on a planar surface on which the part is placed. A central idea is that the force exerted at every point can be explicitly programmed so that these devices can exhibit different behaviors. Böhringer and Donald have coined the term "programmable force fields" to describe the manipulation strategies of these new devices. The new devices have several advantages. Their reprogrammability renders them useful for a number of different parts and several manipulation tasks. Manipulation is non-prehensile and one could hope that it will be appropriate for the handling of very small or very fragile parts. In most cases, manipulation is also sensorless and the need of a clock can be eliminated. Hence, the work can be seen as an effort to execute manipulation tasks with the minimal amount of sensing that is required (see the discussion on minimalistic robotics in [10, 26]).

Devices Many research efforts have been devoted to the construction of non-prehensilve, distributed devices that can implement force fields on a planar surface. From 1993-1988, Böhringer and Donald worked with Noel MacDonald at the Cornell Nanofabrication Facility to develop and test new arrays of MEMS microactuators for programmable force fields [21, 7, 15, 19]. Böhringer and Donald also worked with Greg Kovacs' group at the Center for Integrated Systems at Stanford, to develop a control system for MEMS organic ciliary arrays, and to perform experiments with these arrays to manipulate IC dice using array-induced force fields [20, 54]. Hiroyuki Fujita and his group have been developing distributed actuators by IC-compatible micromachining, simulators and manipulation schemes for micromanipulation [37, 38]. Böhringer and Donald have worked with Ken Goldberg at Berkeley and Vivek Bhatt at Cornell to generalize the theory to macroscopic devices, by developing algorithms for transversely vibrating plates in order to implement programmable force fields [6, 9]. Ken Goldberg worked with John Canny and Dan Reznik at Berkeley, to continue research on using vibrating plates for manipulation, showing that longitudinal vibrations can generate a rich vocabulary of programmable force fields [52]. In addition, John Canny and Dan Reznik developed sophisticated dynamic models and dynamic simulators for both MEMS devices and macroscopic vibrating plates and have investigated the design of devices with a few degrees of actuation freedom for part manipulation [50, 51, 53]. Peter Will and his colleagues at USC-ISI have explored a number of different MEMS array designs, as well as algorithms and analysis for programmable force fields [28, 29, 42]. Andy Berlin, David Biegelsen, Warren Jackson and colleagues at Xerox PARC have developed a novel MEMS microactuator array based on controllable airjets, with integrated control and sensing circuitry [5, 27, 4]. Working at CMU, Bill Messner and Jonathan Luntz developed a small room whose floor is tiled with controllable, programmable, macroscopic wheels that can be driven and steered to manipulate large objects such as boxes [43]. Their system employed distributed, local controllers to implement programmable force fields. Together with Howie Choset, they analyzed the resulting dynamical system to obtain interesting results on controllability and developed programmable force field algorithms based on conservative vs. non-conservative fields [44, 45, 46]. Mark Yim at Xerox PARC [56, 57] and Daniela Rus at Dartmouth [39] are also building reconfigurable devices that can execute complex gaits and could be used to implement certain force fields. Working with the Berkeley Sensor & Actuator Center (BSAC), Karl Böhringer and Ken Goldberg explored how MEMS devices employing electrostatic fringing fields can be used to implement programmable force fields for parts manipulation and self-assembly [24]. Many more references in distributed manipulation devices can be found in [11].

Algorithms Since 1994 Böhringer and Donald have advocated that that there is a wealth of algorithmic problems on programmable force fields. Indeed the amount of work in this area and the results produced have justified their claim. A lot of recent work has been on the problem of positioning and orienting a part which is the topic of this paper. Hence we survey results related to finding stable equilibrium configurations of a part. Our list below is non exhaustive. For more information the reader is referred to [21, 15, 18, 36, 13]. In a programmable force field, every point in the plane is associated with a force vector in the plane. When a part is placed into force field, it experiences a translation and re-orientation until an equilibrium is reached.

- A unit squeeze field is defined as $\mathbf{f}(x, y) = -sign(x)(1, 0)$. When a part is placed into a squeeze field, it experiences a translation and re-orientation until a predictable equilibrium is reached [21]. Böhringer and Donald [21] showed that given a polygonal part P with n vertices, there exist $O(n^2k)$ stable equilibrium orientations for P when placed in $\mathbf{f}$ (k is the number of combinatorially distinct bisector placements for P. For details on combinatorially distinct bisector placements see [18].). This result was used to generate strategies for unique parts posing (up to symmetry) by reducing the problem to a parts feeding algorithm developed by Goldberg [33]. The strategies have length $O(n^2k)$ and can be generated in $O(n^4k^2)$ time.
- In later work the same researchers improved the length of their manipulation strategies. In [15] the length was reduced to $O(nk)$ and the planning time $O(n^2k^2)$, by employing combined squeeze and unit radial fields. Unit radial fields are defined as $\mathbf{r}(x, y) = (-1/\sqrt{x^2 + y^2})(x, y)$.
- Using elliptic force fields $\mathbf{f}(x, y) = (-\alpha x, -\beta y)$ such that $0 < \alpha < \beta$, Kavraki [36] reduced the number of stable equilibrium configurations for most parts to a constant number (2) independent of n. In the same paper, an investigation of the effect of control uncertainty on the stability of equilibria was performed. There is an interesting tradeoff made with elliptic fields. For an added complexity in the implementation of the fields (varying magnitude), one is able to reduce the length of the manipulation strategy to a single step eliminating the need of a clock.
- Böhringer and Donald conjectured in [15] that a field which combines a unit radial and constant field $\mathbf{r} + \delta\mathbf{c}$ ($\mathbf{c}(x, y) = (0, -1)$ and δ is a small positive constant), has the property of

uniquely orienting and positioning parts. We call this field the unit radial-constant field. This result was not proven until much later in [13] by Böhringer, Donald, Kavraki and Lamiraux. They showed that for any non-symmetric part, there is a unit radial-constant field inducing exactly one stable equilibrium. A non-symmetric part is defined as a part whose center of mass does not end at the origin under the the unit radial field alone. The paper left open the question of what happens in the case of symmetric parts.

Contributions of This Paper Recent work on symmetric parts under the influence of a combined unit radial and constant field includes [41, 40]. For symmetric parts the center of mass of the part ends up at the origin under the unit-radial field only. Rectangle or regular polygons are instances of symmetric parts. The present paper unifies all previous results on the unit radial-constant fields and exhaustively examines the case of symmetric parts. We present an algorithm that commutes all stable equilibrium orientations for a part, symmetric or not. Our work could be used to build a *universal parts feeder* (inspired by the "universal gripper" as proposed by Abell and Erdmann [1]). In contrast to the universal manipulator fields proposed in [51], such a device could uniquely position a part without the need of a clock, sensors, or programming.

Our analysis is based on a treatment of the problem through the use of potential fields. Former work on force fields mostly defines equilibrium configurations from a mechanical point of view, using force and torque. Potential fields however, are a helpful tool to characterize stable equilibrium configurations as local minima of a potential function. This tool was first introduced in [15] where the authors showed that if the force field in the plane derives from a potential field, this potential function, as well as the force field, can be "lifted" to the configuration space. They proved that any path integral of the lifted force field between two configurations is equal to the difference of the lifted potential field between these configurations.

In this paper, we first reformulate the results in [15] using a different approach based on partial derivatives instead of path integrals. We show in Section 5, that for *any* part in *any* potential field, the lifted potential field is C^1. Therefore, the stable equilibrium configurations and local minima of the lifted potential field are the same. These results constitute the basis for determining the equilibrium configurations of a part placed in the combination of a unit radial and a constant force fields. The basic properties of a combined field are established in Section 6. These properties enable us to devise, in Section 7, an algorithm that determines the set of equilibrium configurations of any part subjected to combined unit radial and small constant field.

3 Notation

In this section, we provide the core notation and definitions used throughout the paper. A table of all symbols used is available at the end of the paper.

If E is a set, we denote $\mathrm{int}(E)$, $\overline{E}$, and ∂E respectively the interior, the closure and the boundary of E.

Let us consider a part in the plane occupying surface S with center of mass G at $(0,0)$ in a reference configuration $\mathbf{q}_0$. Suppose that S is a compact set and that its boundary ∂S is a zero-measure subset of the plane. The configuration space of the part is $\mathcal{C} = \mathbf{R}^2 \times \mathbf{S}^1$, where $\mathbf{S}^1$ is the unit circle. A configuration $\mathbf{q} = (x, y, \theta) \in \mathcal{C}$ corresponds to a rigid-body transformation $\varphi_{\mathbf{q}}$ in the plane transforming $\mathbf{r} = (\xi, \eta)$ into

$$\varphi_{\mathbf{q}}(\mathbf{r}) = \begin{pmatrix} x + \cos\theta\xi - \sin\theta\eta \\ y + \sin\theta\xi + \cos\theta\eta \end{pmatrix}.$$

If E is a subset of $\mathbf{R}^2$, we denote by $E_q = \varphi_{\mathbf{q}}(E)$ the image of E by $\varphi_{\mathbf{q}}$. For instance, $S_{\mathbf{q}}$ is the subset occupied by the part in configuration $\mathbf{q}$. We also denote by $d_{\mathcal{C}}(\mathbf{q}, \mathbf{q}') = \sqrt{(x-x')^2 + (y-y')^2} + |\theta - \theta'|$ the distance between $\mathbf{q}$ and $\mathbf{q}' = (x', y', \theta)$.

4 Conditions for Equilibrium Configurations: The Force and Torque Approach

We investigate the conditions for equilibrium for a part of uniform support in the presence of a force field $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^2$. Without loss of generality, we assume that the the origin of the reference frame in the plane can be chosen as the center of mass of the part.
When a part is placed on a force field $\mathbf{f}(\mathbf{r})$, the resultant force and torque it is subjected to are the following:

$$\begin{aligned} \mathbf{F}(\mathbf{q}) &= \int_{S_{\mathbf{q}}} \mathbf{f}(\mathbf{r}) d\mathbf{r}, & (1) \\ M(\mathbf{q}) &= \int_{S_{\mathbf{q}}} (\mathbf{r} - \mathbf{r}_G) \times \mathbf{f}(\mathbf{r}) d\mathbf{r}, & (2) \end{aligned}$$

where $\mathbf{r}_G = (x, y)$ is the position of the center of mass G, when the part is at configuration $\mathbf{q}$. We say that $\mathbf{q}$ is an *equilibrium configuration* if the resultant force and torque vanish at $\mathbf{q}$:

$$\begin{aligned} \mathbf{F}(\mathbf{q}) &= 0, \\ M(\mathbf{q}) &= 0. \end{aligned}$$

Moreover, an equilibrium configuration $\mathbf{q}$ is said to be *stable* if, subjected to a small perturbation, the part stays in the neighborhood of $\mathbf{q}$.

5 Conditions for Equilibrium Configurations: The Potential Field Approach

Another way to analyze equilibrium configurations is through the theory of potential fields. Potential fields have been extensively used in mechanics and electrostatics. They are very helpful in studying stable equilibrium configurations of a particle because these configurations are exactly the local minima of a potential function. Potential fields can also be used to study the equilibrium configurations of a part and this was first suggested in [15, 19]. We expand and generalize that work.

To avoid confusion of notation, we will use ξ and η as coordinates of a point in $\mathbf{R}^2$. Let us recall that x and y denote the coordinates of the center of mass of the part. Let us consider now a force field $\mathbf{f}(\xi, \eta)$ over the plane, continuous almost everywhere (*i.e.*, everywhere except on a subset of measure zero). We say that $\mathbf{f}(\xi, \eta)$ derives from a potential function $u(\xi, \eta)$ if u is continuous over $\mathbf{R}^2$ and $\mathbf{f} = -(\frac{\partial u}{\partial \xi}, \frac{\partial u}{\partial \eta})$ almost everywhere in $\mathbf{R}^2$. Notice that the force field does not need to be continuous everywhere to define the potential function u.

In [15] a potential field $U(\mathbf{q})$ in the configuration space is defined by integrating the potential function u in the plane over the space $S_{\mathbf{q}}$ occupied by the part in configuration $\mathbf{q}$. The authors show that for polygonal parts, this field is continuously differentiable and its partial derivatives are the opposite of the resultant force and torque that the part is subjected to. Such a correspondence establishes an equivalence between stable equilibrium configurations of a part and local minima of the lifted potential field U.

In this section, we generalize the above correspondence to non-polygonal parts. In this way, we will be able to use the theory of potential fields for reasoning about the equilibria of any part in the plane. Most importantly, we show that the above correspondence is a consequence of the commutativity between two operators acting on scalar functions in $\mathbf{R}^2$: integral and partial derivatives. Hence, the property is intrinsic to the field and does not depend on the system of coordinates over the configuration space.

To ensure the existence of the resultant force and torque (1) and (2) for any compact part in any configuration, we assume that u, $\frac{\partial u}{\partial \xi}$ and $\frac{\partial u}{\partial \eta}$ are integrable over any bounded set.

5.1 Lifted Potential Field

As we are reasoning on rigid parts and not on single particles, we need to "lift" the potential field from the plane to the configuration space. We use the lifted potential field U as introduced in [15].

