The Thousand Faces of Constraint Propagation

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Outline

Constraint Programming

- Solving hard combinatorial problems by Decomposition

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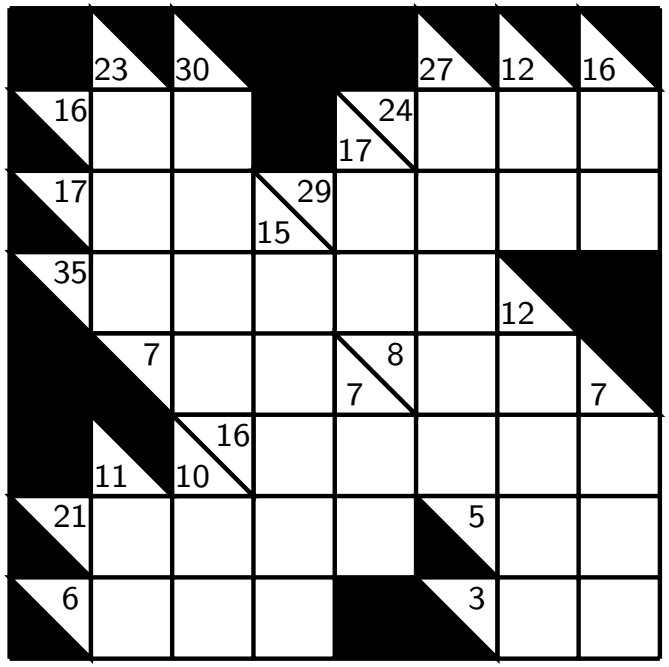

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- Constraints: $C_1, \ldots, C_k$
  - Relations = set of solutions
- Domains: $D_1, \ldots, D_n$
  - Domain representation $\simeq$ relaxation of the problem's solutions
    - ★ $D_1 \times D_2 \times \ldots \times D_n$ is a super set of the solutions

Domain Representation (discrete domains)

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- Minimal Relaxation (discrete domain)

A	A	A	A
A	K	K	A
A	L	L	A
A	N	N	A
A	T	T	A
I	A	A	I
I	K	K	I
I	L	L	I
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Domain Representation (bounds)

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A	K	K	A
A	N	N	A
I	L	L	I
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O	T	T	O

[A,. . . ,O] [A,. . . ,T]

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A	A	A	A
A	B	B	A
A	C	C	A
A	D	D	A
A	E	E	A
A	F	F	A
A	G	G	A
A	H	H	A
A	I	I	A
A	J	J	A
A	K	K	A
A	L	L	A
A	M	M	A
A	N	N	A
A	O	O	A
A	P	P	A
A	Q	Q	A
A	R	R	A
A	S	S	A
.	.	.	.

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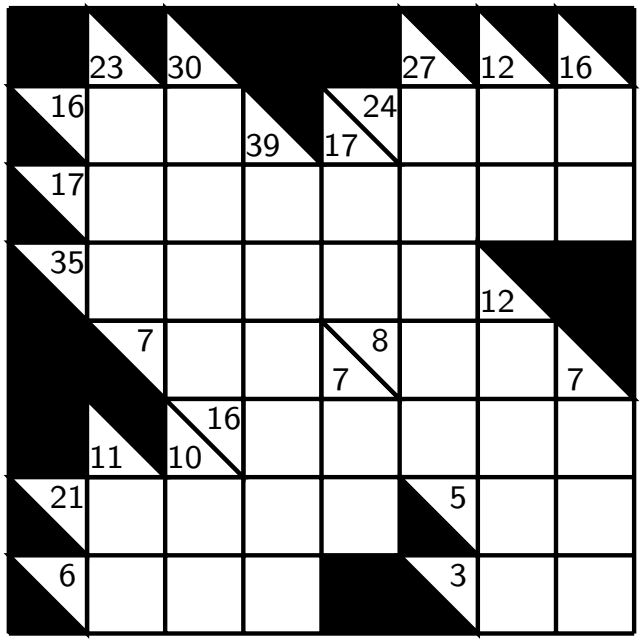
  - Also:
    - ★ Multi-valued Decision Diagram MDD Consistency [Hooker, Hadzic, van Hoeve 2007]
    - ★ Set variables [Puget 1992, Gervet 1997], Length-Lex representation [Gervet and Van Hentenryck 2006]
    - ★ Graph variables, Function variables, ...

Example: Kakuro



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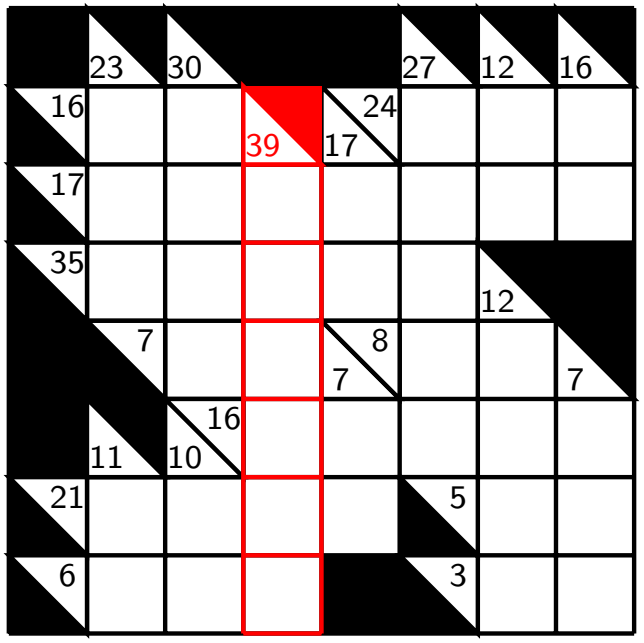

- $\sum_{i=1}^{6} x_i = 39$
- $\textsc{AllDifferent}(\{x_1, \ldots, x_6\})$

x_1 :	$\{$							8	9$\}$
x_2 :	$\{1$	2			6	7		8	9$\}$
x_3 :	$\{$							8	9$\}$
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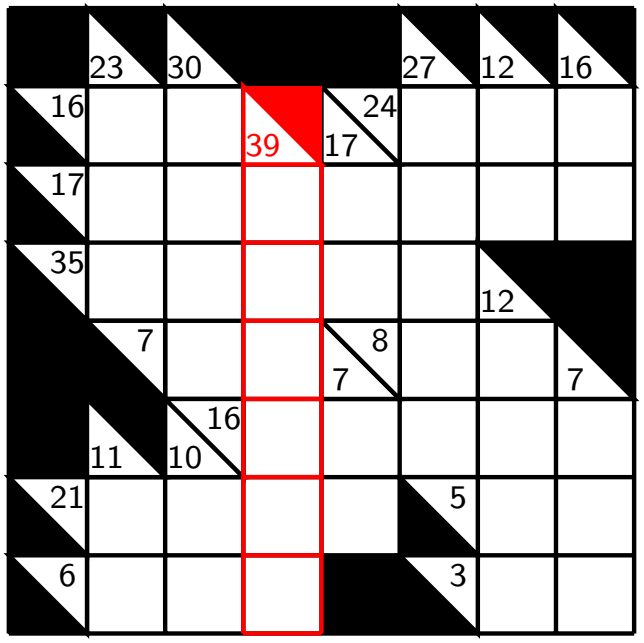