Definition 1. (Lifted potential field) *Let u be a potential field. The function*

$$U(\mathbf{q}) = \int_{S_{\mathbf{q}}} u(\mathbf{r}) d\mathbf{r} = \int_{S} u(\varphi_{\mathbf{q}}(\mathbf{r})) d\mathbf{r}$$

over the configuration space is called the lifted potential field *induced by u.*

Our goal in this section is to show that stable equilibrium configurations and local minima of the potential field U are the same. For that, we first show that $U(\mathbf{q})$ is of class C^1 and that the resultant force and torque F_x, F_y and M are respectively the partial derivatives of U w.r.t. x, y and θ. Under these conditions, if $\mathbf{q}$ is an equilibrium configuration, the partial derivatives of U vanish at $\mathbf{q}$ by definition. Moreover, $\mathbf{q}$ being a local minimum ensures the stability of a small neighborhood around $\mathbf{q}$.

Proposition 1. *U is of class C^1 and*

$$\frac{\partial U}{\partial x}(\mathbf{q}) = -F_x(\mathbf{q}), \tag{3}$$

$$\frac{\partial U}{\partial y}(\mathbf{q}) = -F_y(\mathbf{q}), \tag{4}$$

$$\frac{\partial U}{\partial \theta}(\mathbf{q}) = -M(\mathbf{q}). \tag{5}$$

Proof. In [15, 19], the authors prove using path integrals that

- relations (3), (4), and (5) are satisfied if the right hand sides of these equations are continuous w.r.t. $\mathbf{q}$,

- the right hand sides of (3), (4), and (5) are continuous if the part is a polygon.

We need only to prove the continuity of $F_x(\mathbf{q})$, $F_y(\mathbf{q})$, and $M(\mathbf{q})$ for a general (non polygonal) part. We do the proof only for $F_x(\mathbf{q})$; $F_y(\mathbf{q})$ and $M(\mathbf{q})$ can be treated similarly.

First, let us point out that for any bounded subset $B \subset \mathbf{R}^2$, there exists a positive constant b such that:

$$\forall \mathbf{r} \in B, \|\varphi_{\mathbf{q}}(\mathbf{r}) - \varphi_{\mathbf{q}'}(\mathbf{r})\| \leq b d_{\mathcal{C}}(\mathbf{q}, \mathbf{q}'). \tag{6}$$

b can be chosen as $\max\{\|\mathbf{r}\|, \mathbf{r} \in B\} + 1$, for instance. This inequality means that if two configurations are close, displacements of points between these configurations are small. Notice that

$$\begin{aligned} \left|F_x(\mathbf{q}') - F_x(\mathbf{q})\right| &= \left|\int_{S_{\mathbf{q}'}} \frac{\partial u}{\partial \xi}(\mathbf{r}) d\mathbf{r} - \int_{S_{\mathbf{q}}} \frac{\partial u}{\partial \xi}(\mathbf{r}) d\mathbf{r}\right| && (7) \\ &\leq \int_{(S_{\mathbf{q}} \cup S_{\mathbf{q}'}) \setminus (S_{\mathbf{q}} \cap S_{\mathbf{q}'})} \left|\frac{\partial u}{\partial \xi}(\mathbf{r})\right| d\mathbf{r}. && (8) \end{aligned}$$

We want to show that when $d_{\mathcal{C}}(\mathbf{q}, \mathbf{q}')$ tends toward 0, so does the second term of the former inequality. Let us denote by $\partial S_{\mathbf{q}}$ the boundary of $S_{\mathbf{q}}$. We are going to show that there exists $b > 0$ such that if $d_{\mathcal{C}}(\mathbf{q}, \mathbf{q}') \leq \ell$,

$$(S_{\mathbf{q}} \cup S_{\mathbf{q}'}) \setminus (S_{\mathbf{q}} \cap S_{\mathbf{q}'}) \subset (\partial S_{\mathbf{q}})^{b\ell}, \tag{9}$$

where $(S_{\mathbf{q}})^{b\ell}$ denotes the set of points at a distance not greater than $b\ell$ from $S_{\mathbf{q}}$.

Let us consider $\mathbf{p} \in (S_{\mathbf{q}} \cup S_{\mathbf{q}'}) \setminus (S_{\mathbf{q}} \cap S_{\mathbf{q}'})$. Then,

$$\begin{array}{ccc} \mathbf{p} \in S_{\mathbf{q}} & \text{and} & \mathbf{p} \notin S_{\mathbf{q}'}, \\ & \text{or} & \\ \mathbf{p} \notin S_{\mathbf{q}} & \text{and} & \mathbf{p} \in S_{\mathbf{q}'}. \end{array}$$

In the first case, we define $\mathbf{r} = \varphi_{\mathbf{q}'}^{-1}(\mathbf{p})$. Then $\mathbf{p} = \varphi_{\mathbf{q}'}(\mathbf{r})$ and $\mathbf{r} \notin S$ since $\mathbf{p} \notin S_{\mathbf{q}'}$ (see Figure 3). Let us now consider $\mathbf{p}' = \varphi_{\mathbf{q}}(\mathbf{r})$. $\mathbf{p}' \notin S_{\mathbf{q}}$ since $\mathbf{r} \notin S$. Without loss of generality, we can assume that $d_{\mathcal{C}}(\mathbf{q}, \mathbf{q}')$ is bounded by a constant and therefore constrain $\mathbf{r}$ to remain in a bounded set B. Then, we can apply (6) to get $\|\mathbf{p} - \mathbf{p}'\| \leq b\ell$. As $\mathbf{p} \in S_{\mathbf{q}}$ and $\mathbf{p}' \notin S_{\mathbf{q}}$, we can easily show that the line segment $[\mathbf{p}, \mathbf{p}']$ crosses $\partial S_{\mathbf{q}}$ and therefore that $\mathrm{dist}(\mathbf{p}, \partial S_{\mathbf{q}}) < b\ell$, which is equivalent to $\mathbf{p} \in (\partial S_{\mathbf{q}})^{b\ell}$.

In the second case, there exists $\mathbf{r} \in S$ such that $\mathbf{p} = \varphi_{\mathbf{q}'}(\mathbf{r})$. If we consider now $\mathbf{p}' = \varphi_{\mathbf{q}}(\mathbf{r})$, $\mathbf{p}' \in S_{\mathbf{q}}$ and again $\|\mathbf{p} - \mathbf{p}'\| \leq b\ell$. For the same reason as above, $\mathbf{p} \in (\partial S_{\mathbf{q}})^{b\ell}$.
(8) and (9) together imply that

$$|F_x(\mathbf{q}') - F_x(\mathbf{q})| \leq \int_{(\partial S_{\mathbf{q}})^{b\ell}} |\frac{\partial u}{\partial \xi}(\mathbf{r})| d\mathbf{r}.$$

$(\partial S_{\mathbf{q}})^{b\ell}$ is an increasing family of sets, that is $\ell < \ell' \Rightarrow (\partial S_{\mathbf{q}})^{b\ell} \subset (\partial S_{\mathbf{q}})^{b\ell'}$.
As $\partial S_{\mathbf{q}}$ is compact, $\cap_{\ell > 0} (\partial S_{\mathbf{q}})^{b\ell} = \partial S_{\mathbf{q}}$ and

$$\lim_{\mathbf{q}' \to \mathbf{q}} |F_x(\mathbf{q}') - F_x(\mathbf{q})| = \int_{\partial S_{\mathbf{q}}} |\frac{\partial u}{\partial \xi}(\mathbf{r})| d\mathbf{r}.$$

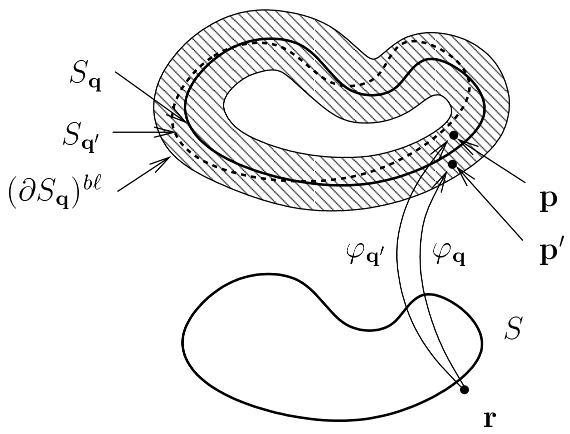

Figure 3: $\mathbf{p}$ and $\mathbf{p}'$ are the images of the same point $\mathbf{r}$ by respectively $\varphi_{\mathbf{q}'}$ and $\varphi_{\mathbf{q}}$. Therefore, if $d_C(\mathbf{q}, \mathbf{q}') \leq l$, the distance between $\mathbf{p}$ and $\mathbf{p}'$ is less than $b\ell$, where b is defined in equation (6).

The right hand side of this equation is 0 since $\partial S_{\mathbf{q}}$ is a set of measure zero and $\frac{\partial u}{\partial \xi}$ is locally integrable. Thus, $F_x(\mathbf{q})$ is continuous. The same proof applies as well to $F_y(\mathbf{q})$ and $M(\mathbf{q})$. $\square$

In the next two sections, we give some interesting and original interpretations of Proposition 1.

5.2 Physical Interpretation of Proposition 1

According to the fundamental principles of mechanics, along a trajectory of the part, the variation of the kinetic energy E_k is equal to the power the force field communicates to the part:

$$\begin{aligned}
\frac{dE_k}{dt} &= F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + M \frac{d\theta}{dt} \\
&= -\left(\frac{\partial U}{\partial x}\frac{dx}{dt} + \frac{\partial U}{\partial y}\frac{dy}{dt} + \frac{\partial U}{\partial \theta}\frac{d\theta}{dt}\right) \\
&= -\frac{dU}{dt}.
\end{aligned}$$

Thus, $U + E_k$ is constant along the trajectories of the part. This constant is called the mechanical energy of the system. Proposition 1 establishes an equivalence between local minimum of U and stable equilibrium configuration. Indeed if $\mathbf{q}$ is a local minimum and if the part starts from $\mathbf{q}$ with a small kinetic energy ε, the part necessarily stays in the neighborhood of $\mathbf{q}$ where $U(\mathbf{q}) \leq U \leq U(\mathbf{q}) + \varepsilon$. This neighborhood can be made as small as desired by decreasing ε.

5.3 Commutativity of Integrals and Partial Derivatives

Beyond the former physical interpretation of Proposition 1, we would like to point out the mathematical meaning of this result. Definition 1 defines $U(\mathbf{q})$ by integrating the potential field over the domain occupied by the part in configuration $\mathbf{q}$. Notice that this definition is independent from the coordinate system used to describe the configuration $\mathbf{q}$. If we rewrite $\mathbf{q} = (q_1, q_2, q_3)$, a quick calculation shows that equations (3-5) are equivalent to

$$\frac{\partial U}{\partial q_i} = \frac{\partial}{\partial q_i}\int_S u(\varphi_{\mathbf{q}}(\mathbf{r}))d\mathbf{r} = \int_S \frac{\partial}{\partial q_i}(u(\varphi_{\mathbf{q}}(\mathbf{r})))d\mathbf{r}, \qquad i = 1, 2, 3.$$

In other words, the partial derivatives commute with the integral. This property is the counterpart of the commutativity pointed out in [19] between path integrals and surface integrals. This latter formulation however, is subordinate to the coordinate system (x, y, θ) over the configuration space.

Formulated with partial derivatives, the above commutativity can be generalized to any parameterization of the configuration space. Indeed, if (q'_1, q'_2, q'_3) is another system of coordinates around $\mathbf{q}$, we can write

$$\begin{aligned}
\frac{\partial U}{\partial q'_i}(\mathbf{q}) &= \sum_{j=1}^{3} \frac{\partial q_j}{\partial q'_i}(\mathbf{q}) \frac{\partial U}{\partial q_j}(\mathbf{q}) \\
&= \sum_{j=1}^{3} \frac{\partial q_j}{\partial q'_i}(\mathbf{q}) \int_S \frac{\partial}{\partial q_j}(u(\varphi_{\mathbf{q}}(\mathbf{r})))d\mathbf{r} \\
&= \int_S \sum_{j=1}^{3} \frac{\partial q_j}{\partial q'_i}(\mathbf{q}) \frac{\partial}{\partial q_j}(u(\varphi_{\mathbf{q}}(\mathbf{r})))d\mathbf{r} \\
&= \int_S \frac{\partial}{\partial q'_i}(u(\varphi_{\mathbf{q}}(\mathbf{r})))d\mathbf{r}.
\end{aligned}$$

Thus the commutativity property is true for any system of coordinates over the configuration space.

The next section introduces in detail the unit radial field. We will use a system of coordinates that exploits the radial symmetry of the field. In this system of coordinates, the lifted potential field depends only of two configuration variables. The above commutativity property will play crucial role in our proofs.

6 Combination of a Unit Radial Field with a Constant Field

In the previous section, we pointed out the equivalence between local minima of the potential field and stable equilibrium configurations. In this section, we investigate a class of potential fields obtained by adding a unit radial field and a small constant field. We give conditions under which a part can be *uniquely* positioned in such a combined field. If these conditions are not fulfilled, we provide an algorithm to determine all stable equilibrium configurations. Some parts of this work have been presented in [13, 41, 40]. In particular the conditions under which unique orientation is obtained and the proof where also given in [13]. We need to repeat them here for completeness reasons and for making possible the discussion of what happens when the conditions for equilibrium are not fulfilled. However, in this paper our proofs are more detailed, unify previous results and cover cases that we not covered in previous work in [13, 41, 40].