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$\textsc{AllDifferent}(\{x_1, \ldots, x_6\}) \cap D$

```
888888888888888888888888888888888888999999999999999999999999999999999999
111111122222222222266666667777777777711111112222222222266666667777777777
999999999999999999999999999999999999988888888888888888888888888888888888
555666611115556666111155511115556666555666611115556666111155511115556666
267227766771671177227712722661261122267227766771671177227712722661261122
444454545454444545454544445454444545454444545454544445454545444454544444545
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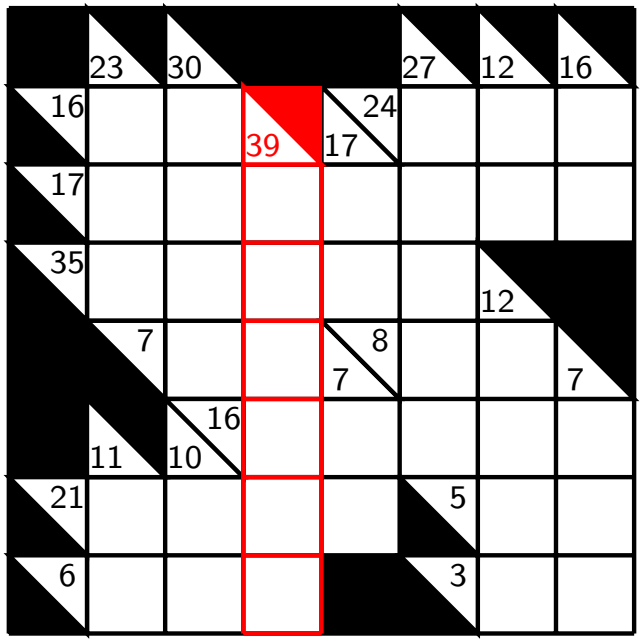

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888888888888888888888888888888888888999999999999999999999999999999999999999     {8, 9}
111111122222222222266666667777777777711111112222222222226666666777777777777 {1, 2, 6, 7}
999999999999999999999999999999999999988888888888888888888888888888888888888     {8, 9}
555666611115556666111155511115556666555666611115556666111155511115556666   {1, 5, 6}
267227766771671177227712722661261122267227766771671177227712722661261122 {1, 2, 6, 7}
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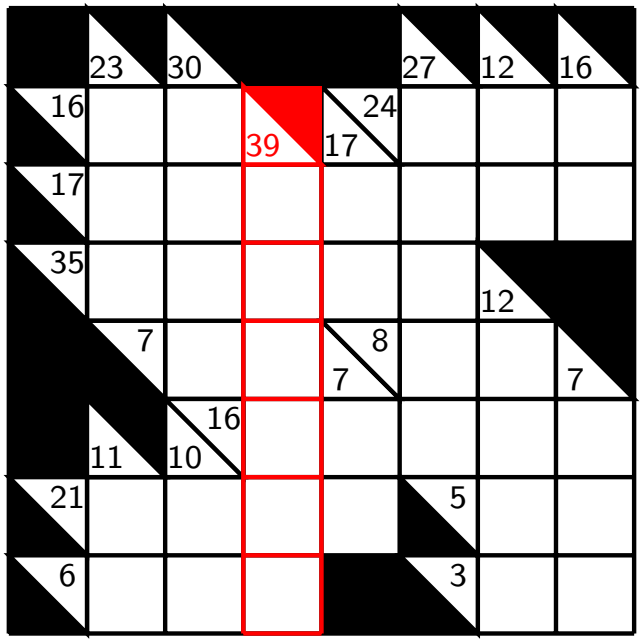

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6	6	6	6	6	6	7	7	7	7	6	6	6	6	7
8	8	8	9	9	9	8	8	8	9	8	8	8	9	8
5	6	6	5	5	6	5	5	6	5	5	5	6	5	5
7	6	7	6	7	6	6	7	6	6	6	7	6	6	6
5	5	4	5	4	4	5	4	4	4	5	4	4	4	4

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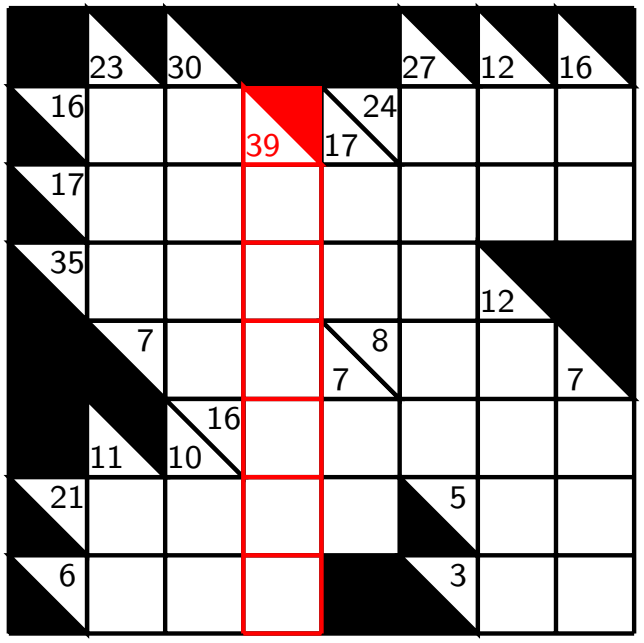

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8	8	8	9	9	9	8	8	8	9	8	8	8	9	8	$\{8, 9\}$
5	6	6	5	5	6	5	5	6	5	5	5	6	5	5	$\{5, 6\}$
7	6	7	6	7	6	6	7	6	6	6	7	6	6	6	$\{6, 7\}$
5	5	4	5	4	4	5	4	4	4	5	4	4	4	4	$\{4, 5\}$