6.1 Unit Radial Field

We first establish the notation and give some definitions relative to the unit radial field. We also prove some properties of the lifted potential field associated to the unit radial field.

We call *unit radial field* the force field of unit magnitude everywhere, pointing toward the origin $(0, 0)$:

$$\mathbf{f}(\mathbf{r}) = -\frac{\mathbf{r}}{\|\mathbf{r}\|}.$$

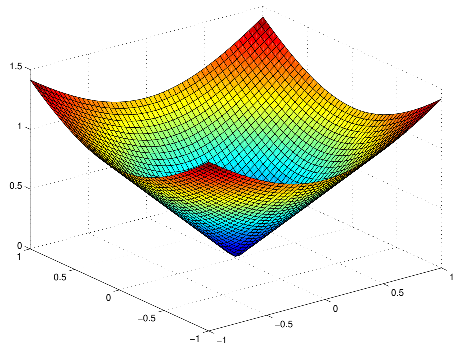

Figure 4: Unit radial potential field: $v(\xi, \eta) = \sqrt{\xi^2 + \eta^2}$.

This force field derives from the potential field $v(\mathbf{r}) = \|\mathbf{r}\| = \sqrt{\xi^2 + \eta^2}$. This potential field is smooth (C^∞) everywhere except at the origin (see Figure 4). v is clearly symmetric by rotation about the origin. Thus if $\mathbf{q}'$ is obtained from $\mathbf{q}$ by a rotation about the origin, the values of the lifted potential field a $\mathbf{q}$ and $\mathbf{q}'$ are the same. To take fully advantage of this property, we are going to use a new system of coordinates (X, Y, θ) for the configuration of the part, illustrated in Figure 5 and defined by:

$$
\begin{aligned}
X &= \cos\theta x + \sin\theta y, \\
Y &= -\sin\theta x + \cos\theta y.
\end{aligned}
$$

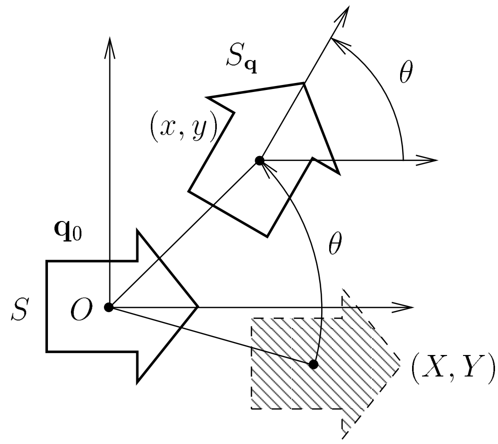

Figure 5: Parameterization of the configuration space with the system of coordinates $\mathbf{q} = (X, Y, \theta)$. $\varphi_{\mathbf{q}}$ corresponds to a translation of (X, Y) followed by a rotation of angle θ about the origin.

Expressed in this system of coordinates, the lifted potential field V corresponding to v depends only on X and Y and can be written as follows:

$$V(X, Y, \theta) = \int_S v(X + \xi, Y + \eta) d\xi d\eta.$$

Because of the independence on θ, we will consider V as a function of (X, Y) only. Notice that when $\theta = 0$, $X = x$ and $Y = y$. $V(X, Y) = V(x, y, 0)$ can thus be considered as the lifted potential field of the part in translation. This interpretation can be helpful to understand some of the forthcoming developments.

We now point out some useful properties of the radial field. First, we show that $V(X,Y)$ has a unique local minimum. For that, we show that V is of class C^2 and that its Hessian Hess $V(X,Y)$ is positive definite everywhere. The results of Section 5 establish that V is C^1 and that its first order partial derivatives are the integrals of the partial derivatives of v. The following proposition states that the commutativity between integrals and partial derivatives extend to the second order partial derivatives.

Proposition 2. *V is of class C^2 and*

$$\begin{aligned}
\frac{\partial V}{\partial X}(X,Y) &= \int_S \frac{\partial v}{\partial \xi}(X+\xi, Y+\eta)d\xi d\eta && (10)\\
\frac{\partial V}{\partial Y}(X,Y) &= \int_S \frac{\partial v}{\partial \eta}(X+\xi, Y+\eta)d\xi d\eta && (11)\\
\frac{\partial^2 V}{\partial X^2}(X,Y) &= \int_S \frac{\partial^2 v}{\partial \xi^2}(X+\xi, Y+\eta)d\xi d\eta && (12)\\
\frac{\partial^2 V}{\partial Y^2}(X,Y) &= \int_S \frac{\partial^2 v}{\partial \eta^2}(X+\xi, Y+\eta)d\xi d\eta && (13)\\
\frac{\partial^2 V}{\partial X \partial Y}(X,Y) &= \int_S \frac{\partial^2 v}{\partial \xi \partial \eta}(X+\xi, Y+\eta)d\xi d\eta. && (14)
\end{aligned}$$

Proof. The proof of Proposition 2 requires some technical developments given in the Appendix. We give here only the main ideas.

First notice that (10) and (11) are direct consequences of Proposition 1. The standard results of the theory of integration lead easily to the following statement: if v is C^2 and if all its partial derivatives up to second order are integrable, then V is C^2 and its partial derivatives are obtained by integrating the corresponding partial derivatives of $v(X+\xi, Y+\eta)$. In the case of the unit radial field v, the partial derivatives of v up to second order are integrable. However, v is not C^2 at the origin. To overcome this difficulty, we define C^2 approximations v_h of v equal to v everywhere except on the disc centered at the origin and of radius $h > 0$. The corresponding lifted potential field V_h is thus C^2 and its partial derivatives are obtained by differentiating in the integral. It can be then easily stated that as long as the disc of radius h remains completely inside or completely outside the part, the difference between V and V_h is constant and thus on these subsets, V is C^2. By making h tend toward 0, we ensure that the partial derivatives up to second order of V are continuous even when the origin of the radial field crosses the boundary of the part. $\square$

The unit radial field is smooth everywhere except at the origin. Thus, when the part is not above the origin, we should expect the lifted potential field V to be smooth. We will show that in fact, if the part is above the origin, the lifted potential field is still smooth. The singularity affects the lifted potential function only when the origin crosses the boundary of the part.

The boundary of the part divides the configuration space into 3 subsets:

- the set $\mathcal{C}^{out}$ of configurations $\mathbf{q}$ such that the origin of the radial field is outside $S_{\mathbf{q}}$: $(0,0) \notin S_{\mathbf{q}}$,
- the set $\mathcal{C}^{in}$ of configurations $\mathbf{q}$ such that the origin of the radial field is inside $S_{\mathbf{q}}$: $(0,0) \in \text{int}(S_{\mathbf{q}})$,

- and the set $\mathcal{C}^{bound}$ of configurations $\mathbf{q}$ such that the origin of the radial field is on the boundary of $S_{\mathbf{q}}$: $(0,0) \in \partial S_{\mathbf{q}}$.

A straightforward calculation shows that whether a configuration $\mathbf{q}$ is in $\mathcal{C}^{out}$, $\mathcal{C}^{in}$, or $\mathcal{C}^{bound}$ depends only on (X, Y). Indeed,

$$\begin{array}{lcl} \mathbf{q} \in \mathcal{C}^{in} & \Leftrightarrow & (-X,-Y) \in \mathrm{int}(S), \\ \mathbf{q} \in \mathcal{C}^{out} & \Leftrightarrow & (-X,-Y) \notin S, \\ \mathbf{q} \in \mathcal{C}^{bound} & \Leftrightarrow & (-X,-Y) \in \partial S. \end{array}$$

Figure 5 provides intuition of these equivalences: $(-X,-Y) \in \mathrm{int}(S)$ means that the origin is inside the part translated by (X, Y). Then, the rotation of the part about the origin does not affect the position of the origin w.r.t. the part. The following proposition is essential for the analysis in the rest of this section.

Proposition 3. *$V(X,Y)$ is smooth over $\mathcal{C}^{in}$ and over $\mathcal{C}^{out}$.*

Proof. In the proof of Proposition 2, given in the Appendix, we constructed a family v_h of C^2 approximations of v. Following the same idea, and using the same notation, we now consider another family of potential fields $v_h(\mathbf{r}), h > 0$ such that

- $v_h(\mathbf{r})$ is smooth over $\mathbf{R}^2$ and
- $v_h(\mathbf{r}) = v(\mathbf{r})$ for any $\mathbf{r}$ verifying $\|\mathbf{r}\| \geq h$.

$v_h(\mathbf{r})$ can be constructed by multiplying $v(\mathbf{r})$ by a smooth function $\rho_h(\mathbf{r})$ equal to 1 outside the disc of radius h and to 0 inside the disc of radius $h/2$. See [32] for a method to build the functions ρ_h.

Let V_h be the lifted potential field corresponding to v_h. If we denote by $D_h(X', Y')$ the disc of radius h centered on (X', Y'), for any $(X, Y) \in \mathbf{R}^2$, we have

$$V_h(X,Y) - V(X,Y) = \int_S (v_h - v)(X+\xi, Y+\eta) d\xi d\eta \qquad (15)$$

$$= \int_{S \cap D_h(-X,-Y)} (v_h - v)(X+\xi, Y+\eta) d\xi d\eta \qquad (16)$$

since by definition, the integrand is zero outside $D_h(-X,-Y)$.

Let us consider $\mathbf{q}' = (X', Y', \theta')$ a configuration in $\mathcal{C}^{in}$. As stated above, $(-X', -Y') \in \mathrm{int}(S)$. Therefore, there exists $\alpha > 0$ such that $D_\alpha(-X', -Y') \subset \mathrm{int}(S)$.

If $h < \frac{\alpha}{2}$, for any $(X, Y) \in D_h(X', Y')$, $D_h(-X,-Y) \subset S$ and (16) becomes

$$\begin{aligned} V_h(X,Y) - V(X,Y) &= \int_{D_h(-X,-Y)} (v_h - v)(X+\xi, Y+\eta) d\xi d\eta \\ &= \int_{D_h(0,0)} (v_h - v)(\xi, \eta) d\xi d\eta, \end{aligned}$$

using the change of variable $(\xi, \eta) \leftarrow (X+\xi, Y+\eta)$. We notice then that $V_h(X,Y) - V(X,Y)$ is independent of (X, Y) in a neighborhood of (X', Y'). Moreover, as v_h is smooth, so is V_h. Therefore, V is also smooth in a neighborhood of (X', Y').

The case $\mathbf{q}' \in \mathcal{C}^{out}$ is similar: for h small enough, S and $D_h(-X,-Y)$ are disjoint, so that (16) is equal to 0 and V_h being smooth implies that V is also smooth. $\square$

Minimum of V and Pivot Point

Although the unit radial field is unable to orient a part, due to its symmetry, it has the property to make the part converge toward the center of the field. The set of all the equilibrium configurations is stable by rotation of the part about the center of the field. Thus, for any of these configurations, a point of the part remains at the center of the field. This point is called the *pivot point* of the part and will be denoted by P. This point was first defined in [15]. We give in this section a different proof of its existence and uniqueness using our potential field formalism. Parts of the rest of this section were given in [13] but applied to a more restricted case.

With our notation, the existence and uniqueness of the pivot point is a consequence of the following proposition.

Proposition 4. *V verifies the following properties:*

(i) The Hessian of V Hess $V(X,Y)$ is positive definite everywhere in $\mathbf{R}^2$,

(ii) V has a unique local minimum over $\mathbf{R}^2$.

Similar to [13] . As Hess V is a 2 by 2 matrix, (i) is equivalent to

$$\forall (X,Y) \in \mathbf{R}^2, \quad \begin{array}{c} \text{tr Hess } V(X,Y) > 0, \\ \text{det Hess } V(X,Y) > 0, \end{array}$$

where tr and det are respectively the trace and determinant operators. According to (12-14), the second order partial derivatives of V are

$$\begin{aligned} \frac{\partial^2 V}{\partial X^2}(X,Y) &= \int_S \frac{(Y+\eta)^2}{((X+\xi)^2+(Y+\eta)^2)^{3/2}} d\xi d\eta, \\ \frac{\partial^2 V}{\partial Y^2}(X,Y) &= \int_S \frac{(X+\xi)^2}{((X+\xi)^2+(Y+\eta)^2)^{3/2}} d\xi d\eta, \\ \frac{\partial V}{\partial X \partial Y}(X,Y) &= \int_S \frac{-(X+\xi)(Y+\eta)}{((X+\xi)^2+(Y+\eta)^2)^{3/2}} d\xi d\eta. \end{aligned}$$

It is straightforward from these expressions that tr Hess $V = \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2}$ is positive everywhere. The determinant of the Hessian of V

$$\det \text{Hess } V(X,Y) = \left(\frac{\partial^2 V}{\partial X^2} \frac{\partial^2 V}{\partial Y^2} - (\frac{\partial^2 V}{\partial X \partial Y})^2 \right)(X,Y)$$

is the sum of two terms, each of which is a product of two integrals over S. Replacing these products by integrals over the Cartesian product $S = S \times S$:

$$\left(\int_S f(\xi,\eta) d\xi d\eta \right) \left(\int_S g(\xi,\eta) d\xi d\eta \right) = \int_S f(\xi_1,\eta_1) g(\xi_2 \eta_2) d\xi_1 d\eta_1 d\xi_2 d\eta_2$$

and condensing the notation as follows, $X_i = (X+\xi_i)$, $Y_i = (Y+\eta_i)$ for $i = 1, 2$, $dS = d\xi_1 d\eta_1 d\xi_2 d\eta_2$, we get

$$\begin{aligned}
\det \operatorname{Hess} V(X,Y) &= \int_S \frac{X_1^2 Y_2^2}{(X_1^2+Y_1^2)^{3/2}(X_2^2+Y_2^2)^{3/2}} dS \\
&- \int_S \frac{X_1 Y_1 X_2 Y_2}{(X_1^2+Y_1^2)^{3/2}(X_2^2+Y_2^2)^{3/2}} dS \\
&= \int_{S^2} \frac{Y_1^2 X_2^2 - X_1 Y_1 X_2 Y_2}{(X_1^2+Y_1^2)^{3/2}(X_2^2+Y_2^2)^{3/2}} dS.
\end{aligned}$$