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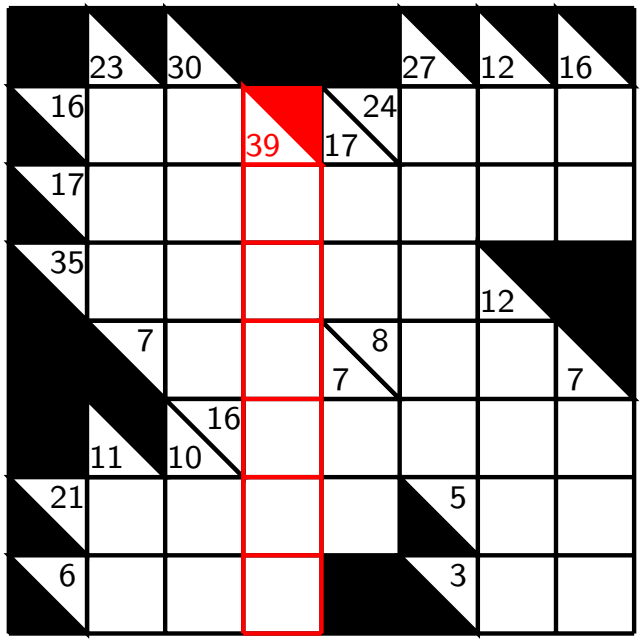

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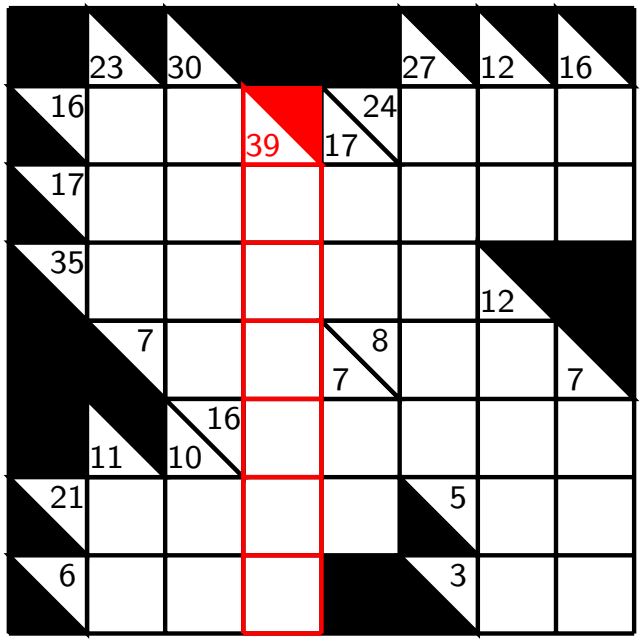

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    - ★ Propagating in $O(K)$, then solving in $O(nK)$

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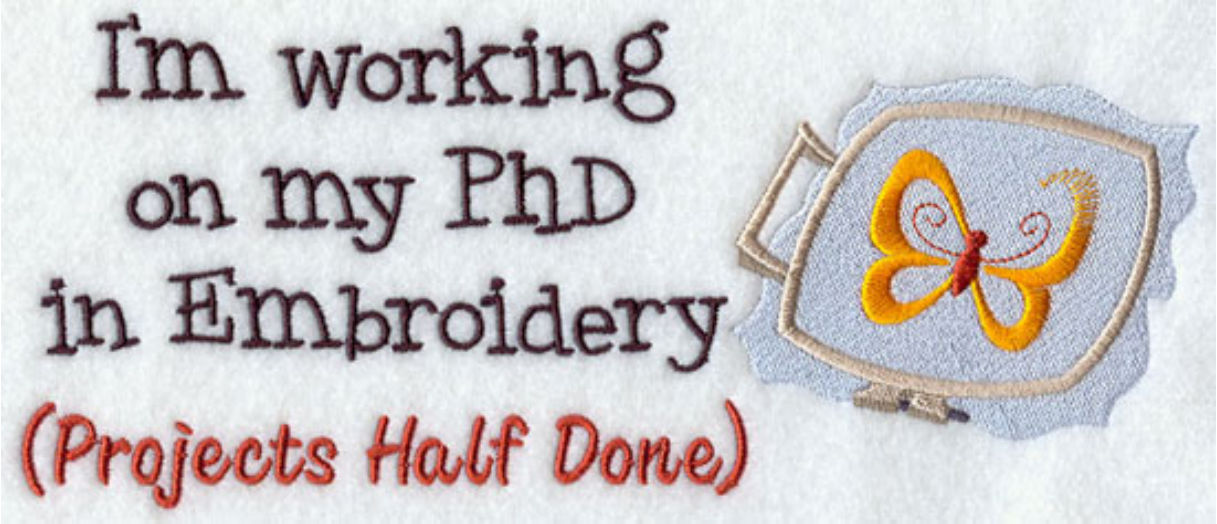
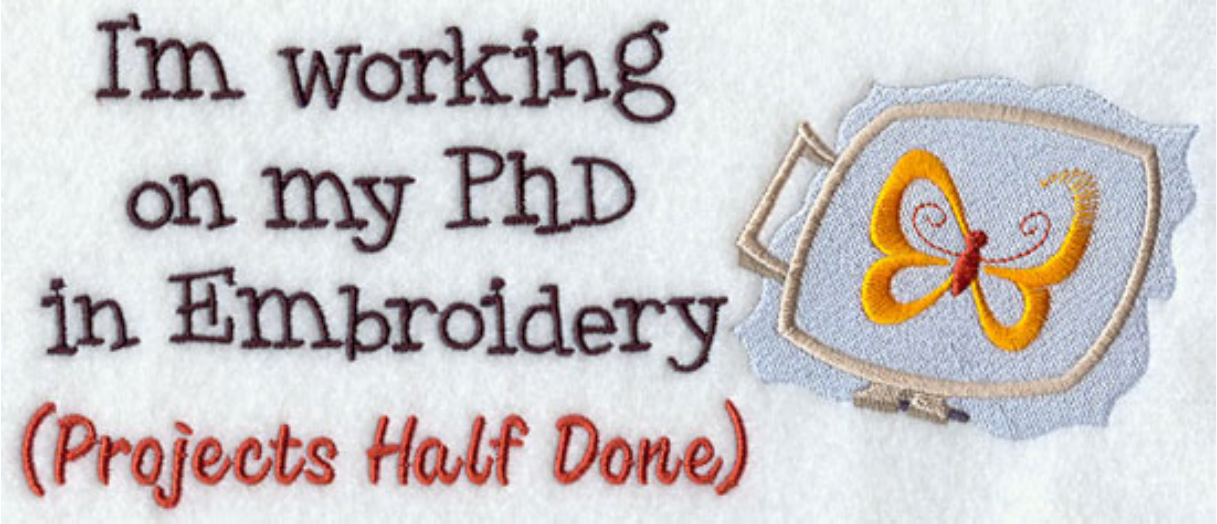
    - ★ Solving in $O(K)$, then propagating in $O(ndK)$
    - ★ Propagating in $O(K)$, then solving in $O(nK)$
  - In practice:
    - ★ Same complexity class, but different algorithms
    - ★ Propagating is harder

An Example of Propagator: the SWITCH Constraint



- A set of n garments to embroid
  - Each garment require a subset of m available colors
- A machine with k reels of thread



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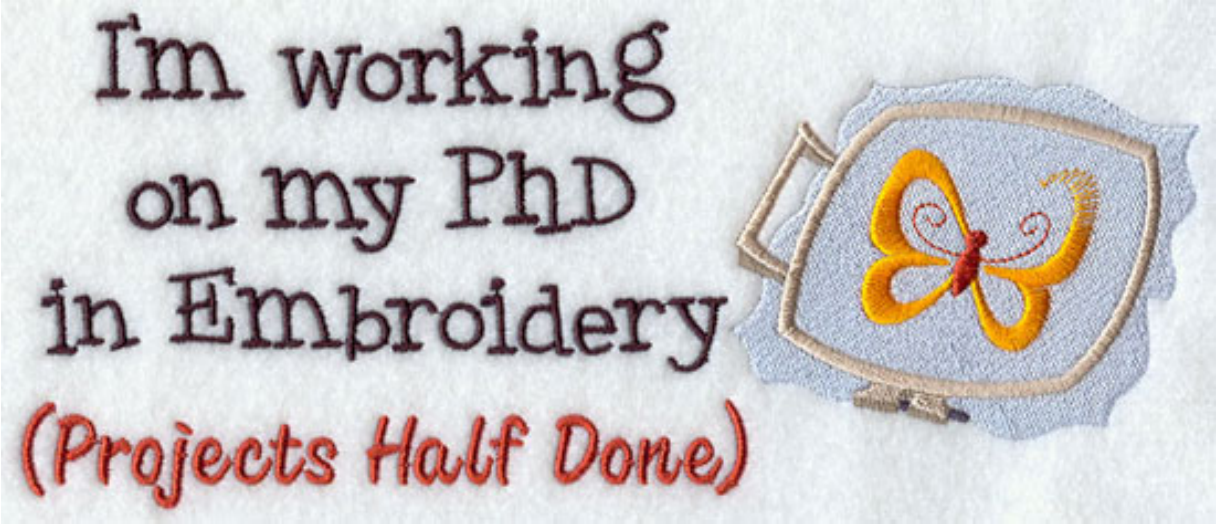
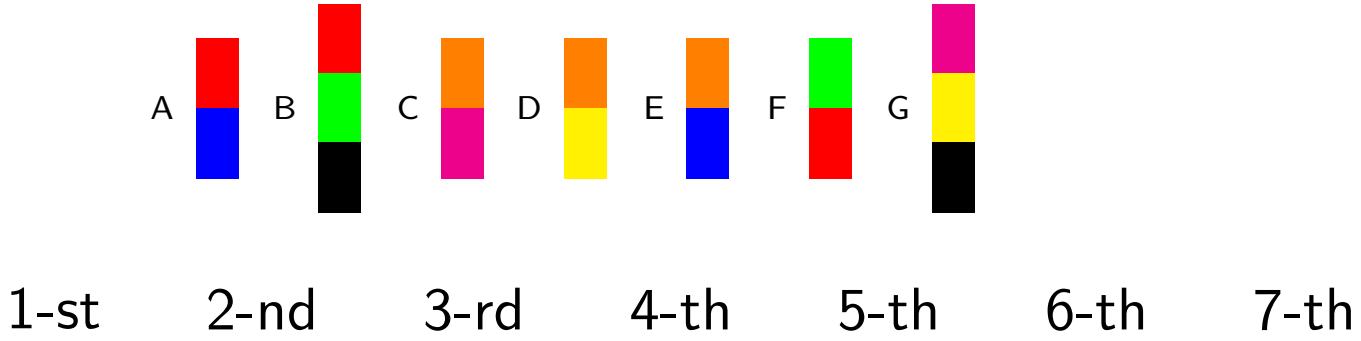


- A set of n garments to embroid
  - Each garment require a subset of m available colors
- A machine with k reels of thread
- Replacing a reel costs time
- What is the best sequence of garments?

Embroidery Scheduling



Reel 1

Reel 2

Reel 3

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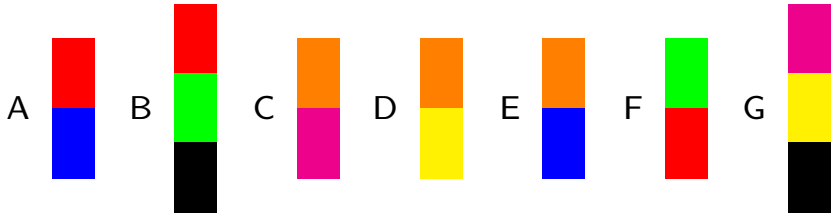

1-st	2-nd	3-rd	4-th	5-th	6-th	7-th
A	*B*	*C*	*D*	*E*	*F*	*G*

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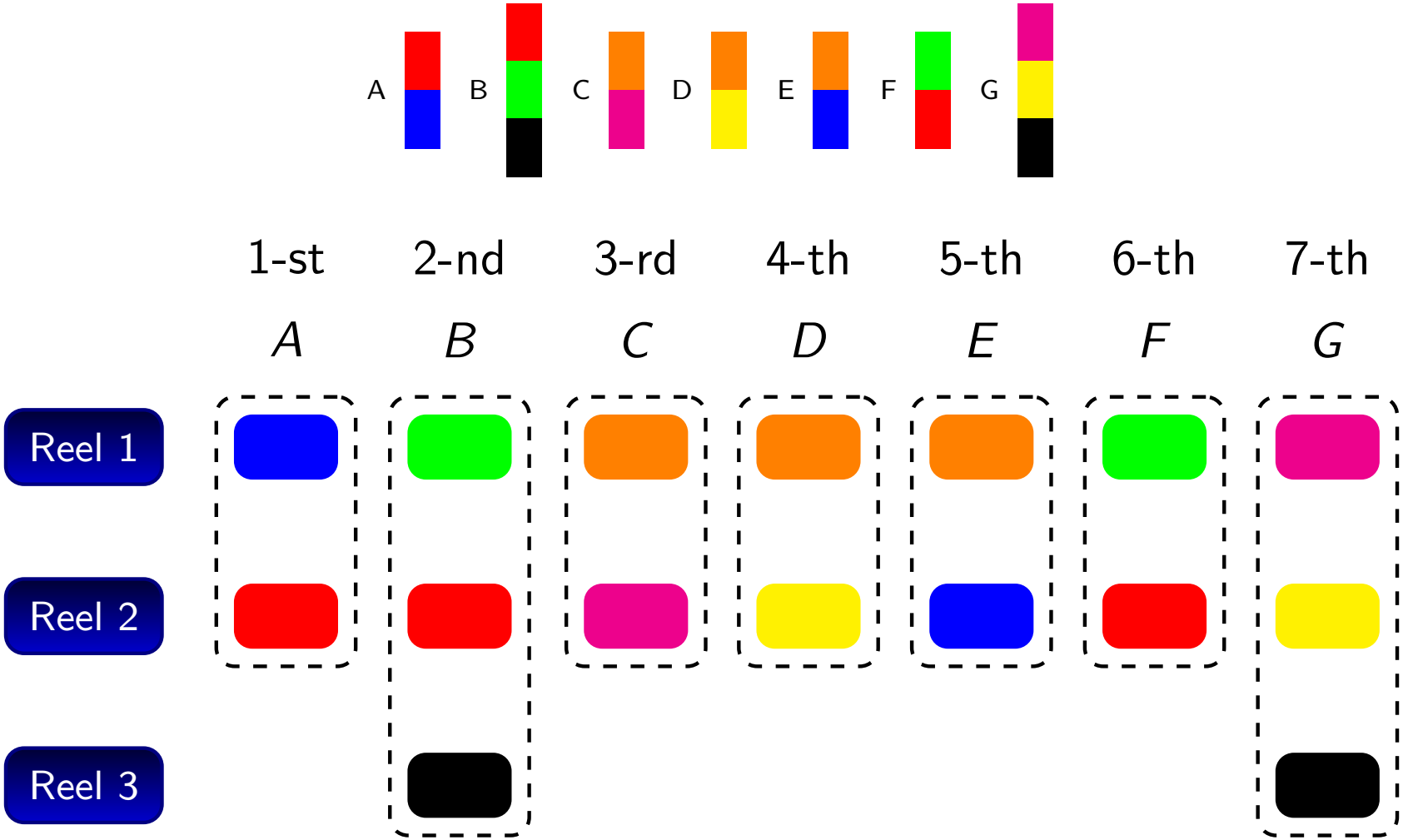

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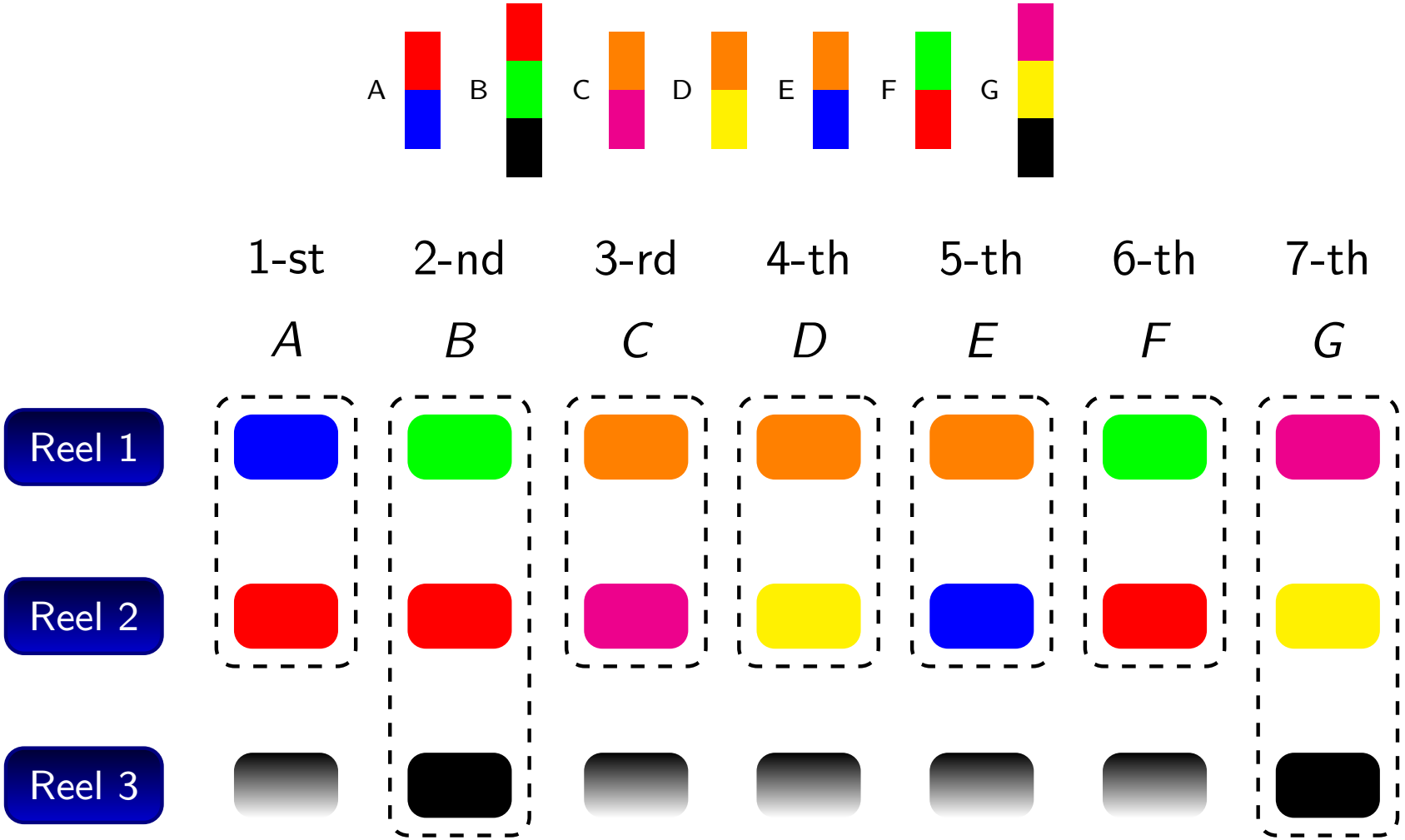

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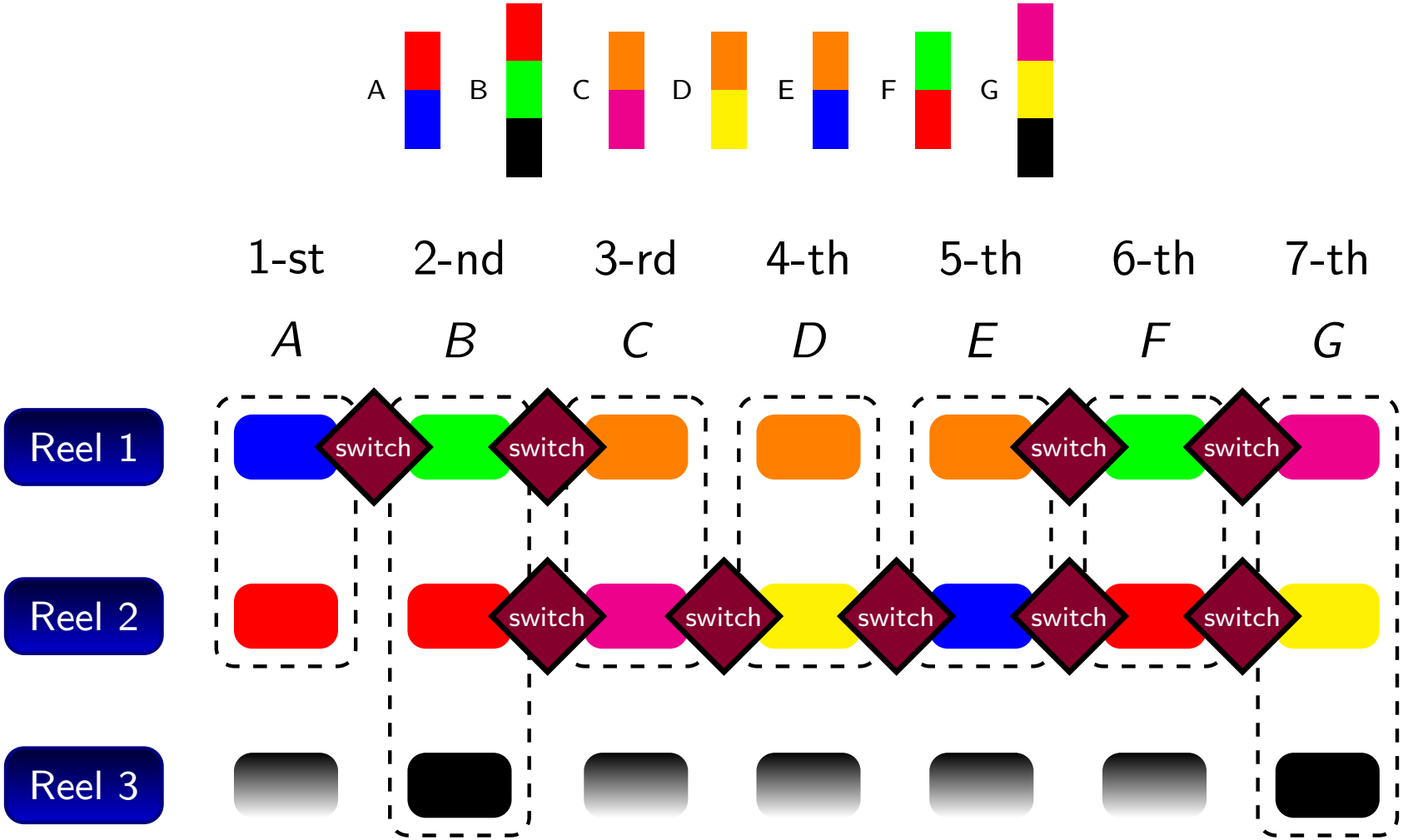

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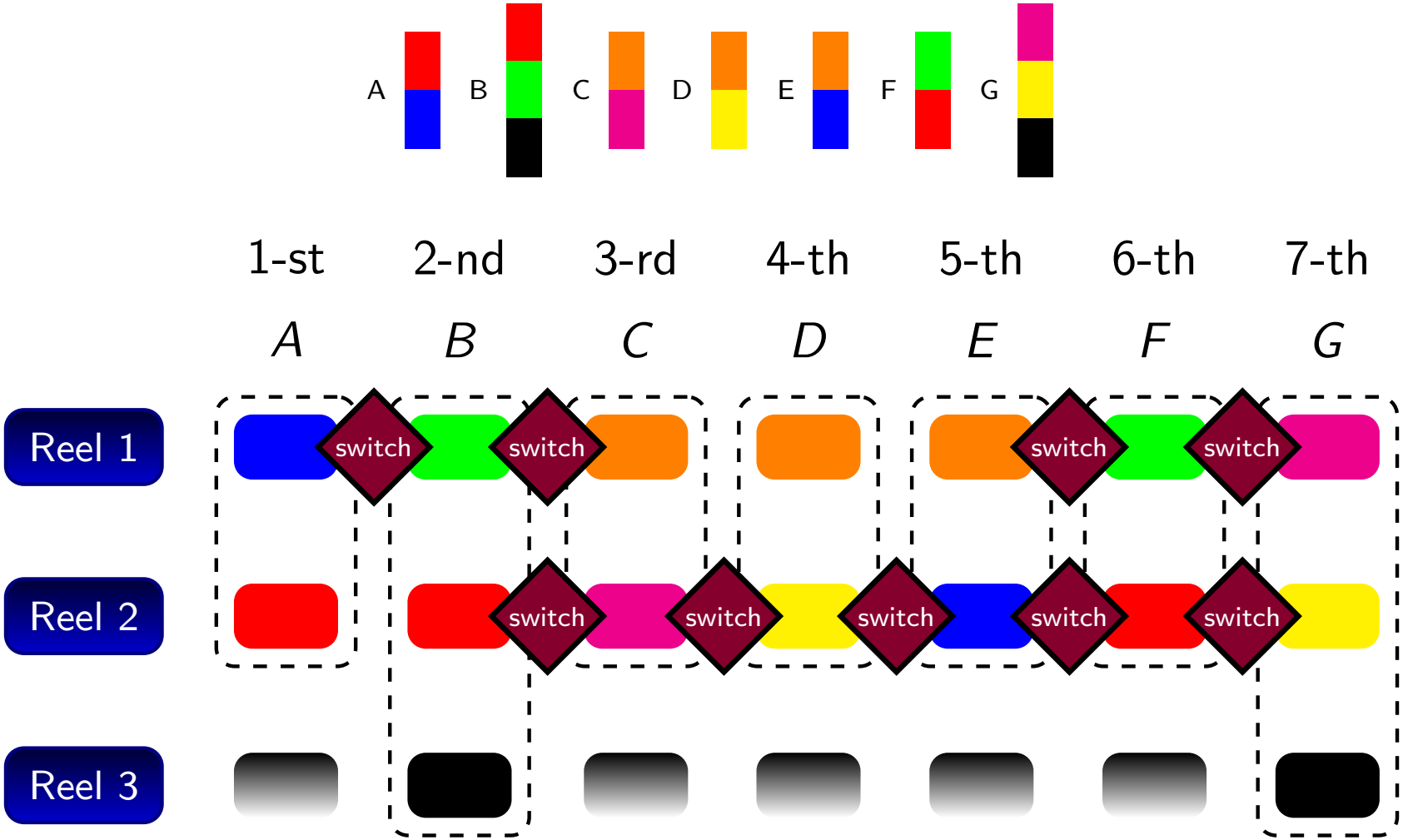