In the first integral, (X_1, Y_1) and (X_2, Y_2) have a symmetric role and can be switched so that $X_1^2 Y_2^2$ can be replaced by $\frac{1}{2}(X_1^2 Y_2^2 + X_2^2 Y_1^2)$ and

$$\begin{aligned}
\det \operatorname{Hess} V(X,Y) &= \frac{1}{2}\int_{S^2} \frac{X_1^2 Y_2^2 + X_2^2 Y_1^2 - 2X_1 Y_1 X_2 Y_2}{(X_1^2+Y_1^2)^{3/2}(X_2^2+Y_2^2)^{3/2}} dS \\
&= \frac{1}{2}\int_{S^2} \frac{(X_1 Y_2 - X_2 Y_1)^2}{(X_1^2+Y_1^2)^{3/2}(X_2^2+Y_2^2)^{3/2}} dS \\
&> 0.
\end{aligned}$$

Thus Hess V is positive definite everywhere. This ensures us that if V has a local minimum, it is unique. Moreover, as $v(\mathbf{r})$ tends toward infinity when $\|\mathbf{r}\|$ tends toward infinity, $V(X,Y)$ also tends toward infinity as (X,Y) diverges. This property implies the existence of a local minimum of V. $\square$

We denote by (X_0, Y_0) the unique minimum of V. The set of equilibrium configurations of the part under the radial field is the following $\{(X_0, Y_0, \theta), \theta \in \mathbf{S}^1\}$. Let us express this curve in the classical system of coordinates:

$$\begin{aligned}
x &= X_0 \cos\theta - Y_0 \sin\theta \\
y &= X_0 \sin\theta + Y_0 \cos\theta.
\end{aligned}$$

As expected, the set of equilibrium configurations is obtained by rotation of the part about the pivot point. As V does not depend on θ, we can consider the case $\theta = 0$, where $x = X$ and $y = Y$. In the corresponding equilibrium configuration, the center of mass is thus translated to (X_0, Y_0) and P is located at the origin. As a consequence, in configuration $\mathbf{q}_0$, P is at $(-X_0, -Y_0)$ (Figure 6).

6.2 Unit Radial Combined With a Constant Field

The former section established the existence and uniqueness of the pivot point. In this section, we perturb the radial field by adding a small constant field in order to break the symmetry. Using the results of the previous sections, we are going to show that for each orientation θ of the part, the corresponding lifted potential field has a unique minimum in (X,Y) and that when θ varies, the curve of these minima is continuous and smooth if the pivot point is not on the boundary of the part. We call this curve *the equilibrium curve.* Then we will give a characterization of the local minima of the lifted potential using the equilibrium curve. This characterization will reveal crucial in the following section to determine the set of stable equilibrium configurations.

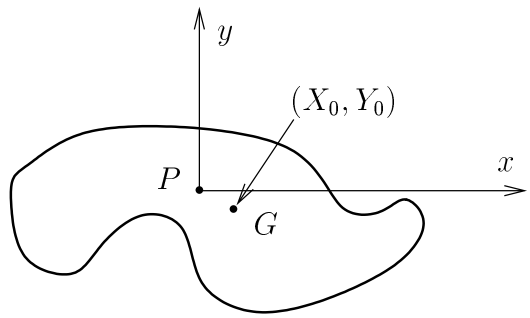

Figure 6: In the equilibrium configuration corresponding to $\theta = 0$, the center of mass is at (X_0, Y_0).

We now consider the potential function (Figure 7):

$$u(\mathbf{r}) = v(\mathbf{r}) + \delta\eta,$$

where v is the unit radial field and δ a positive constant. The second term $\delta\eta$ corresponds to the constant force field $-\delta(0, 1)$. The lifted potential field for a given value of δ can be expressed as follows in the (X, Y, θ) system of coordinates:

$$U_\delta(X, Y, \theta) = V(X, Y) + \delta|S|(\sin\theta X + \cos\theta Y), \tag{17}$$

where $|S|$ is the area of S. For clarity purposes, we define the following functions

$$U_{\theta,\delta}(X, Y) = U(X, Y, \theta, \delta) = U_\delta(X, Y, \theta),$$

where the variables we put in the subscript are considered constant.

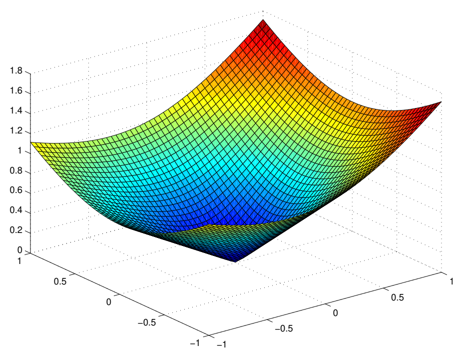

Figure 7: Radial-constant potential field.

Minimum of $U_{\theta,\delta}$ and Equilibrium Curve

The second term of the right hand side of Expression (17) is linear in (X, Y). For this reason, $U_{\theta,\delta}$ has the same second order partial derivatives as V and Hess $U_{\theta,\delta} =$ Hess V is positive definite (Proposition 4). For any fixed value of θ and $\delta < 1$, $U_{\theta,\delta}(X, Y)$ tends toward infinity with (X, Y).

Thus, $U_{\theta,\delta}$ has a unique local minimum. We denote by $(X^*(\theta,\delta), Y^*(\theta,\delta))$ this local minimum. We can express it in the standard system of coordinates by:

$$x^*(\theta,\delta) = \cos\theta\; X^*(\theta,\delta) - \sin\theta\; Y^*(\theta,\delta), \tag{18}$$
$$y^*(\theta,\delta) = \sin\theta\; X^*(\theta,\delta) + \cos\theta\; Y^*(\theta,\delta). \tag{19}$$

For each value of $\delta < 1$, these local minima define a curve of parameter θ that we call *equilibrium curve*. We are going to show now that this curve is of class C^1 and smooth for small values of δ if the pivot point is not on the boundary of the part.

Notice that $(X^*, Y^*)(\theta, 0) = (X_0, Y_0)$ is the minimum of V and therefore is independent of θ.

Proposition 5. *The equilibrium curve is smooth.*

(i) X^*, Y^*, x^* and y^* are continuously differentiable.

(ii) if $(-X_0, -Y_0) \notin \partial S$ (i.e., the pivot point is not on the boundary of the part), there exists δ_0 such that X^*, Y^*, x^* and y^* are smooth over $\mathbf{S}^1 \times [0, \delta_0]$.

Proof. (i) This proposition is a direct result of the implicit function theorem. Indeed, if we define the following function from $\mathbf{R}^4$ into $\mathbf{R}^2$

$$F : (X, Y, \theta, \delta) \to \begin{pmatrix} \frac{\partial U}{\partial X}(X, Y, \theta, \delta) \\ \frac{\partial U}{\partial Y}(X, Y, \theta, \delta) \end{pmatrix}.$$

From (17), as V is of class C^2 (Proposition 2), F is of class C^1. By definition, the equilibrium curve minimizes the lifted potential field for fixed θ and δ and therefore fits the following implicit representation:

$$F(X^*, Y^*, \theta, \delta) = 0.$$

The differential of the partial function $F_{\theta,\delta}$ of the variables (X, Y) is exactly the Hessian of V. From Proposition 4, this differential is invertible everywhere. According to the implicit function theorem, these conditions imply that X^* and Y^* can be expressed as C^1 functions of (θ, δ). As the equilibrium curve is unique, these C^1 functions are necessarily the formerly defined $X^*(\theta,\delta)$ and $Y^*(\theta,\delta)$.

(ii) If the pivot point is not on the boundary of the part, $-(X^*(\theta,0), Y^*(\theta,0)) \notin \partial S$, by the continuity of X^* and Y^*, there exists a δ_0 such that for any $\theta \in \mathbf{S}^1$ and $0 \leq \delta \leq \delta_0$, $-(X^*(\theta,\delta), Y^*(\theta,\delta)) \notin \partial S$. In other words, if we follow the equilibrium curve for a small δ, the origin of the field remains completely inside or completely outside the part and $(X^*, Y^*, \theta, \delta)$ remains in a domain where F is smooth (from Proposition 3). Therefore, according to the implicit function theorem, X^* and Y^* are also smooth. Relations (18) and (19) state that x^* and y^* have the same differentiability properties as X^* and Y^*. $\square$

From now on, we will assume that the pivot point is not on the boundary of the part, so that the partial derivatives of the equilibrium curves are all defined for small δ.

We now point out a property of the equilibrium curves that will constitute the basis of our method to determine the local minima of U. For a fixed value of δ the local minima of U are obviously on the equilibrium curve associated to δ. We are going to show that these local minima are the points where (x^*, y^*) crosses the y axis from $x < 0$ to $x > 0$. For that, we define

$$U_\delta^*(\theta) = U_\delta(X^*(\theta,\delta), Y^*(\theta,\delta), \theta),$$

the minimum value of the lifted potential field for given θ and δ. The variation of $U_\delta^*(\theta)$ along the equilibrium curve is given by the following proposition.

Proposition 6. *For any $\theta \in \mathbf{S}^1$,*

$$\frac{dU_\delta^*}{d\theta}(\theta) = \delta |S| x^*(\theta, \delta).$$

From [13]. For clarity, we omit δ in the notation of this proof. By definition $U_\delta^*(\theta) = U_\delta(X^*(\theta), Y^*(\theta), \theta)$. Differentiating this expression w.r.t. to θ leads to

$$\begin{aligned}
\frac{dU_\delta^*}{d\theta}(\theta) &= \frac{\partial U_\delta}{\partial X}(X^*(\theta), Y^*(\theta), \theta)\frac{dX^*}{d\theta}(\theta) + \frac{\partial U_\delta}{\partial Y}(X^*(\theta), Y^*(\theta), \theta)\frac{dY^*}{d\theta}(\theta) + \\
&\quad \frac{\partial U_\delta}{\partial \theta}(X^*(\theta), Y^*(\theta), \theta) \\
&= \frac{\partial U_\delta}{\partial \theta}(X^*(\theta), Y^*(\theta), \theta) \\
&= \delta|S|(\cos\theta\ X^*(\theta) - \sin\theta\ Y^*(\theta)) \\
&= \delta|S| x^*(\theta),
\end{aligned}$$

using the expression (17) and the fact that the partial derivatives of U_δ w.r.t. X and Y vanish at (X^*, Y^*). □

This proposition leads directly to the following property.

Proposition 7. *For any fixed value of δ, the two following properties are equivalent:*
(i) (X, Y, θ) *is a local minimum of* U_δ*,*
(ii) $X = X^*(\theta, \delta)$*,* $Y = Y^*(\theta, \delta)$ *and the equilibrium curve crosses the y-axis from left to right when θ increases:* $x^*(\theta, \delta) = 0$*,* $x^*(\theta^-, \delta) < 0$*,* $x^*(\theta^+, \delta) > 0$*.*

7 Computation of All Stable Configurations of a Part Under the Combined Unit Radial and Constant Field

Proposition 7 provides a characterization of the stable equilibrium configurations of a part subjected to a combination of a unit radial field and a constant field, in terms of intersection of the equilibrium curve with the y-axis. For a general part and a given value of δ, these configurations can be computed numerically. In [12], such simulations are carried out and the results of these simulations are consistent with the conclusions of our theoretical developments. However, numerical simulation is time-consuming and may yield results difficult to interpret, especially for limit values of δ where the equilibrium curve becomes tangent to the y-axis. In this section, we give a method to predict the equilibrium configurations for small values of δ, using the fact that x^* is smooth.

7.1 Partial Derivatives of x^* and Local Minima of U_δ

In general, we do not have an expression of $x^*(\theta, \delta)$. Otherwise, stable equilibrium configurations would be determined by solving $x^*(\theta, \delta) = 0$ and by applying Proposition 7. However, for small values of δ, we can exploit the continuity of the partial derivatives of x^* using the following idea. Let us assume that we are interested in the sign of a smooth function $f(\delta)$ for small values of δ. If $f(0) \neq 0$ there exists an interval of δ containing 0 over which $f(\delta)$ has the same sign as $f(0)$. If $f(0) = 0$, and if $\frac{\partial^n f}{\partial \delta^n}(0)$ is the first non-zero derivative of f in 0, we can approximate f around 0 by $f(\delta) \sim \frac{\partial^n f}{\partial \delta^n}(0)\frac{\delta^n}{n!}$ (Taylor expansion) and conclude that there exists a small interval of δ containing 0 over which $f(\delta)$ has the same sign as $\frac{\partial^n f}{\partial \delta^n}(0)$.