- A,B,C,D,E,F,G: 9 switches

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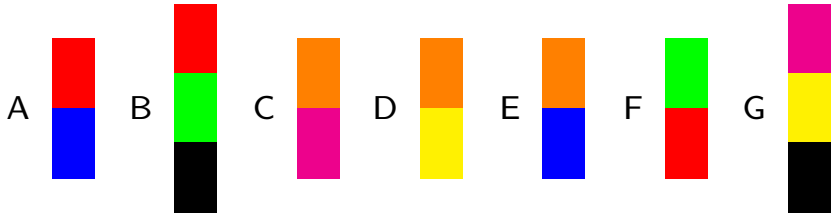

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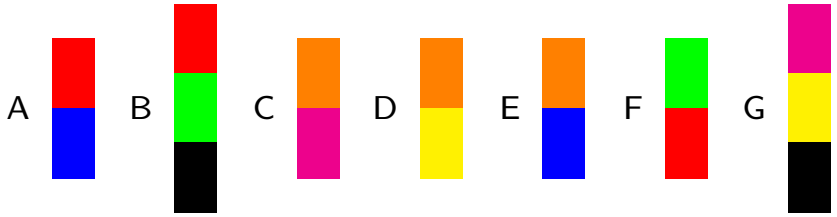

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B	*F*	*A*	*E*	*D*	*C*	*G*

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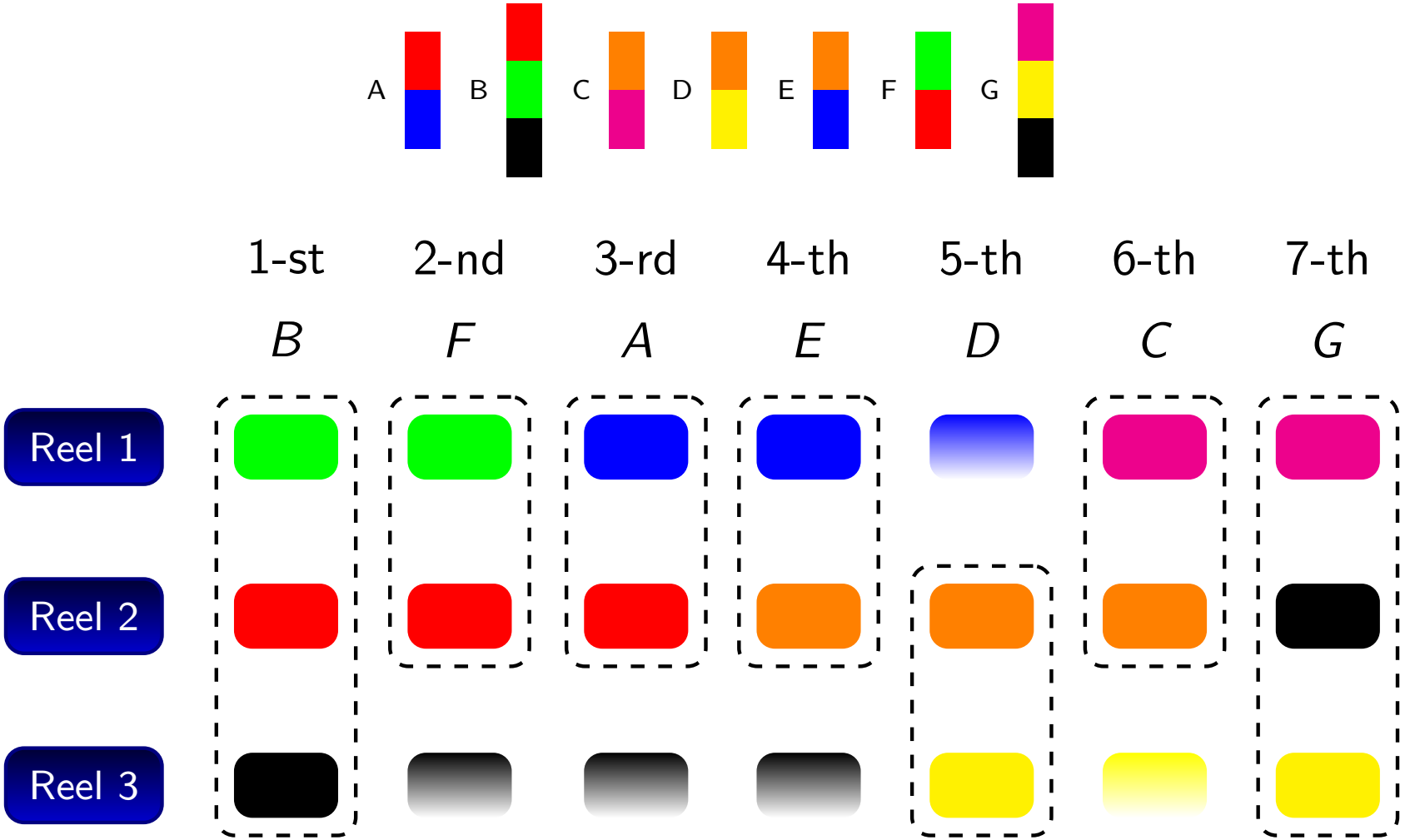

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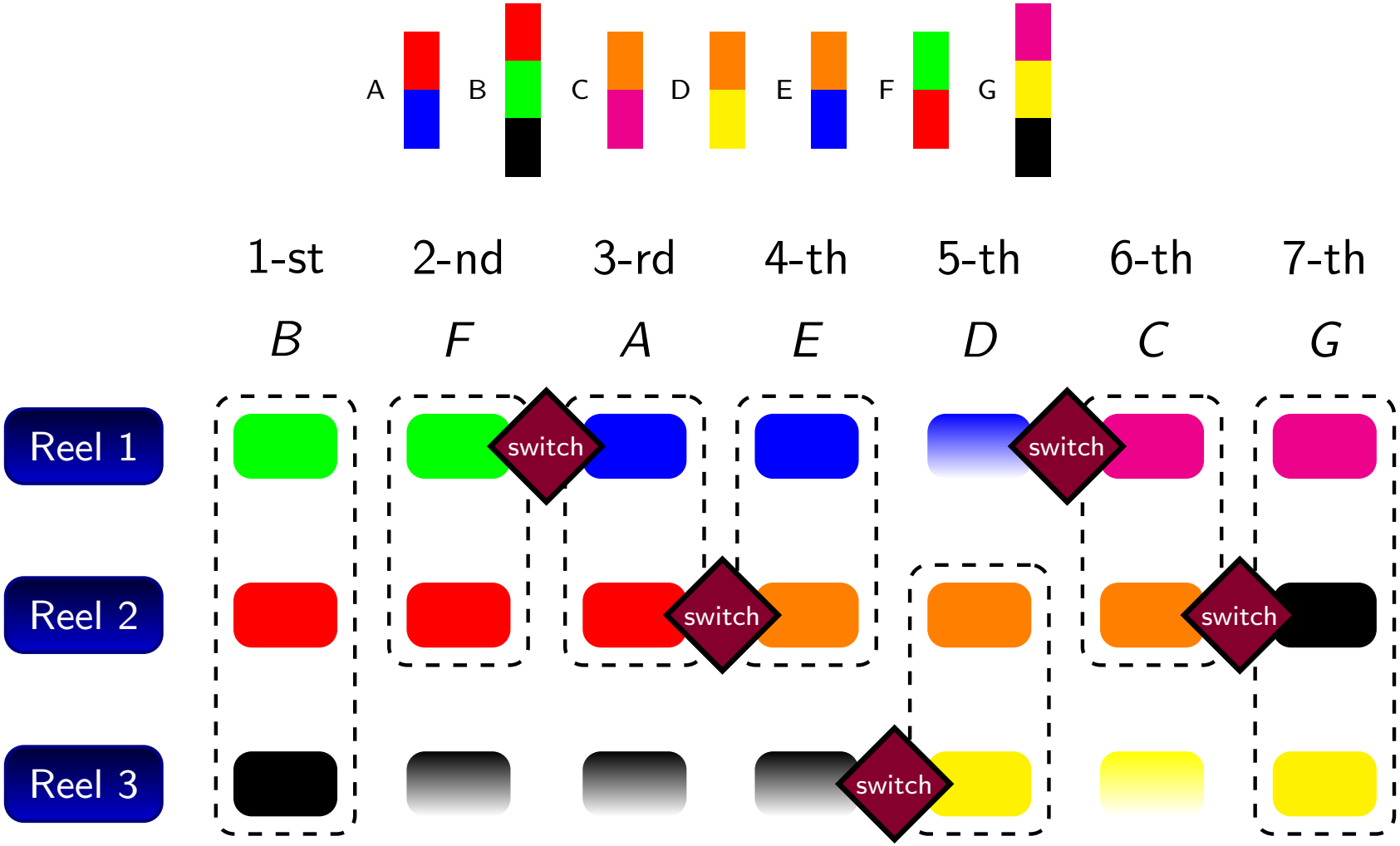
Embroidery Scheduling



- A,B,C,D,E,F,G: 9 switches
- B,F,A,E,D,C,G: 5 switches

The Switch Constraint

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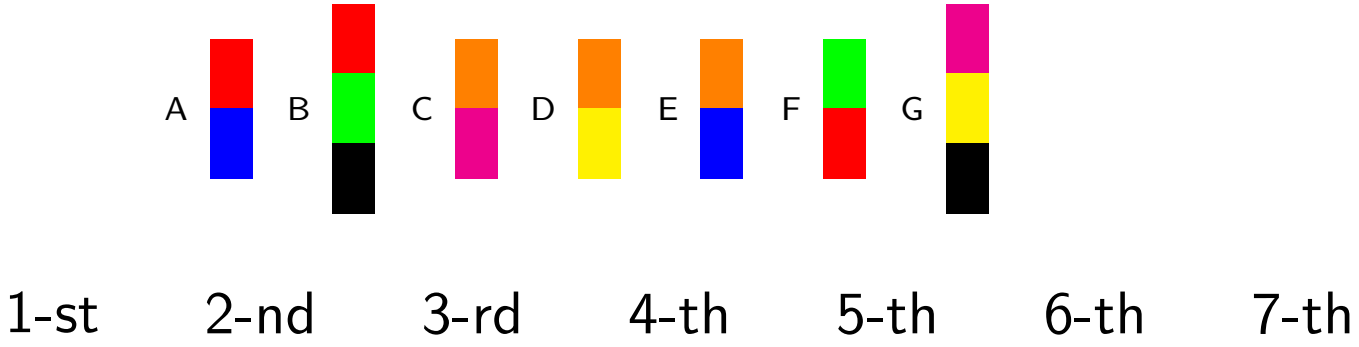
- Switch turned up in 3/4 of the industrial projects I was involved in!

Propagating the SWITCH Constraint

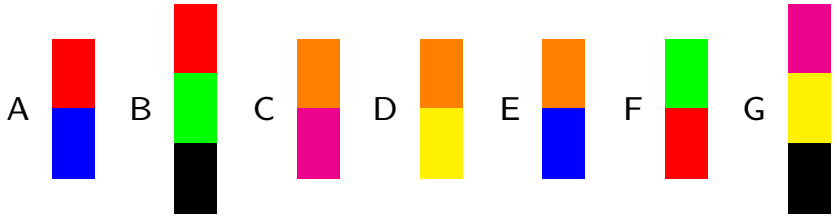


Reel 1

Reel 2

Reel 3