The following theorem states that this property applies to $x^*(\theta, \delta)$ for small δ and uniformly over θ, that is $x^*(\theta, \delta)$ changes sign around the same values of θ as the first non-uniformly zero $\frac{\partial^n x^*}{\partial \delta^n}(\theta, 0)$ (see Figure 8).

Theorem 1. *If there exists an integer $n \geq 0$ such that*

- *(i) for any k such that $0 \leq k \leq n-1$, $\frac{\partial^k x^*}{\partial \delta^k}(\theta, 0) = 0$ uniformly over $\mathbf{S}^1$,*
- *(ii) $\frac{\partial^n x^*}{\partial \delta^n}(\theta, 0)$ vanishes only at a finite number of points $(\theta_1, ..., \theta_{2m})$, and*
- *(iii) for any $1 \leq l \leq m$, $\frac{\partial^{n+1} x^*}{\partial \theta \partial \delta^n}(\theta_{2l}, 0) > 0$, $\frac{\partial^{n+1} x^*}{\partial \theta \partial \delta^n}(\theta_{2l-1}, 0) < 0$,*

then for small values of δ, the part has exactly m stable equilibrium configurations, and these configurations converge toward $(x^*(\theta_{2l}, 0), y^*(\theta_{2l}, 0), \theta_{2l})$, $1 \leq l \leq m$, when δ tends toward 0.

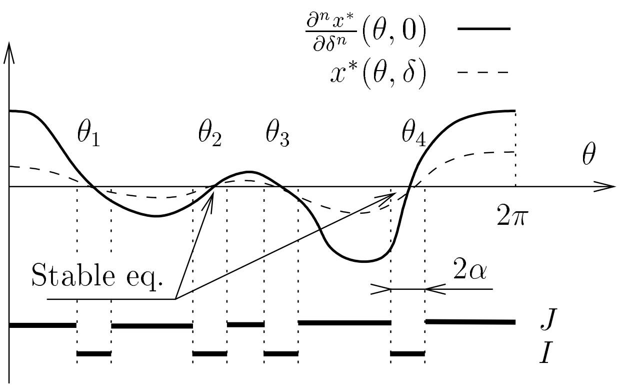

Figure 8: The first non-uniformly zero partial derivative of x^* w.r.t. δ provides all the information about the local minima of U_δ for small values of δ.

Proof. Let us first notice that $\frac{\partial^{n+1} x^*}{\partial \theta \partial \delta^n}(\theta_{2k}, 0)$ represents the slope at θ_{2k} of $\frac{\partial^n x^*}{\partial \delta^n}(\theta, 0)$ seen as a function of θ. Therefore condition (iii) means that $\frac{\partial^n x^*}{\partial \delta^n}(\theta, 0)$ changes sign at each θ_l for $1 \leq l \leq 2m$.

All the partial derivatives of x^* are continuous. Thus, from condition (iii), there exist two positive numbers α and δ_1 such that for any positive real number $\delta < \delta_1$ and any integer l between

1 and m,

$$\begin{array}{ll} \forall\theta \in (\theta_{2l}-\alpha, \theta_{2l}+\alpha), & \frac{\partial^{n+1}x^*}{\partial\theta\partial\delta^n}(\theta,\delta) > 0, \\ \forall\theta \in (\theta_{2l-1}-\alpha, \theta_{2l-1}+\alpha), & \frac{\partial^{n+1}x^*}{\partial\theta\partial\delta^n}(\theta,\delta) < 0. \end{array} \tag{20}$$

Let us denote by I the union of the intervals of θ defined above

$$I = \bigcup_{1\leq l\leq 2m} (\theta_l - \alpha, \theta_l + \alpha)$$

and by $J = \mathbf{S}^1 \setminus I$ the complement of I (see Figure 8). The proof consists of two parts:

1. We first prove that over each interval constituting I, for small fixed δ, the slope $\frac{\partial x^*}{\partial\theta}$ of x^* keeps a constant sign and therefore x^* vanishes only once over each of these intervals.
2. Then, we show that, for small δ, x^* does not vanish over J.

Let us consider each part separately. 1. Differentiating the equation defined in (i) w.r.t. θ, we get that, for any $k \leq n-1$,

$$\frac{\partial^{k+1}x^*}{\partial\theta\partial\delta^k}(\theta,0) = 0 \tag{21}$$

uniformly over $\mathbf{S}^1$. If we take a fixed θ in the interval $(\theta_{2l}-\alpha, \theta_{2l}+\alpha)$, and we consider $\frac{\partial x^*}{\partial\theta}(\theta,\delta)$ as a function of δ that we temporarily denote by $f(\delta)$, (21) can be rewritten

$$\frac{\partial^k f}{\partial\delta^k}(0) = 0 \quad \text{for} \quad 0 \leq k \leq n-1. \tag{22}$$

Moreover, (20) implies that $\forall\delta \in [0,\delta_1]$,

$$\frac{\partial^n f}{\partial\delta^n}(\delta) > 0. \tag{23}$$

Therefore, using Taylor-Lagrange formula, for any $\delta \in [0,\delta_1]$, there exists β, $0 \leq \beta \leq 1$ such that

$$f(\delta) = \sum_{k=0}^{n-1} \frac{\partial^k f}{\partial\delta^k}(0)\frac{\delta^k}{k!} + \frac{\partial^n f}{\partial\delta^n}(\beta\delta)\frac{\delta^n}{n!} \tag{24}$$

$$= \frac{\partial^n f}{\partial\delta^n}(a\delta)\frac{\delta^n}{n!} > 0. \tag{25}$$

This establishes that for any $\delta < \delta_1$ and any $\theta \in (\theta_{2l}-\alpha, \theta_{2l}+\alpha)$, $\frac{\partial x^*}{\partial\theta}(\theta,\delta) > 0$. Thus for a fixed $\delta \in [0,\delta_1]$, the function $x^*(\theta,\delta)$ of θ is increasing over $(\theta_{2l}-\alpha, \theta_{2l}+\alpha)$ and cannot vanish more than once over this interval. Using the same reasoning, we can establish that $x^*(\theta,\delta)$ is decreasing over the intervals $(\theta_{2l-1}-\alpha, \theta_{2l-1}+\alpha)$ and cannot vanish more than once in each of them either.

2. From condition (ii), when θ remains in J, $\frac{\partial^n x^*}{\partial\delta^n}(\theta,0)$ does not vanish. As J is compact, $\left|\frac{\partial^n x^*}{\partial\delta^n}(\theta,0)\right|$ admits a positive lower bound over J, that we denote by m

$$m = \min\left\{\left|\frac{\partial^n x^*}{\partial\delta^n}(\theta,0)\right|, \theta \in J\right\} > 0.$$

From the uniform continuity of $\frac{\partial^n x^*}{\partial \delta^n}(\theta,\delta)$ over the compact set $J \times [0,\delta_1]$, there exists δ_2, $0 < \delta_2 \leq \delta_1$ such that for any $\delta \in [0,\delta_2]$ and any $\theta \in J$,

$$\left|\frac{\partial^n x^*}{\partial \delta^n}(\theta,0)\right| > m \qquad \text{and} \qquad \left|\frac{\partial^n x^*}{\partial \delta^n}(\theta,\delta) - \frac{\partial^n x^*}{\partial \delta^n}(\theta,0)\right| < m,$$

thus

$$\left|\frac{\partial^n x^*}{\partial \delta^n}(\theta,\delta)\right| > 0. \tag{26}$$

We can apply again Taylor-Lagrange relation. To keep the same notation, $f(\delta)$ denotes now $x^*(\theta,\delta)$ considered as a function of δ for a fixed value of $\theta \in J$, we get from condition (i) that condition (22) is satisfied and from (26) that $\frac{\partial^n f}{\partial \delta^n}(\delta)$ does not vanish over $[0,\delta_2]$ and thus keeps a constant sign over this interval. Condition (23) (or its counterpart $\frac{\partial^n f}{\partial \delta^n}(\delta) < 0$) is thus satisfied over $[0,\delta_2]$. We can reuse (24) and (25) (replacing $>$ by $<$ if $\frac{\partial^n f}{\partial \delta^n}(\delta) < 0$) to conclude that $x^*(\theta,\delta)$ does not vanish over $[0,\delta_2]$ and has the same sign as $\frac{\partial^n x^*}{\partial \delta^n}(\theta,0)$.

If we now consider a fixed $\delta < \delta_2$ and if we recall that $\frac{\partial^n x^*}{\partial \delta^n}(\theta,0)$ changes sign between two subintervals constituting J, we can conclude that $x^*(\theta,\delta)$ vanishes *exactly* once over each $(\theta_l - \alpha, \theta_l + \alpha)$, $1 \leq l \leq 2m$, defining a stable equilibrium configuration in each $(\theta_{2l} - \alpha, \theta_{2l} + \alpha)$, $1 \leq l \leq m$. As α can be chosen as small as desired, as δ tends toward 0 the stable equilibrium configurations converge toward $(x^*(\theta_{2l},0),, y^*(\theta_{2l},0), \theta_{2l})$ since x^* and y^* are continuous. □

Theorem 1 changes the problem of determining the sign of $x^*(\theta,\delta)$ to the problem of determining the sign of the first non-uniformly zero $\frac{\partial^n x^*}{\partial \delta^n}(\theta,0)$ $(n \geq 0)$. In the following section, we will devise an algorithm that computes the successive expressions of $\frac{\partial^k x^*}{\partial \delta^k}(\theta,0)$ w.r.t. the partial derivatives of the radial field V at $(0,0)$. But first, let us apply the result of Theorem 1 in the simple cases where $n = 0$ and $n = 1$. We will see that if the pivot point and the center of mass of a part are distinct, the part can be uniquely positioned by a radial-constant field. If the pivot point and the center of mass are the same, an additional condition ensures that the part has exactly two stable equilibrium configurations.

7.2 Unique Stable Equilibrium Configuration

The simplest application of Theorem 1 arises when $n = 0$. In this case, we do not need to compute any partial derivative of x^* w.r.t. δ. Indeed, from (18), we get

$$x^*(\theta,0) = \cos\theta X_0 - \sin\theta Y_0.$$

Thus x^* verifies the conditions of Theorem 1 with $n = 0$ iff $(X_0, Y_0) \neq (0,0)$, that is if the pivot point is distinct from the center of mass. Under this condition, we can rewrite $(X_0, Y_0) = (R\cos\varphi, R\sin\varphi)$, with $R > 0$ and φ two constants, and get

$$x^*(\theta,0) = R\cos(\theta + \varphi).$$

We can thus apply Theorem 1 to get the following corollary.

Corollary 1. *If the pivot point P and the center of mass G are different, that is $(X_0, Y_0) \neq (0,0)$, then for small δ the part has a unique equilibrium configuration. When δ tends toward 0, the pivot point gets closer to the origin and the direction $\vec{PG}$ aligns with the constant force field.*

7.3 Two Stable Equilibrium Configurations

If the pivot point is the same as the center of mass $(X_0, Y_0) = (0, 0)$, Theorem 1 cannot be applied with $n = 0$. This situation happens for symmetric parts, like rectangles and regular polygons for instance. We need thus to get an expression of $\frac{\partial x^*}{\partial \delta}$. For that we use the implicit representation of X^* and Y^*. Since X^* and Y^* minimize U for fixed θ and δ,

$$\begin{aligned} \frac{\partial U}{\partial X}(X^*(\theta,\delta), Y^*(\theta,\delta), \theta, \delta) &= 0, \\ \frac{\partial U}{\partial Y}(X^*(\theta,\delta), Y^*(\theta,\delta), \theta, \delta) &= 0. \end{aligned}$$

Using the expression (17) of U, we can rewrite these equations as

$$\begin{aligned} \frac{\partial V}{\partial X}(X^*(\theta,\delta), Y^*(\theta,\delta)) + \delta|S|\sin\theta &= 0, \\ \frac{\partial V}{\partial Y}(X^*(\theta,\delta), Y^*(\theta,\delta)) + \delta|S|\cos\theta &= 0. \end{aligned}$$

Differentiating these expressions w.r.t. δ and omitting variables in the functions leads to

$$\frac{\partial^2 V}{\partial X^2}(X^*, Y^*)\frac{\partial X^*}{\partial \delta} + \frac{\partial^2 V}{\partial X \partial Y}(X^*, Y^*)\frac{\partial Y^*}{\partial \delta} + |S|\sin\theta = 0, \tag{27}$$

$$\frac{\partial^2 V}{\partial X \partial Y}(X^*, Y^*)\frac{\partial X^*}{\partial \delta} + \frac{\partial^2 V}{\partial Y^2}(X^*, Y^*)\frac{\partial Y^*}{\partial \delta} + |S|\cos\theta = 0. \tag{28}$$

Now, taking $\delta = 0$, by hypothesis, $X^* = 0$, $Y^* = 0$ and we recognize the Hessian of V at $(0, 0)$ in the above equations:

$$\text{Hess } V(0,0) \begin{pmatrix} \frac{\partial X^*}{\partial \delta}(\theta, 0) \\ \frac{\partial Y^*}{\partial \delta}(\theta, 0) \end{pmatrix} + |S| \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = 0. \tag{29}$$

To make notation more compact, let us denote by R_ψ the rotation matrix of angle ψ.