Propagating the SWITCH Constraint



1-st	2-nd	3-rd	4-th	5-th	6-th	7-th
$\{B, F\}$	$\{A\}$	$\{B.F\}$	$\{C, D, E\}$	$\{C, D, E\}$	$\{G\}$	$\{C, D, E\}$

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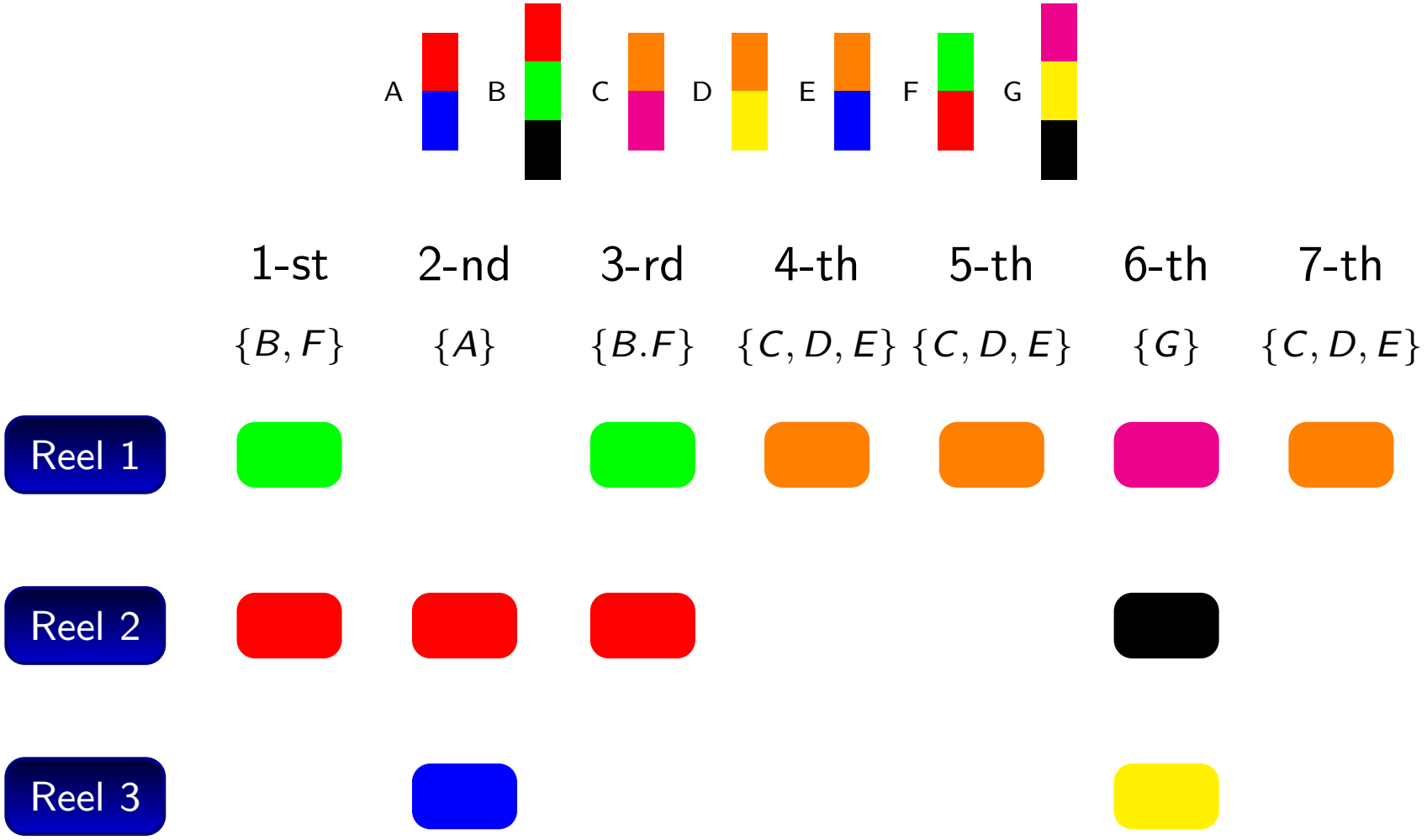
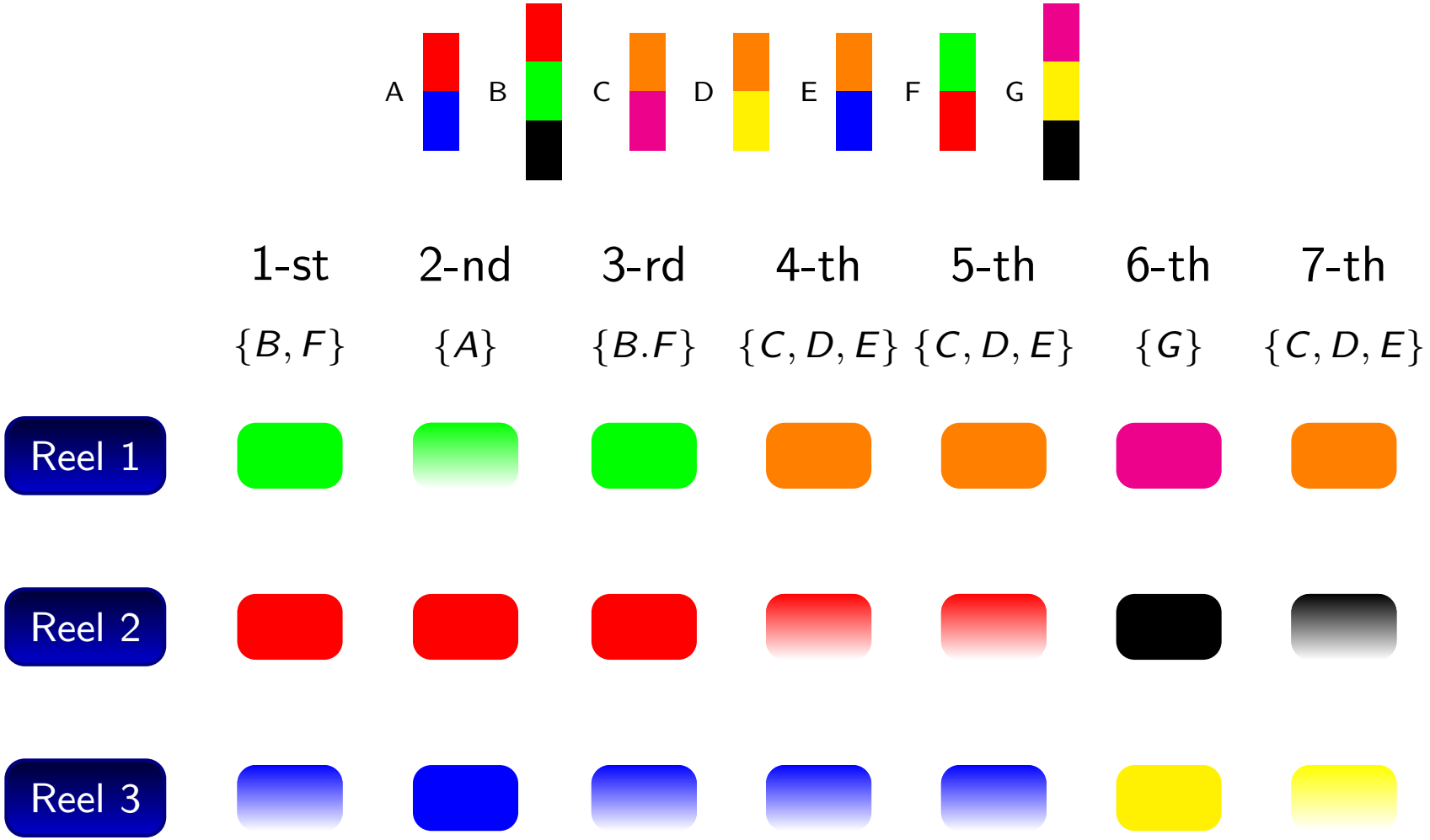


Propagating the SWITCH Constraint



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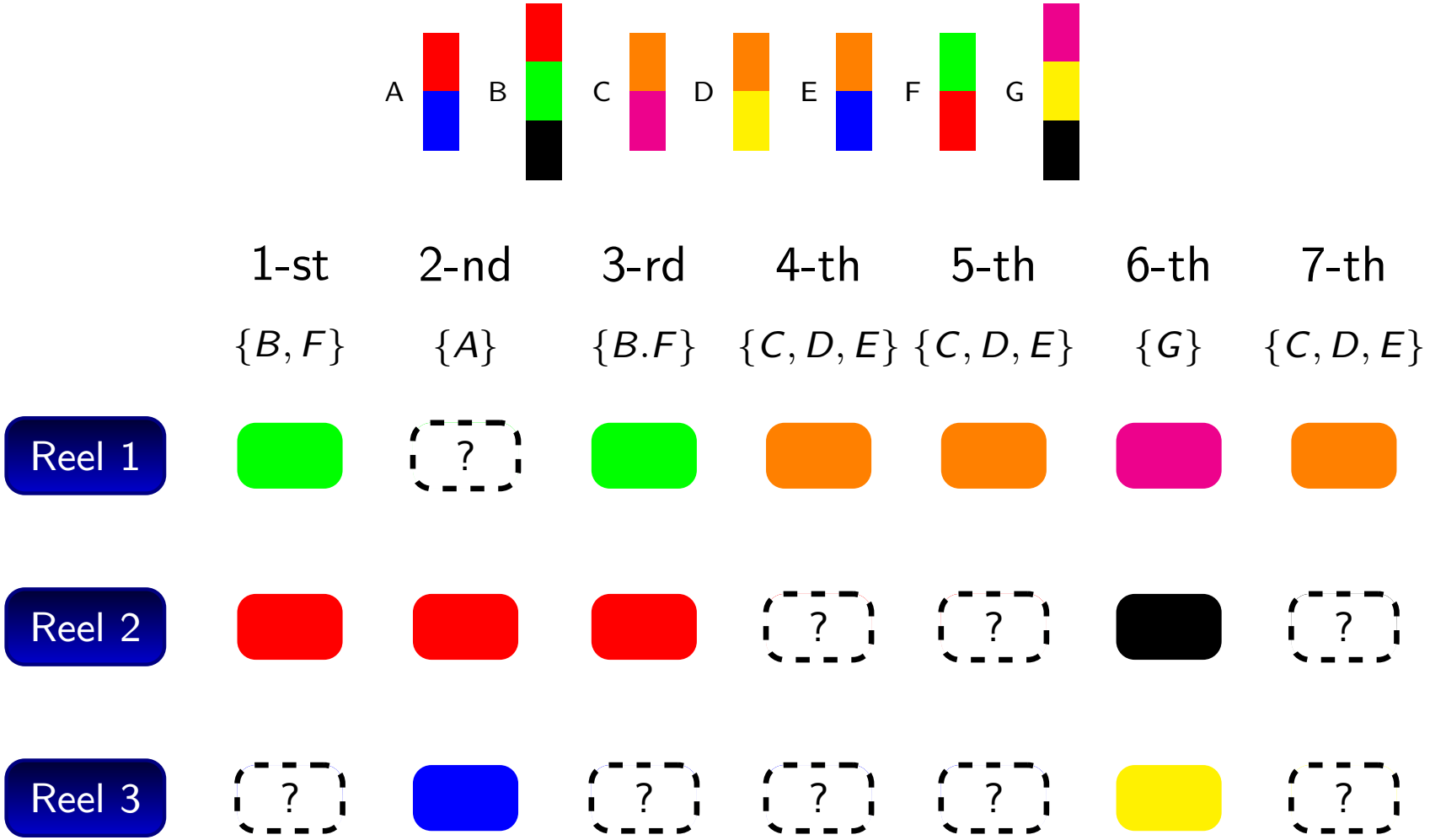
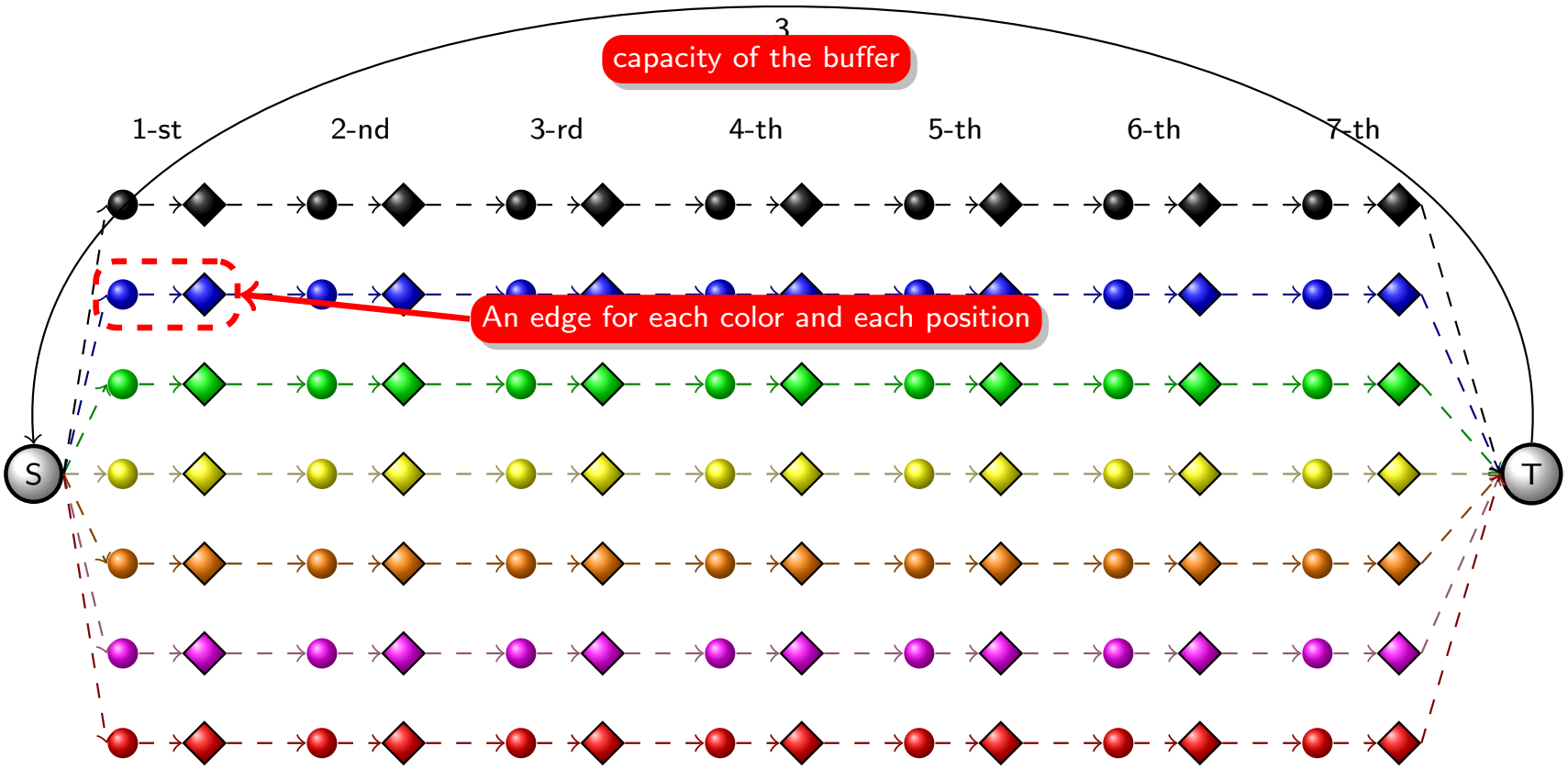


- How many switches at least? (compute a lower bound for M)
- What are the possible colors (given the upper bound on M)

Propagating the Switch Constraint

The Switch Constraint



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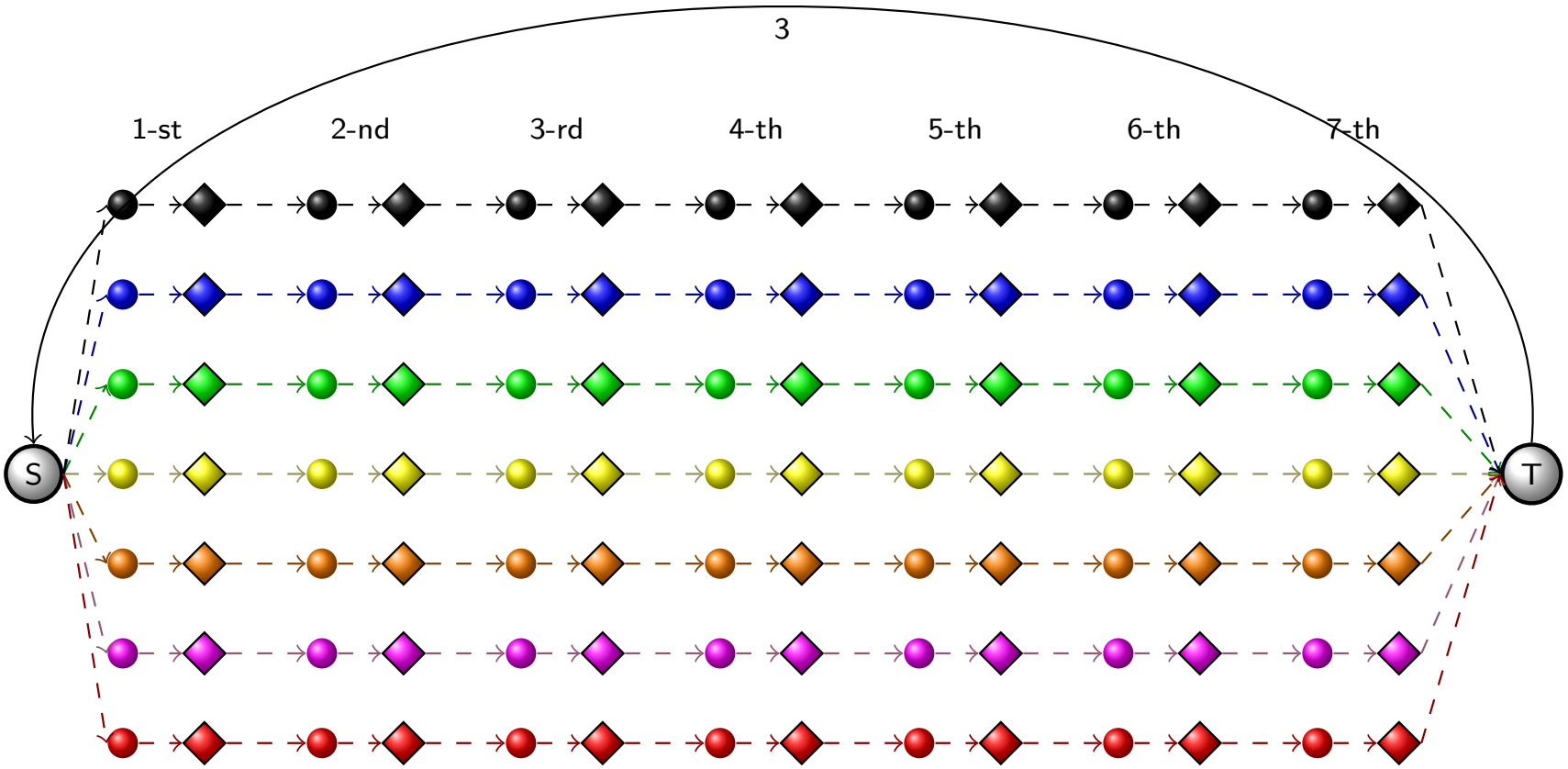

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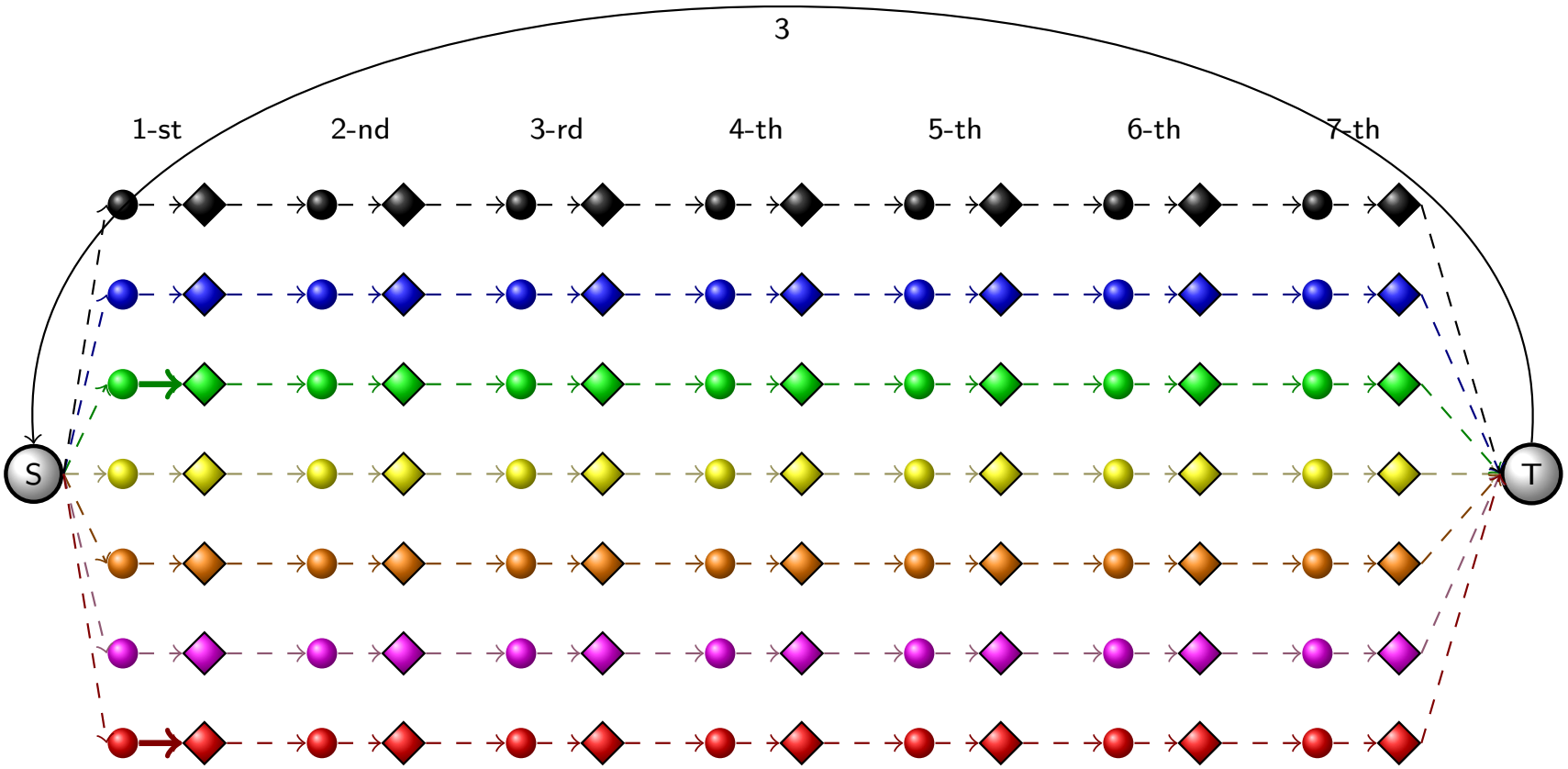