Let us recall that the Hessian of $V(0, 0)$ is symmetric and positive definite. For this reason, it is diagonalizable, *i.e.*, there exists a rotation matrix R_ψ and two positive eigenvalues $1/\lambda_1$, $1/\lambda_2$ such that $\lambda_1 \leq \lambda_2$ and

$$\text{Hess } V(0,0) = R_{-\psi} \begin{pmatrix} \frac{1}{\lambda_1} & 0 \\ 0 & \frac{1}{\lambda_2} \end{pmatrix} R_\psi. \tag{30}$$

Then from (29)

$$\begin{pmatrix} \frac{\partial X^*}{\partial \delta}(\theta, 0) \\ \frac{\partial Y^*}{\partial \delta}(\theta, 0) \end{pmatrix} = [\text{Hess } V(0,0)]^{-1} \begin{pmatrix} -|S|\sin\theta \\ -|S|\cos\theta \end{pmatrix} \tag{31}$$

$$= R_{-\psi} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R_\psi \begin{pmatrix} -|S|\sin\theta \\ -|S|\cos\theta \end{pmatrix}. \tag{32}$$

Using rotation matrix notation, (18) and (19) yield

$$\begin{pmatrix} \frac{\partial x^*}{\partial \delta} \\ \frac{\partial y^*}{\partial \delta} \end{pmatrix} = R_\theta \begin{pmatrix} \frac{\partial X^*}{\partial \delta} \\ \frac{\partial Y^*}{\partial \delta} \end{pmatrix},$$

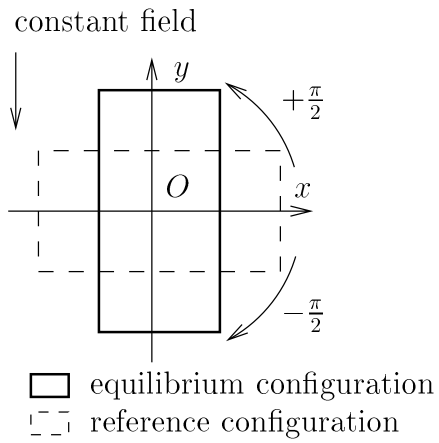

Figure 9: A rectangle put in the radial constant field rotates to align its long edges with the constant component of the field.

and (32) leads to

$$\begin{pmatrix} \frac{\partial x^*}{\partial \delta}(\theta,0) \\ \frac{\partial y^*}{\partial \delta}(\theta,0) \end{pmatrix} = R_{\theta-\psi} \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} R_\psi \begin{pmatrix} -|S|\sin\theta \\ -|S|\cos\theta \end{pmatrix}.$$

Expanding the first line of this latest equation gives the following expression

$$\frac{\partial x^*}{\partial \delta}(\theta,0) = \frac{1}{2}|S|(\lambda_2 - \lambda_1)\sin(2(\theta+\psi)). \quad (33)$$

From this expression, we can deduce the following proposition.

Proposition 8. *If the pivot point P and the center of mass G are the same and if Hess $V(0,0)$ has two different eigenvalues, then the part has two stable equilibrium configurations for small δ. In these configurations, the axis of the part coinciding in configuration $\mathbf{q}_0$ with the x-axis aligns with the eigenvector of Hess $V(0,0)$ of larger eigenvalue.*

Example Let us consider the rectangle defined by the following domain $S = [-2,2]\times[-1,1]$ and described in Figure 9. Because of the symmetries, both the center of mass and the pivot point are located at the center of the rectangle. Let us notice that for polygonal parts, the partial derivatives of V can be computed exactly. We will see how later in the paper. For this rectangle we find

$$\begin{aligned} \frac{\partial^2 V}{\partial X^2}(0,0) &= 8\,\mathrm{argsinh}\tfrac{1}{2} \approx 3.85, \\ \frac{\partial^2 V}{\partial X \partial Y}(0,0) &= 0, \\ \frac{\partial^2 V}{\partial Y^2}(0,0) &= 4\,\mathrm{argsinh}2 \approx 5.77. \end{aligned}$$

The Hessian is already in diagonal form. The larger eigenvalue is $\frac{\partial^2 V}{\partial Y^2}(0,0)$ and the associated eigenvectors are along the y-axis. Thus, for small δ, the small side of the rectangle aligns with the x-axis.

Notice that the above rectangle is uniquely positioned up to symmetry under the radial constant field. Of course, there might exist non symmetric parts with the same center of mass and pivot point. These parts cannot be positioned uniquely by the radial-constant field. Let us point out that some symmetric parts give rise to a Hessian with one double eigenvalue. For instance, it can easily be established that regular polygons have this property. For such parts, Proposition 8 fails to provide a result since $\frac{\partial x^*}{\partial \delta}(\theta, 0)$ is uniformly 0. Thus, we need to compute further partial derivatives of x^* in order to use Theorem 1.

7.4 Algorithm to Compute All Stable Equilibrium Configurations

In this section, we consider a part with the same center of mass and pivot point. Moreover, we assume that the Hessian of V for this part has one double eigenvalue that we denote $1/\lambda$. The goal of this section is to devise an algorithm that computes an expression of the successive partial derivatives $\frac{\partial^n x^*}{\partial \delta^n}(\theta, 0)$ of the equilibrium curve. Once we find the first order n for which $\frac{\partial^n x^*}{\partial \delta^n}(\theta, 0)$ is not uniformly zero, we can use Theorem 1 to determine the number and location of the local minima of the lifted potential field for small δ. We describe now the main steps of this method.

First of all, differentiating (18) n times with respect to δ, we get

$$\frac{\partial^n x^*}{\partial \delta^n}(\theta, \delta) = \cos\theta \, \frac{\partial^n X^*}{\partial \delta^n}(\theta, \delta) - \sin\theta \, \frac{\partial^n Y^*}{\partial \delta^n}(\theta, \delta). \tag{34}$$

Thus, to compute the successive expressions of $\frac{\partial^n x^*}{\partial \delta^n}$, we only need to find expressions of the partial derivatives of X^* and Y^* w.r.t. δ. For that, we will use Equations (27) and (28), that we will successively differentiate w.r.t. δ and evaluate for $\delta = 0$. Let us condensate these equations into the following form

$$\text{Hess } V(X^*, Y^*) \begin{pmatrix} \frac{\partial X^*}{\partial \delta} \\ \frac{\partial Y^*}{\partial \delta} \end{pmatrix} + |S| \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} = 0, \tag{35}$$

where X^*, Y^* and their derivatives are evaluated at (θ, δ).
Since the pivot point and the center of mass are the same,

$$X^*(\theta, 0) = 0, \quad Y^*(\theta, 0) = 0, \quad \frac{\partial V}{\partial X}(0,0) = \frac{\partial V}{\partial Y}(0,0) = 0,$$

and since Hess V has a double eigenvalue,

$$\frac{\partial^2 V}{\partial X^2}(0,0) = \frac{\partial^2 V}{\partial Y^2}(0,0) = \frac{1}{\lambda}, \quad \frac{\partial^2 V}{\partial X \partial Y}(0,0) = 0.$$

Using these equalities, let us evaluate (35) for $\delta = 0$

$$\begin{aligned} \frac{1}{\lambda} \frac{\partial X^*}{\partial \delta}(\theta, 0) + |S| \sin\theta &= 0, \\ \frac{1}{\lambda} \frac{\partial Y^*}{\partial \delta}(\theta, 0) + |S| \cos\theta &= 0. \end{aligned}$$

These equations give an expression of $\frac{\partial X^*}{\partial \delta}(\theta, 0)$, and $\frac{\partial Y^*}{\partial \delta}(\theta, 0)$.

$$\begin{aligned} \frac{\partial X^*}{\partial \delta}(\theta, 0) &= -\lambda |S| \sin\theta, \\ \frac{\partial Y^*}{\partial \delta}(\theta, 0) &= -\lambda |S| \cos\theta. \end{aligned} \tag{36}$$

If we now differentiate (35) w.r.t. θ, we get

$$\text{Hess } V(X^*,Y^*)\begin{pmatrix} \frac{\partial^2 X^*}{\partial\delta^2} \\ \frac{\partial^2 Y^*}{\partial\delta^2} \end{pmatrix} + \frac{\partial}{\partial\delta}(\text{Hess } V(X^*,Y^*))\begin{pmatrix} \frac{\partial X^*}{\partial\delta} \\ \frac{\partial Y^*}{\partial\delta} \end{pmatrix} = 0, \tag{37}$$

where

$$\begin{aligned} \frac{\partial}{\partial\delta}(\text{Hess } V(X^*,Y^*)) &= \begin{pmatrix} \frac{\partial^3 V}{\partial X^3}(X^*,Y^*) & \frac{\partial^3 V}{\partial X^2\partial Y}(X^*,Y^*) \\ \frac{\partial^3 V}{\partial X^2\partial Y}(X^*,Y^*) & \frac{\partial^3 V}{\partial X\partial Y^2}(X^*,Y^*) \end{pmatrix}\frac{\partial X^*}{\partial\delta} \\ &+ \begin{pmatrix} \frac{\partial^3 V}{\partial X^2\partial Y}(X^*,Y^*) & \frac{\partial^3 V}{\partial X\partial Y^2}(X^*,Y^*) \\ \frac{\partial^3 V}{\partial X\partial Y^2}(X^*,Y^*) & \frac{\partial^3 V}{\partial Y^3}(X^*,Y^*) \end{pmatrix}\frac{\partial Y^*}{\partial\delta}. \end{aligned}$$

These equations may seem complex, but we only need to focus on their overall form to notice that

1. the second term of the left-hand side of (37) is an expression including the third-order partial derivatives of V at $(X^*(\theta,\delta),Y^*(\theta,\delta))$ as well as $\frac{\partial X^*}{\partial\delta}(\theta,\delta)$ and $\frac{\partial Y^*}{\partial\delta}(\theta,\delta)$;
2. evaluated at $\delta = 0$, the first term of (37) becomes

$$\frac{1}{\lambda}\begin{pmatrix} \frac{\partial^2 X^*}{\partial\delta^2}(\theta,0) \\ \frac{\partial^2 Y^*}{\partial\delta^2}(\theta,0) \end{pmatrix}.$$

Thus, $\frac{\partial^2 X^*}{\partial\delta^2}(\theta,0)$ and $\frac{\partial^2 Y^*}{\partial\delta^2}(\theta,0)$ can be expressed w.r.t. the third-order partial derivatives of V evaluated at $(0,0)$, $\frac{\partial X^*}{\partial\delta}(\theta,0)$ and $\frac{\partial Y^*}{\partial\delta}(\theta,0)$ only. Using (36) to replace $\frac{\partial X^*}{\partial\delta}(\theta,0)$ and $\frac{\partial Y^*}{\partial\delta}(\theta,0)$, we get an expression of $\frac{\partial^2 x^*}{\partial\theta^2}(\theta,0)$ w.r.t. θ and the partial derivatives of V up to order 3 evaluated at $(0,0)$.

Differentiating again (37) and repeating these operations, we would obtain expressions of the successive $\frac{\partial^n X^*}{\partial\delta^n}(\theta,0)$ and $\frac{\partial^n Y^*}{\partial\delta^n}(\theta,0)$ w.r.t. the partial derivatives of V up to order $n+1$, evaluated at $(0,0)$. Table 1 presents an algorithm performing these computations.

Remark: It can be checked iteratively that the $\frac{\partial^n X^*}{\partial\delta^n}(\theta,0)$ and $\frac{\partial^n Y^*}{\partial\delta^n}(\theta,0)$ are trigonometric polynomials in θ, the coefficients of which are sum of products of $\frac{\partial^k V}{\partial X^l\partial Y^{k-l}}(0,0)$, for $3 \le k \le n+1$, $0 \le l \le k$. Therefore, $\frac{\partial^n x^*}{\partial\theta^n}(\theta,0)$ can either be uniformly zero over $\mathbf{S}^1$ or vanish at a finite number of points. Among these points, those where the slope $\frac{\partial^{n+1} x^*}{\partial\theta\partial\delta^n}(\theta,0)$ is positive are stable equilibrium orientations, according to Theorem 1.

The above computations express the successive derivatives $\frac{\partial^n x^*}{\partial\delta^n}(\theta,0)$ using the partial derivatives of the lifted radial potential field V at $(0,0)$. In the general case, we do not have closed forms of these partial derivatives. The next section addresses this issue.

7.5 Computation of the Partial Derivatives of V

Again, in this section, we consider symmetric parts: $P = G$. We show how to compute the partial derivatives of V, a task which at first approach seems very difficult.

```
minima ← ∅
If (∂V/∂X(0,0) ≠ 0) or (∂V/∂Y(0,0) ≠ 0)
    write ``Pivot point and center of mass distinct.'';
    exit;
endif;
If (∂²V/∂X²(0,0) ≠ ∂²V/∂Y²(0,0)) or (∂²V/∂X∂Y(0,0) ≠ 0)
    ψ ← diagonalize Hess V(0,0) (equation (30))
    minima ← {(0,0,−ψ), (0,0,−ψ+π)};
    exit;
endif;

λ ← ∂²V/∂X²(0,0);
X[1] ← −λ|S| sin θ;
Y[1] ← −λ|S| cos θ;
n ← 1;
x[1] ← cos θ X[1] − sin θ Y[1];
expr[1] ← equation (35);
while (x[n] ≡ 0) do
    n ← n + 1;
    expr[n] ← differentiate(expr[n − 1],δ);
    equation ← evaluate(expr[n],δ = 0);
    (X[n],Y[n]) ← solve(equation,(∂ⁿX*/∂δⁿ(θ,0), ∂ⁿY*/∂δⁿ(θ,0)));
    x[n] ← cos θ X[n] − sin θ Y[n];
    x_θ ← differentiate(x[n],θ);
od;

(θ_1,...,θ_m) ← solve(x[n] = 0, θ);
for each i between 1 and m do
    If evaluate(x_θi, θ = θ_i) > 0
        minima ← minima ∪ {(0,0,θ_i)}
    endif
od;
```

Table 1: Algorithm that computes the local minima of the lifted radial-constant field U_δ for small δ. If the pivot point and the center of mass are distinct, we know the equilibrium configuration is unique but we cannot express it.

Non-Polygonal Parts For a general part, the only way do compute the partial derivatives of V is numerically. For that, there are two methods. The first one consists in evaluating by numerical integration $V(X,Y)$ for several values of (X,Y) around $(0,0)$ and then to compute the successive partial derivatives of V by finite difference: $\frac{\partial V}{\partial X}(X,Y) = \frac{1}{2h}(V(X+h,Y) - V(X-h,Y))$. The main drawback of this numerical method is that the precision decreases dramatically with the order of differentiation.

The second method is more elaborate but more accurate. If the pivot point $P = G$ is outside the part, the origin is outside S and v is smooth over S. Thus, the partial derivatives of V of any order commute with the integral and expressions (10-14) can be generalized to all the other partial derivatives of V. The right way to compute these partial derivatives is then to integrate numerically the partial derivatives of v over S. If the pivot point is on the boundary of the part, there is nothing we can do since the partial derivatives are not defined. If the pivot point is inside the part, V is smooth but the partial derivatives of v of order 3 or more are not integrable over S and expressions (10-14) cannot be generalized. We can overcome this difficulty by using the same idea as in the proof of Proposition 2 presented in the Appendix. The idea consists in approximating v with a field v_h such that

1. $h > 0$ is such that the disc $D_h(0,0)$ is completely inside $\mathrm{int}(S)$,
2. $v_h(\mathbf{r}) = v(\mathbf{r})$ if $\|\mathbf{r}\| \geq h$ and
3. $v_h(\mathbf{r})$ is of class C^k.

For that, we define v_h as a function of $\|\mathbf{r}\|$: $v_h(\mathbf{r}) = g(\|\mathbf{r}\|)$ where of course $g(r) = r$ if $r \geq h$. For $r < h$, we define $f(r)$ as an even polynomial:

$$g(r) = a_0 + a_2 r^2 + \ldots + a_{2k} r^{2k} \quad r < h.$$

Then we determine the coefficients $a_0, \ldots, a_{2k}$ of the polynomial by solving the following linear system of order $k+1$:

$$\begin{aligned} g(h) &= h \\ g'(h) &= 1 \\ g''(h) &= 0 \\ \vdots \;\; &\vdots \;\; \vdots \\ \frac{\partial^k g}{\partial h^k}(h) &= 0. \end{aligned}$$

Thus, with these constraints, g is of class C^k over $(0, +\infty)$ and by defining $v_h(\mathbf{r}) = g(\|\mathbf{r}\|)$, we preserve the continuity of the partial derivatives up to order k at the boundary $\|\mathbf{r}\| = h$. Moreover, on $D_h(0,0)$, $g(\|\mathbf{r}\|)$ contains only even powers of $\|\mathbf{r}\|$. $g(\|\mathbf{r}\|)$ is in fact a polynomial function of $\|\mathbf{r}\|^2 = \xi^2 + \eta^2$ and is thus smooth over $D_h(0,0)$, even at the origin.
If we denote V_h the lifted potential field associated to v_h, and as long as $D_h(-X,-Y)$ remains completely in S,

$$\begin{aligned} V(X,Y) - V_h(X,Y) &= \int_{D_h(-X,-Y)} v(X+\xi, Y+\eta) - v_h(X+\xi, Y+\eta) d\xi d\eta \\ &= \int_{D_h(0,0)} u(\xi,\eta) - v_h(\xi,\eta) d\xi d\eta. \end{aligned}$$

Thus for h small enough, $V(X,Y) - V_h(X,Y)$ is constant in a neighborhood of $(0,0)$. Therefore the partial derivatives of V are the same as the partial derivatives of V_h and the partial derivatives of V_h up to order k commute with the integral since v_h is of class C^k. Therefore,

$$\frac{\partial^k V}{\partial X^p \partial Y^{k-p}}(X,Y) = \int_S \frac{\partial^k v_h}{\partial X^p \partial Y^{k-p}}(X+\xi, Y+\eta) d\xi d\eta, \quad 0 \leq p \leq k.$$

These partial derivatives can be computed by integrating numerically the partial derivatives of v_h. The numerical error comes from the integration and does not increase with the order of differentiation.

Polygonal Parts If the part is polygonal, the partial derivatives of V can be expressed exactly. We explain here how.

Once we have an expression of $\frac{\partial V}{\partial X}$ and $\frac{\partial V}{\partial Y}$, the higher order derivatives follow by formal derivation. By partitioning the polygon into stripes (see Figure 10), $\frac{\partial V}{\partial X}$ can be decomposed into a sum of terms of the form

$$\frac{\partial V}{\partial X} = \sum \int_{\eta_0}^{\eta_1} \int_{a_0\eta+b_0}^{a_1\eta+b_1} \frac{X+\xi}{\sqrt{(X+\xi)^2+(Y+\eta)^2}} d\xi d\eta,$$

where $\xi = a_0\eta + b_0$ and $\xi = a_1\eta + b_1$ are the equations of the sides of each stripe. Thanks to

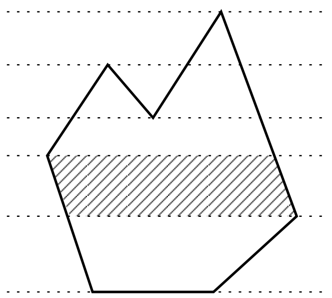

Figure 10: Computation of $\frac{\partial V}{\partial X}$ by partitioning the polygonal part into stripes.

our system of coordinates (X, Y, θ), the integrand of the above integral is the partial derivative of $\sqrt{(X+\xi)^2+(Y+\eta)^2}$ w.r.t. X but also w.r.t. ξ, so that each of these integral can be rewritten as follows:

$$\int_{\eta_0}^{\eta_1} \sqrt{(X+a_1\eta+b_1)^2+(Y+\eta)^2} - \sqrt{(X+a_0\eta+b_0)^2+(Y+\eta)^2} d\eta.$$

Using the appropriate change of variables, the above integrals can be computed exactly and expressed with the coordinates of the vertices of the polygon. $\frac{\partial V}{\partial Y}$ is computed the same way, by switching the role of ξ and η.

8 Examples

We have implemented the algorithm presented in Table 1 using Maple and we have applied it to a number of examples. In this section we compute the stable equilibrium configurations of several parts, focusing our attention on regular polygons. We report the relevant values of the partial derivatives of V and the successive $\frac{\partial^k x^*}{\partial \delta^k}(\theta, 0)$.

Equilateral Triangle

$$\begin{array}{ll}
\frac{\partial^2 V}{\partial X^2}(0,0) = \frac{\partial^2 V}{\partial Y^2}(0,0) = \frac{3}{2}\text{argsinh}(\sqrt{3}) & \frac{\partial^2 V}{\partial X \partial Y}(0,0) = 0 \\
\frac{\partial^3 V}{\partial X^3}(0,0) = \frac{3\sqrt{3}}{2} - \frac{3}{2}\,\text{argsinh}(\sqrt{3}) & \frac{\partial^3 V}{\partial X^2 \partial Y}(0,0) = 0 \\
\frac{\partial^3 V}{\partial X \partial Y^2}(0,0) = -\frac{3\sqrt{3}}{2} + \frac{3}{2}\,\text{argsinh}(\sqrt{3}) & \frac{\partial^3 V}{\partial Y^3}(0,0) = 0
\end{array}$$

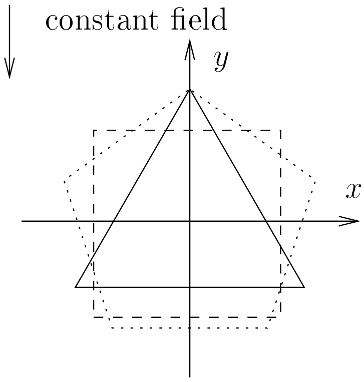

Figure 11: Equilibrium configurations of regular polygons subjected to the combination of a unit radial field and a small constant field.

and

$$\frac{\partial x^*}{\partial \delta}(\theta,0) = 0, \qquad \frac{\partial^2 x^*}{\partial \delta^2}(\theta,0) = \frac{3}{4}\frac{\mathrm{argsinh}(\sqrt{3}) - \sqrt{3}}{\mathrm{argsinh}(\sqrt{3})^3}\cos(3\theta).$$

Since, the coefficient in front of $\cos(3\theta)$ is positive, for small δ, the triangle has 3 equilibrium positions corresponding to the orientations $\pi/2 + 2k\pi/3$, $k = 1, 2$ or 3.

Square

$$\begin{array}{lll}
\frac{\partial^2 V}{\partial X^2}(0,0) = \frac{\partial^2 V}{\partial Y^2}(0,0) = 2\sqrt{2}\mathrm{argsinh}(1) & & \frac{\partial^2 V}{\partial X \partial Y}(0,0) = 0 \\
\frac{\partial^3 V}{\partial X^3}(0,0) = \frac{\partial^3 V}{\partial X^2 \partial Y}(0,0) = \frac{\partial^3 V}{\partial X \partial Y^2}(0,0) = & & \frac{\partial^3 V}{\partial Y^3}(0,0) = 0 \\
\frac{\partial^4 V}{\partial X^4}(0,0) = -4, & \frac{\partial^4 V}{\partial X^3 \partial Y}(0,0) = 0 & \frac{\partial^4 V}{\partial X^2 \partial Y^2}(0,0) = 0 \\
\frac{\partial^4 V}{\partial X \partial Y^3}(0,0) = 0, & \frac{\partial^4 V}{\partial Y^4}(0,0) = -4 &
\end{array}$$

and

$$\frac{\partial x^*}{\partial \delta}(\theta,0) = 0, \quad \frac{\partial^2 x^*}{\partial \delta^2}(\theta,0) = 0, \quad \frac{\partial^3 x^*}{\partial \delta^3}(\theta,0) = \frac{1}{8\,\mathrm{argsinh}(1)^4}\sin(4\theta).$$

The square has thus 4 equilibrium configurations corresponding to $\theta = k\pi/2$, $k = 1, 2, 3$ or 4, for small δ.

Pentagon The expressions of the partial derivatives of $V(X, Y)$ for a regular pentagon are very long. We will only report the final result, namely

$$\frac{\partial x^*}{\partial \delta}(\theta,0) = 0 \quad \frac{\partial^2 x^*}{\partial \delta^2}(\theta,0) = 0 \quad \frac{\partial^3 x^*}{\partial \delta^3}(\theta,0) = 0$$

$$\frac{\partial^4 x^*}{\partial \delta^4}(\theta,0) = -\lambda\cos(5\theta),$$

where $\lambda = \frac{3125(3+\sqrt{5})\frac{\partial^5 V}{\partial X^5}(0,0)}{512\frac{\partial^2 V}{\partial X^2}(0,0)^5} > 0$. From this latter expression, we can conclude that for small δ, the regular pentagon has five stable equilibrium configurations about $\pi/10 + 2k\pi/5$,$k = 0, 1, 2, 3, 4$.

Interpretation Although the expressions of the partial derivatives of x^* w.r.t. the partial derivatives of V are very complex, they simplify dramatically in the case of regular polygons. We checked up to order $n = 6$ vertices and we noticed that x^* has the following properties:

1. the first non-uniformly zero expression $\frac{\partial^k x^*}{\partial \delta^k}(\theta, 0)$ is for $k = n - 1$ and
2. $\frac{\partial^{n-1} x^*}{\partial \delta^{n-1}}(\theta, 0)$ can be simplified to:

$$\frac{\partial^{n-1} x^*}{\partial \delta^{n-1}}(\theta, 0) = \lambda_1 \cos(n\theta + \lambda_2),$$

   where λ_1 and λ_2 are constants.

The above results suggest that regular polygons can be uniquely positioned up to symmetry using a combination of a unit radial field and a small constant field. The equilibrium configurations seem to be those for which an edge of the polygon crosses perpendicularly the negative y-axis (see Figure 11).

9 Discussion

This paper proves that the combination of a unit radial and a constant force field orients symmetric parts to a single stable equilibrium and describes an algorithm to compute all stable configurations for non-symmetric parts. Our work provides further evidence that programmable force fields are a powerful tool for part manipulation as it has been suggested in [21, 15, 18]. To our knowledge this is the first work that provides an exhaustive analysis and computation of all stable equilibrium configurations of any part under a single universal force field. Our work can lead to the design of a new generation of efficient, open-loop parts feeders that can bring a part to a desired orientation from any initial orientation without the need of sensing or a clock.

Practical implementations of force fields are currently limited. While waiting for the technology to yield better devices, we find it worthwhile to study what programmable force fields can and what they can not do. It is our belief that the theory of programmable force fields is at a very early stage. We hope that as the field matures, we will see an increasing number of papers addressing the geometric, algorithmic and implementation problems that emerge in this area. We only point out a few open issues below.

Issues related to the analysis in this paper We showed in this paper that even symmetric parts can be oriented with the combination of a unit radial and a constant force field. One question we have not been able to answer is if the above field can orient any symmetric part upto symmetry. Our calculations in Section 8 suggest that this might be the case. Another issue is that our results are asymptotic for small δ. The study of the same fields for general δ is an open question.

Implementation of the Unit Radial - Constant Field In [13] we discussed practical issues related to building devices that could implement the combination of a unit radial and a constant fields. The models offered by MEMS technology and motor actuator arrays were mentioned. The same discussion applies here and the reader is refered to our earlier paper. The only point that we would like to iterate in this paper is that if an array implementing a unit radial field is built, then the same array can be used to implement the combined field. The idea is that the array can be titled to add a gravity component that will implement the small constant field. Under some simplifying assumptions the magnitute of the constant field equals the tangent of the angle at which the planar array has been titled.

Modeling and rate of convergence A question that arises in the context of this paper, as it arises in the context of all previous published work on this topic [21, 15, 18, 36, 13], is the rate of convergence to the equilibrium configuration. This is an open question. Clearly, the value of δ and the distance of the pivot point and the center of mass of the part affect convergence. A missing component for a thorough analysis is a good model for a device that can implement the programmable force field and a modeling of friction and dynamics.

Other fields that achieve a single equilibrium for parts Another interesting direction is to search for different force fields that can achieve a single equilibrium configuration for most parts and are simpler or easier to implement with a specified technology.

Discretization and uncertainty issues As mentioned above, one very important question is the practical implementation of the proposed field. Our earlier work [13] describes some preliminary ideas. Clearly, it will take a long time before devices that are capable of implementing general force fields are available. It is conceivable that MEMS technology will become so advanced that a huge number of actuators would be fitted in a wafer, or that air jet technology will become so accurate that it could deliver almost a continuous force field. In that case, if the part is relatively big, one may assume that the implemented field is a continuous field. Waiting for such a technology however, it is worth studying discretization issues. Recent research described in [46, 47] studies several issues arising with a discretized implementation of force fields using small mechanical motors. Another interesting topic is to study the stability of the equilibrium orientations for toleranced parts. It is well known that parts are not manufactured with perfect geometry. Ideally, one would like to use force fields in which small changes in the shape of the part do not affect the equilibrium substantially.

Parallelism So far we have considered only the equilibria of one part in a force field. But what happens if two parts are placed into the field simultaneously? It is possible that the parts will settle in predictable configurations. This effect could be exploited for automated assembly. When parts are initially placed far enough apart, it may be possible to implement several unit radial-constant fields next to each other to achieve parallel positioning. This issue is particularly interesting since there is no overhead for parallelism in such a device, as no communication and control is required.

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Table of Symbols

E	subset of $\mathbf{R}^2$
$\mathrm{int}(E)$	interior of E
∂E	boundary of E
E^d	set of points at a distance not more than d from E
$\mathcal{C} = \mathbf{R}^2 \times \mathbf{S}^1$	configuration space of a part in the plane
$\mathbf{q}_0$	reference configuration of a part
S	surface occupied by the part in configuration $\mathbf{q}_0$
G	center of mass of the part
P	pivot point of the part
$\mathbf{q} = (x, y, \theta)$	configuration of the part
$\varphi_{\mathbf{q}}$	rigid-body transformation associated to $\mathbf{q}$
$\mathbf{r} = (\xi, \eta)$	point in the plane
$E_{\mathbf{q}} = \varphi_{\mathbf{q}}(E)$	image of the subset E by $\varphi_{\mathbf{q}}$
$d_{\mathcal{C}}$	distance in the configuration space
$u(\xi, \eta)$	potential field in the plane
$\mathbf{f}(\xi, \eta)$	force field in the plane
$U(\mathbf{q})$	potential field over CS
$\mathbf{F}(\mathbf{q})$	resultant force of a force field over a part
$M(\mathbf{q})$	moment of a force field over a part
$v(\xi, \eta)$	unit-radial potential field
V	potential field over $\mathcal{C}$ associated to v
(X, Y, θ)	system of coordinates over $\mathcal{C}$
(X_0, Y_0)	unique minimum of V
$\mathcal{C}^{in}$	subset of configurations for which $(0,0) \in \mathrm{int}(S_{\mathbf{q}})$
$\mathcal{C}^{out}$	subset of configurations for which $(0,0) \notin S_{\mathbf{q}}$
$\mathcal{C}^{bound}$	subset of configurations for which $(0,0) \in \partial S_{\mathbf{q}}$
det	determinant operator
tr	trace operator
Hess	Hessian operator
δ	magnitude of the constant field
$(x^*(\theta, \delta), y^*(\theta, \delta))$	equilibrium curve in the (x, y, θ) system of coordinates
$(X^*(\theta, \delta), Y^*(\theta, \delta))$	equilibrium curve in the (X, Y, θ) system of coordinates
U_δ, $U_{\theta,\delta}$	radial-constant field over $\mathcal{C}$, the variables in subscript being considered as constant

Appendix: Proof of Proposition 2

We will only prove here that $\frac{\partial^2 V}{\partial X^2}$ exists everywhere and is continuous and we will establish (12). V being C^2 and the expressions of the other partial derivatives can be established by a similar proof.

According to Proposition 1, we already have

$$\frac{\partial V}{\partial X}(X,Y) = \int_S \frac{\partial v}{\partial \xi}(X+\xi, Y+\eta) d\xi d\eta,$$

and we want to get

$$\frac{\partial^2 V}{\partial X^2}(X,Y) = \int_S \frac{\partial^2 v}{\partial \xi^2}(X+\xi, Y+\eta) d\xi d\eta. \tag{38}$$

The standard results of theory of integration make the differentiation of $\frac{\partial v}{\partial \xi}(X+\xi, Y+\eta)$ under the integral possible if three conditions are met

1. $\frac{\partial v}{\partial \xi}(X+\xi, Y+\eta)$ is integrable over S for any $(X,Y) \in K$, where $K \subset \mathbf{R}^2$ is a compact set,
2. $\frac{\partial^2 v}{\partial \xi^2}(X+\xi, Y+\eta)$ exists for any $(X,Y) \in K$ and any $(\xi, \eta) \in S$,
3. there exists a positive integrable function $g(\xi, \eta)$ such that $\left|\frac{\partial^2 v}{\partial \xi^2}(X+\xi, Y+\eta)\right| < g(\xi,\eta)$ for any $(X,Y) \in K$ and any $(\xi,\eta) \in S$.

For the unit radial field, conditions 2 and 3 are not met because of the singularity of v at the origin: $\frac{\partial^2 v}{\partial \xi^2}(X+\xi, Y+\eta)$ does not exist if $X = -\xi$ and $Y = -\eta$ and cannot be bounded by a integrable function independent of (X,Y).

To avoid the singularity of the field at the origin, we define a set of C^2 potential fields equal to $v(\mathbf{r})$ outside the disc $D_h(0,0)$ of radius h centered at the origin. Let us denote $\mathbf{r} = (\xi, \eta)$. Then

$$\begin{array}{rcll} v_h(\mathbf{r}) & = & v(\mathbf{r}) = \sqrt{\xi^2+\eta^2} & \text{if } \|\mathbf{r}\| \geq h \\ v_h(\mathbf{r}) & = & -\frac{1}{8h^3}\|\mathbf{r}\|^4 + \frac{3}{4h}\|\mathbf{r}\|^2 + \frac{3h}{8} & \text{if } \|\mathbf{r}\| < h \end{array} \tag{39}$$

and we define the corresponding lifted potential fields

$$V_h(X,Y) = \int_S v_h(X+\xi, Y+\eta) d\xi d\eta.$$

As v_h is C^2 over the compact set $S \times K$, its partial derivatives up to order 2 are continuous and bounded over this set. Conditions 1,2,3 defined above are thus satisfied, V_h is C^2 and we can write

$$\frac{\partial^2 V_h}{\partial X^2}(X,Y) = \int_S \frac{\partial^2 v_h}{\partial \xi^2}(X+\xi, Y+\eta) d\xi d\eta. \tag{40}$$

Given a configuration of the part, there are three different cases.

1. The center of the radial field is outside the domain of integration S: $(-X,-Y) \notin S$,
2. the center of the radial field is inside S: $(-X,-Y) \in \text{int}(S)$ or

3. the center of the field is on the boundary of S: $(-X,-Y)\in\partial S$.

In the first case, for h sufficiently small, the disc $D_h(-X,-Y)$ where the fields v and v_h are different is completely outside the domain of integration thus $V(X,Y)=V_h(X,Y)$. In the second case, for h small enough, the disc $D_h(-X,-Y)$ is completely inside S and the difference $V(X,Y)-V_h(X,Y)$ does not depend on (X,Y). Indeed,

$$\begin{aligned}V(X,Y)-V_h(X,Y) &= \int_{D_h(-X,-Y)} v(X+\xi,Y+\eta)-v_h(X+\xi,Y+\eta)d\xi d\eta\\ &= \int_{D_h(0,0)} u(\xi,\eta)-v_h(\xi,\eta)d\xi d\eta.\end{aligned}$$

And this equality holds for any (X,Y) such that $D_h(-X,-Y)$ remains completely in S. Therefore, in cases 1 and 2, for h small enough,

$$\frac{\partial^2 V}{\partial X^2}(X,Y)=\frac{\partial^2 V_h}{\partial X^2}(X,Y).$$

To establish (38) outside ∂S, we only need to prove that for any (X,Y) such that $(-X,-Y)\notin\partial S$,

$$\lim_{h\to 0}\int_S \frac{\partial^2 v_h}{\partial\xi^2}(X+\xi,Y+\eta)d\xi d\eta=\int_S\frac{\partial^2 v}{\partial\xi^2}(X+\xi,Y+\eta)d\xi d\eta. \tag{41}$$

For (X,Y) fixed, $\frac{\partial^2 v_h}{\partial\xi^2}(X+\xi,Y+\eta)$ converges toward $\frac{\partial^2 v}{\partial\xi^2}(X+\xi,Y+\eta)$ point-wise almost everywhere. We need to bound $\frac{\partial^2 v_h}{\partial\xi^2}$ by a an integrable function $g(\xi,\eta)$ independent of h in order to use Lebesgue's dominated convergence theorem. For that, we consider a point $\mathbf{r}=(\xi,\eta)$ and we look at $\frac{\partial^2 v_h}{\partial\xi^2}(\mathbf{r})$ when h varies. There are two different cases: $h\le\|\mathbf{r}\|$ and $h>\|\mathbf{r}\|$.
If $h\le\|\mathbf{r}\|$,

$$\left|\frac{\partial^2 v_h}{\partial\xi^2}(\mathbf{r})\right|=\frac{\eta^2}{\|\mathbf{r}\|^3}\le\frac{1}{\|\mathbf{r}\|}.$$

If $h>\|\mathbf{r}\|$, differentiating Expression (39) of $v(\mathbf{r})$, we get

$$\frac{\partial^2 v}{\partial\xi^2}(\mathbf{r})=-\frac{\|r\|^2}{2h}-\frac{\xi^2}{h},$$

and thus,

$$\left|\frac{\partial^2 v}{\partial\xi^2}(\mathbf{r})\right|\le\frac{\|\mathbf{r}\|^2}{2h}+\frac{\|\mathbf{r}\|^2}{h}\le\frac{3\|\mathbf{r}\|^2}{2h}\le\frac{3}{2}\|\mathbf{r}\|,$$

since $\frac{1}{h}\le\frac{1}{\|\mathbf{r}\|}$. Therefore, $\left|\frac{\partial^2 v}{\partial\xi^2}(\mathbf{r})\right|$ is bounded by $\max(\frac{1}{\|\mathbf{r}\|},\frac{3}{2}\|\mathbf{r}\|)$ which is integrable and does not depend on h. Thus, Lebesgue's dominated convergence theorem implies (41) and we have established (38).

We have now to show that $\frac{\partial^2 V}{\partial X^2}$ is continuous. For that, let us notice that

$$\begin{aligned}\left|\frac{\partial^2 V}{\partial X^2}(X,Y)-\frac{\partial^2 V_h}{\partial X^2}(X,Y)\right| &\le \int_{D_h(0,0)}\left|\frac{\partial^2 v}{\partial\xi^2}(\xi,\eta)-\frac{\partial^2 v_h}{\partial\xi^2}(\xi,\eta)\right|d\xi d\eta\\ &\le \int_{D_h(0,0)}\left|\frac{\partial^2 v}{\partial\xi^2}(\xi,\eta)\right|+\int_{D_h(0,0)}\left|\frac{\partial^2 v_h}{\partial\xi^2}(\xi,\eta)\right|\end{aligned}$$

and both integrals tend toward 0 with h: the first one because $\frac{\partial v}{\partial \xi}$ is integrable, the second one because $\left|\frac{\partial^2 v_h}{\partial \xi^2}(\xi,\eta)\right|$ is bounded by an integrable function independent of h. Thus the functions $\frac{\partial^2 V_h}{\partial X^2}$ converge uniformly toward $\frac{\partial^2 V}{\partial X^2}$ as h tends toward 0. Since they are continuous, so is $\frac{\partial^2 V}{\partial X^2}$ everywhere, even across the boundary of S. The same reasoning can be applied to the other partial derivatives of V, up to order 2, to conclude that V is of class C^2 over $\mathbf{R}^2$ and that its partial derivatives are expressed by (10-14).