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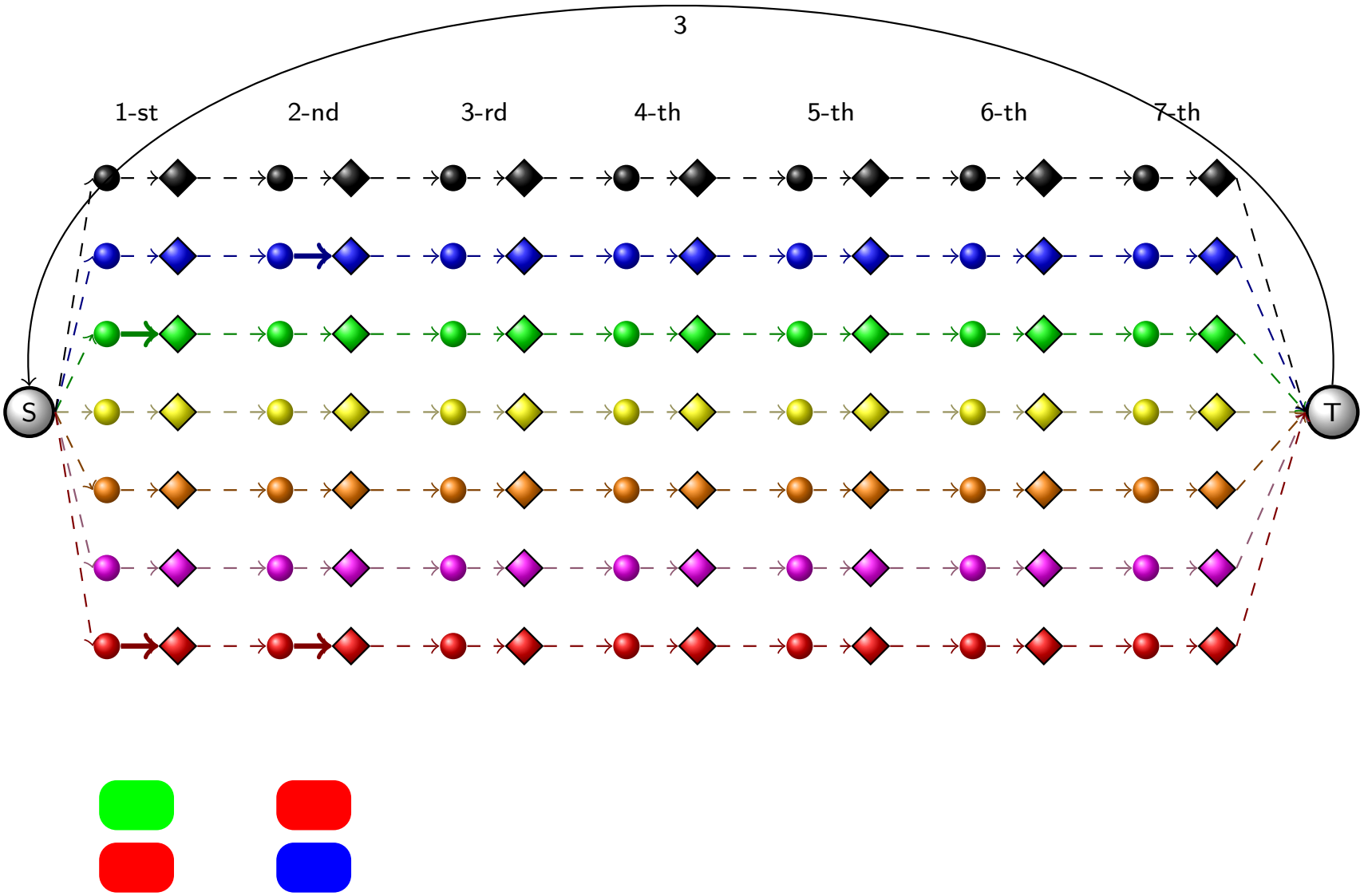

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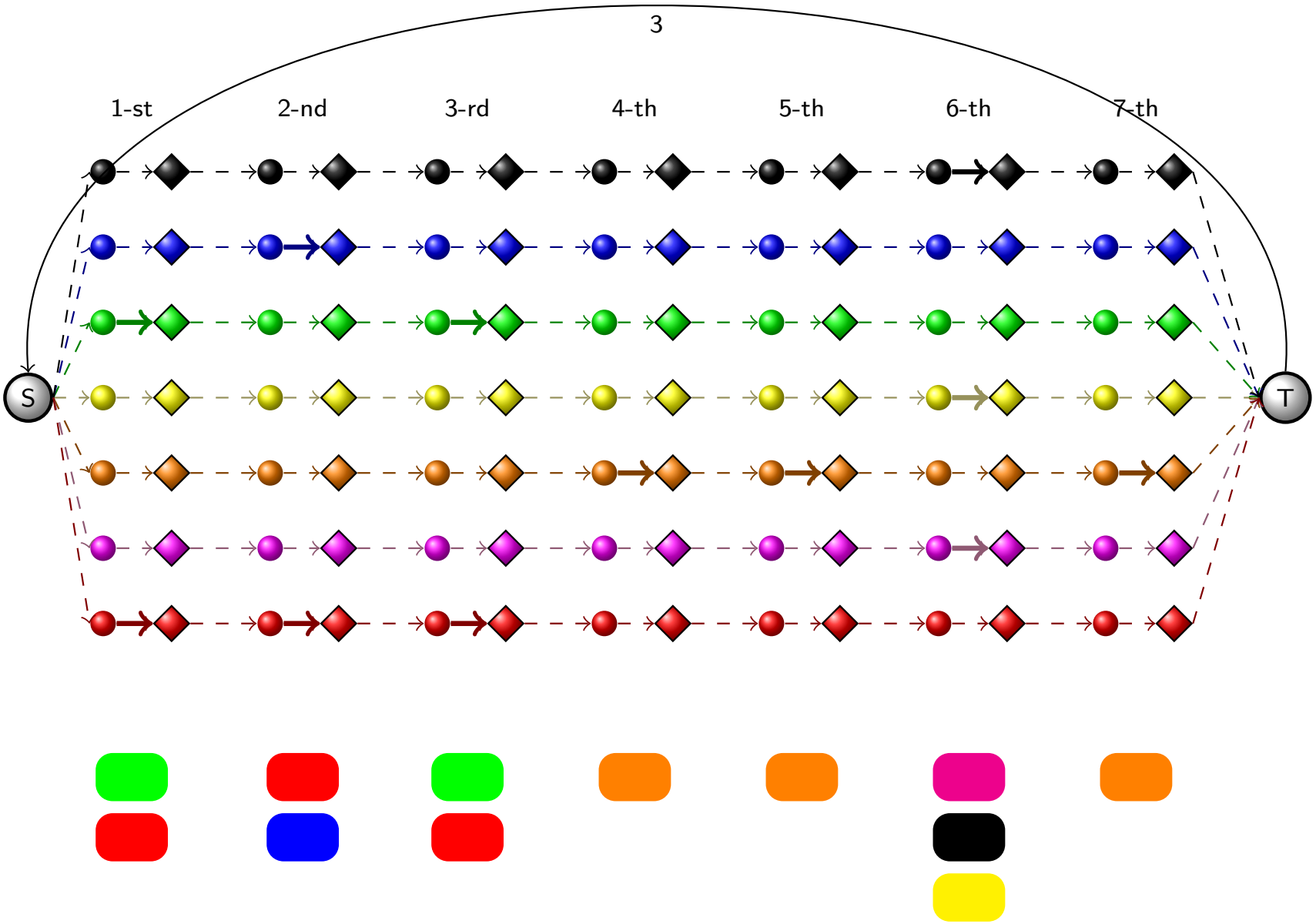

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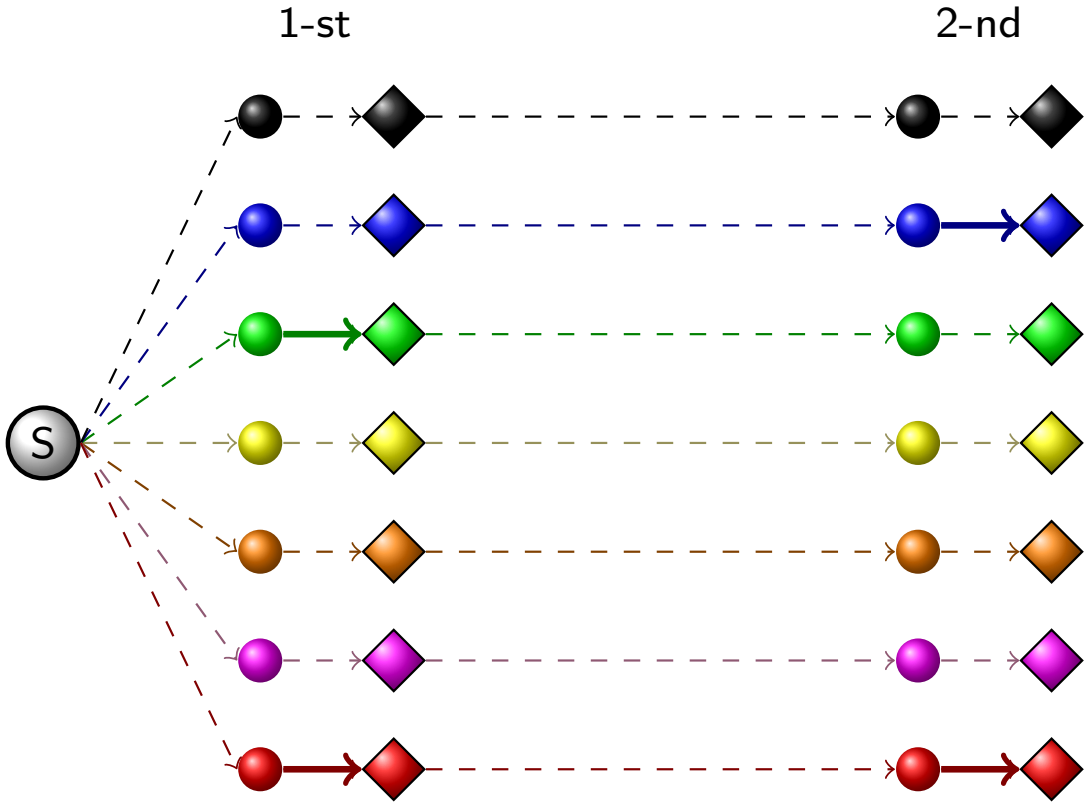

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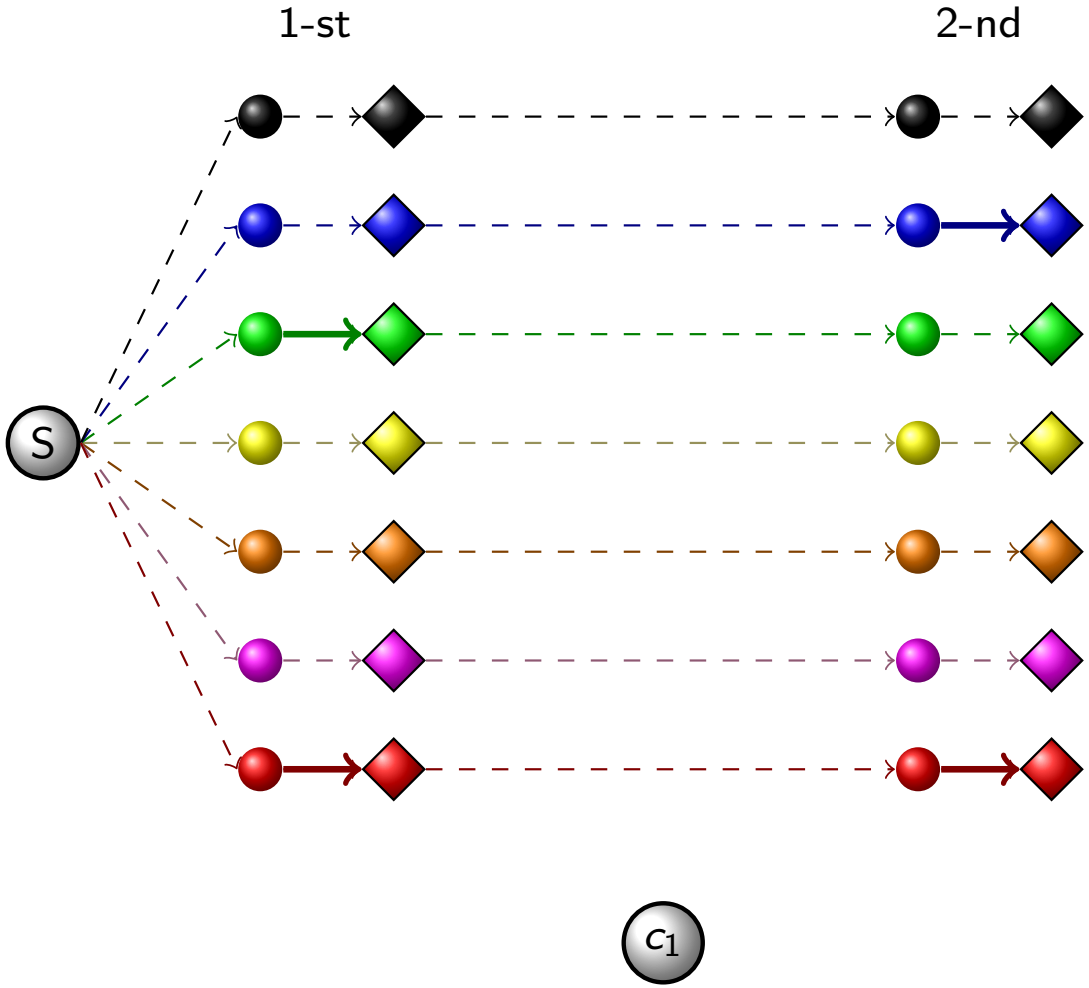

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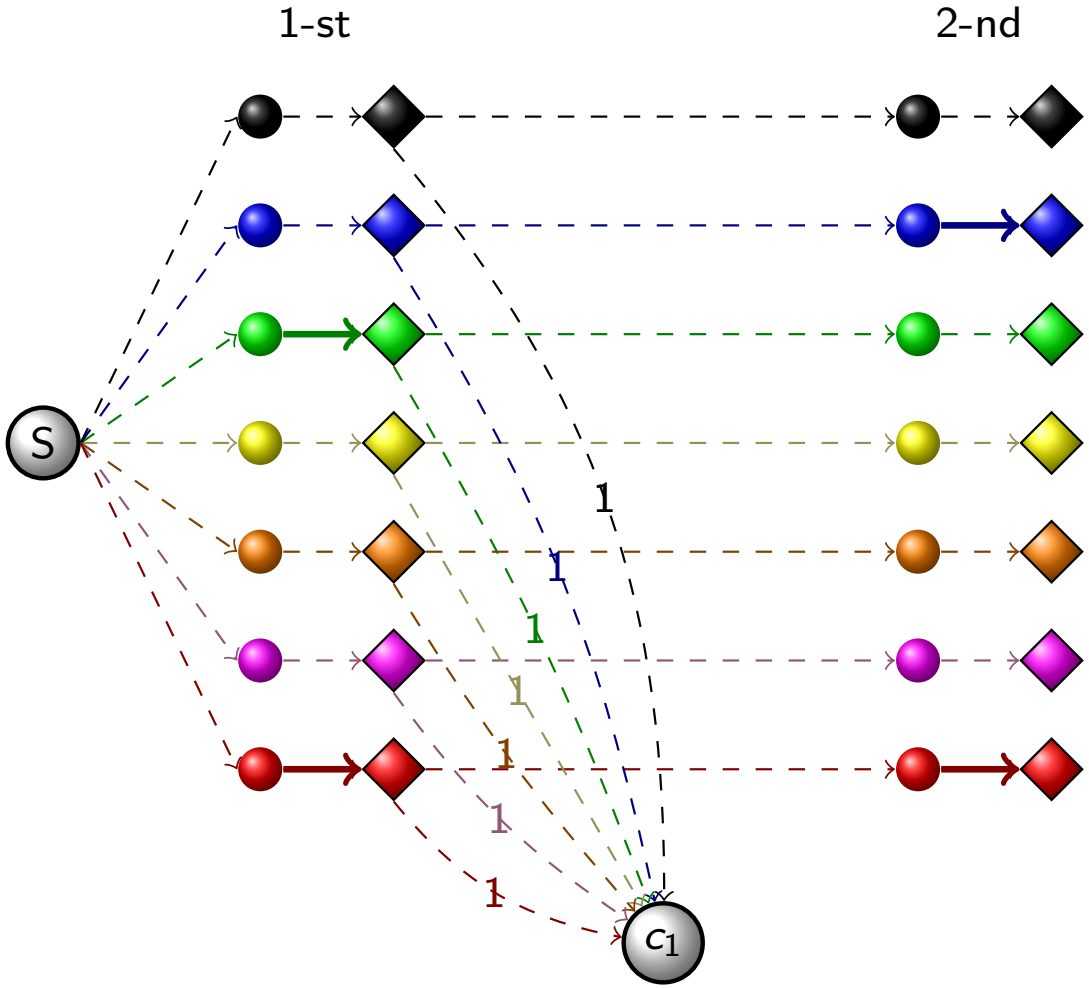

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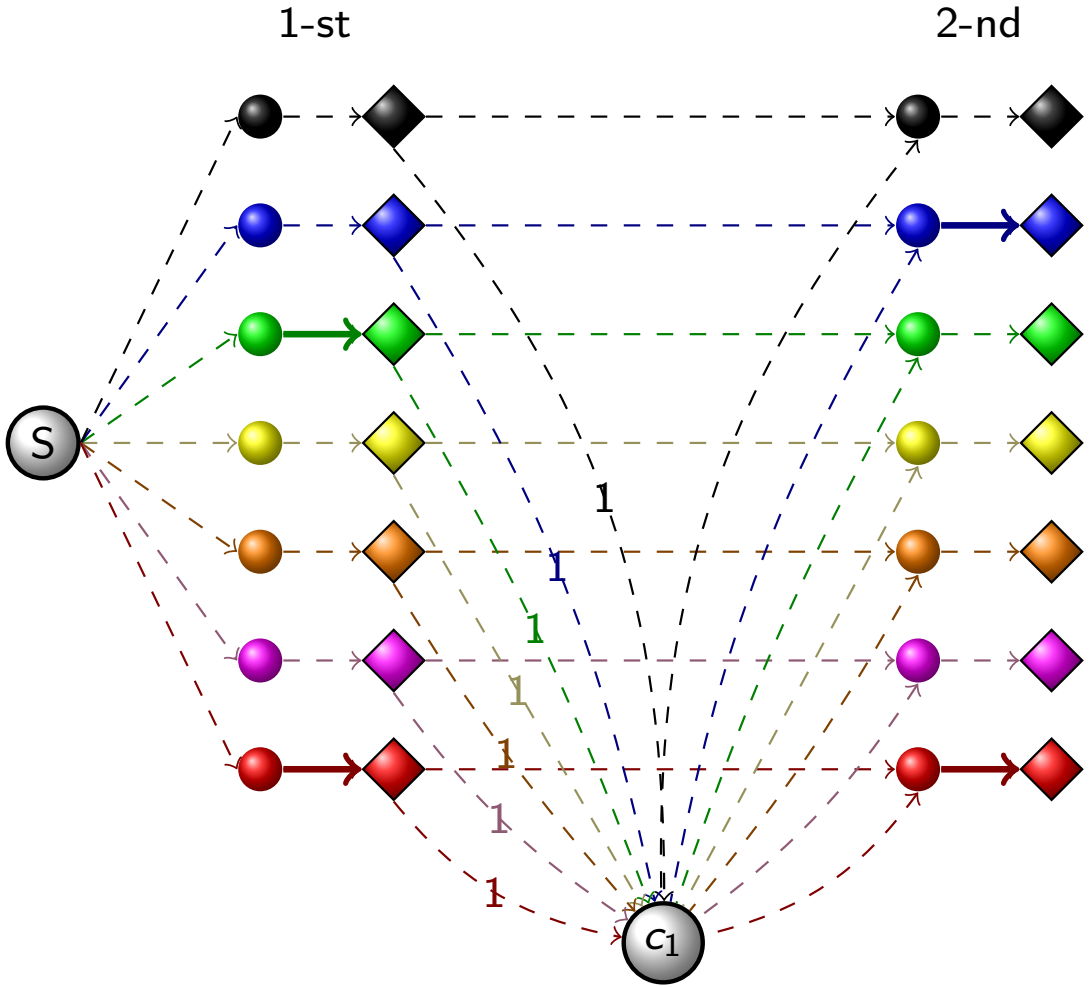

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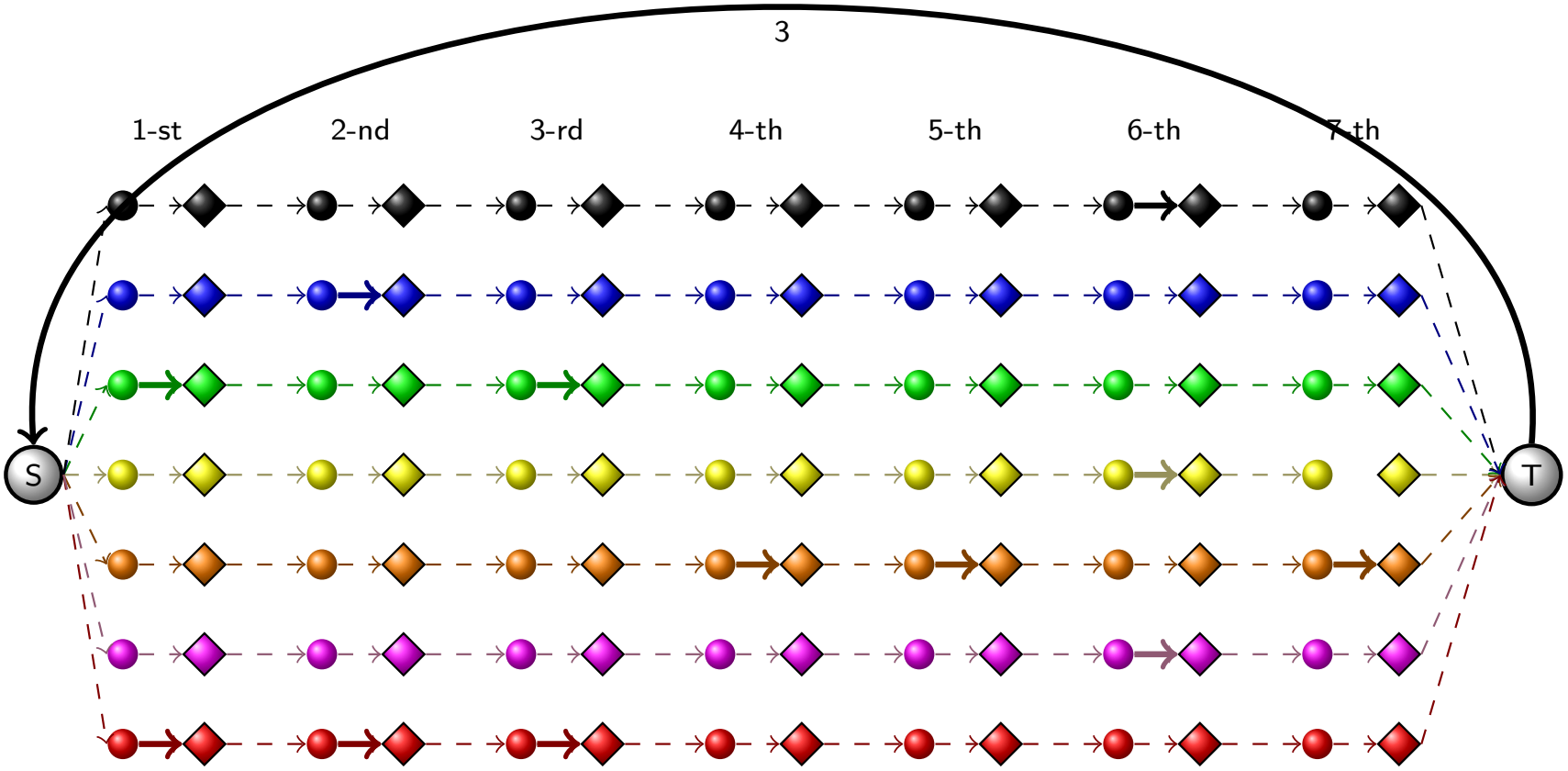

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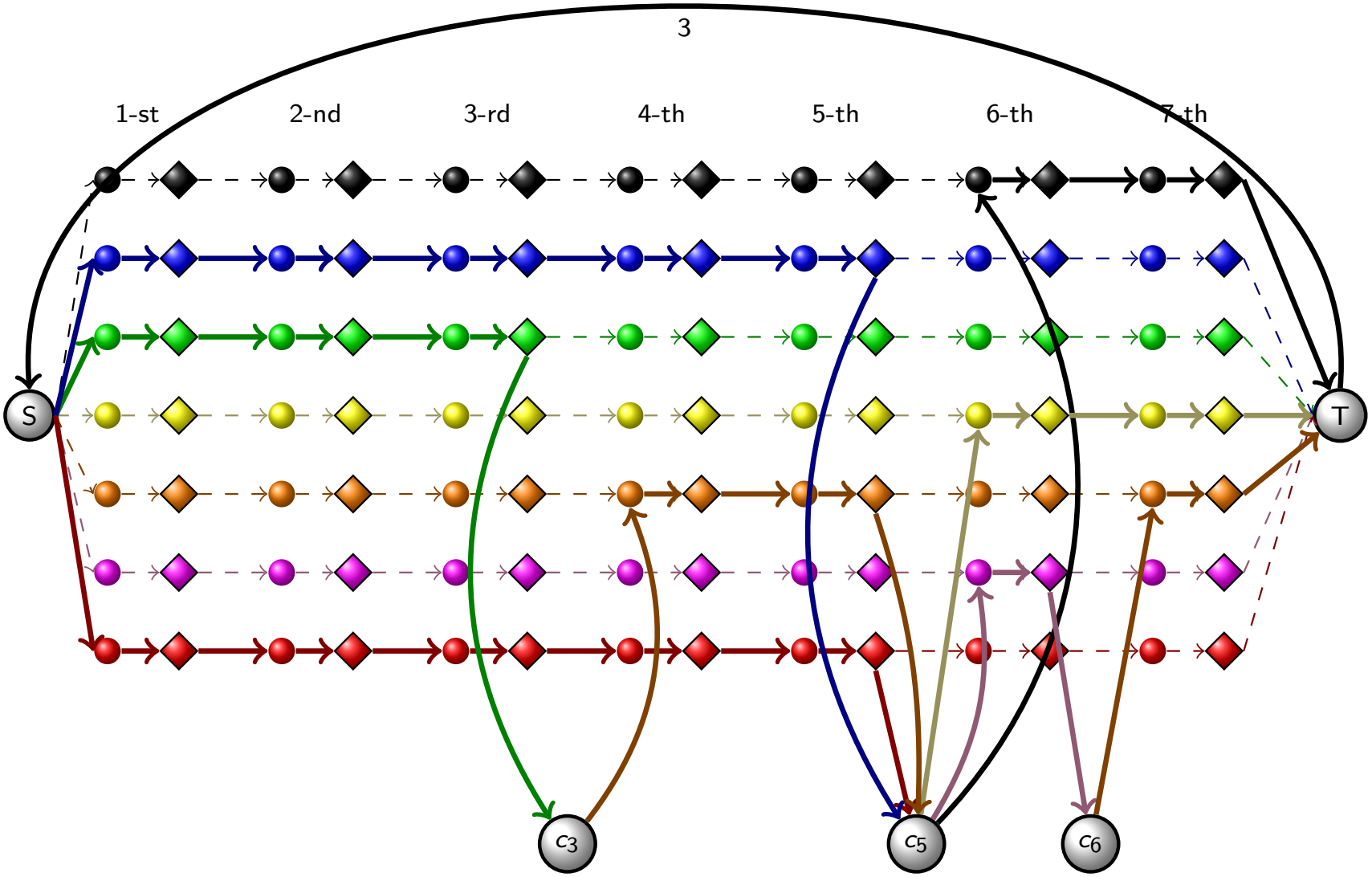

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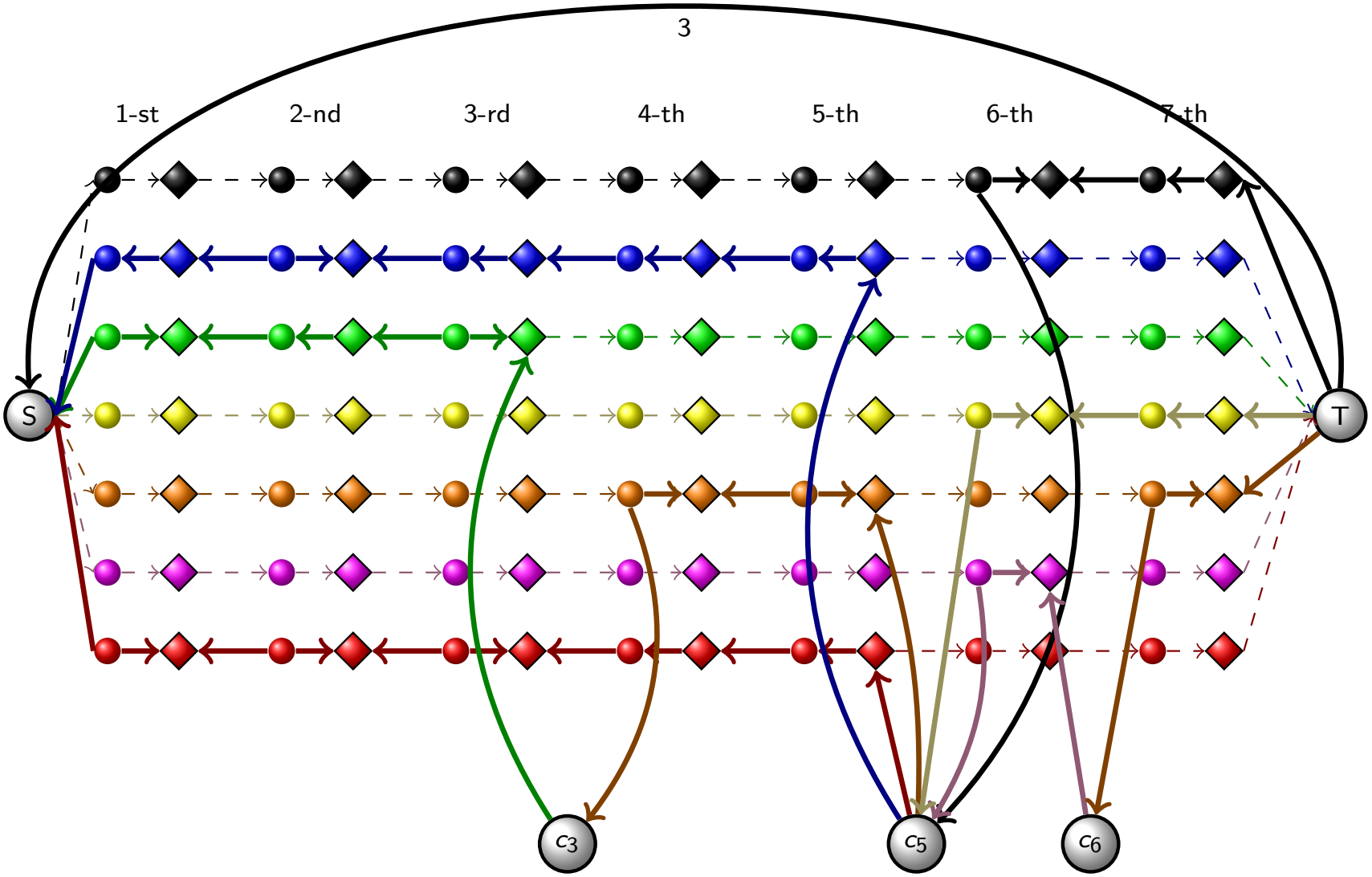

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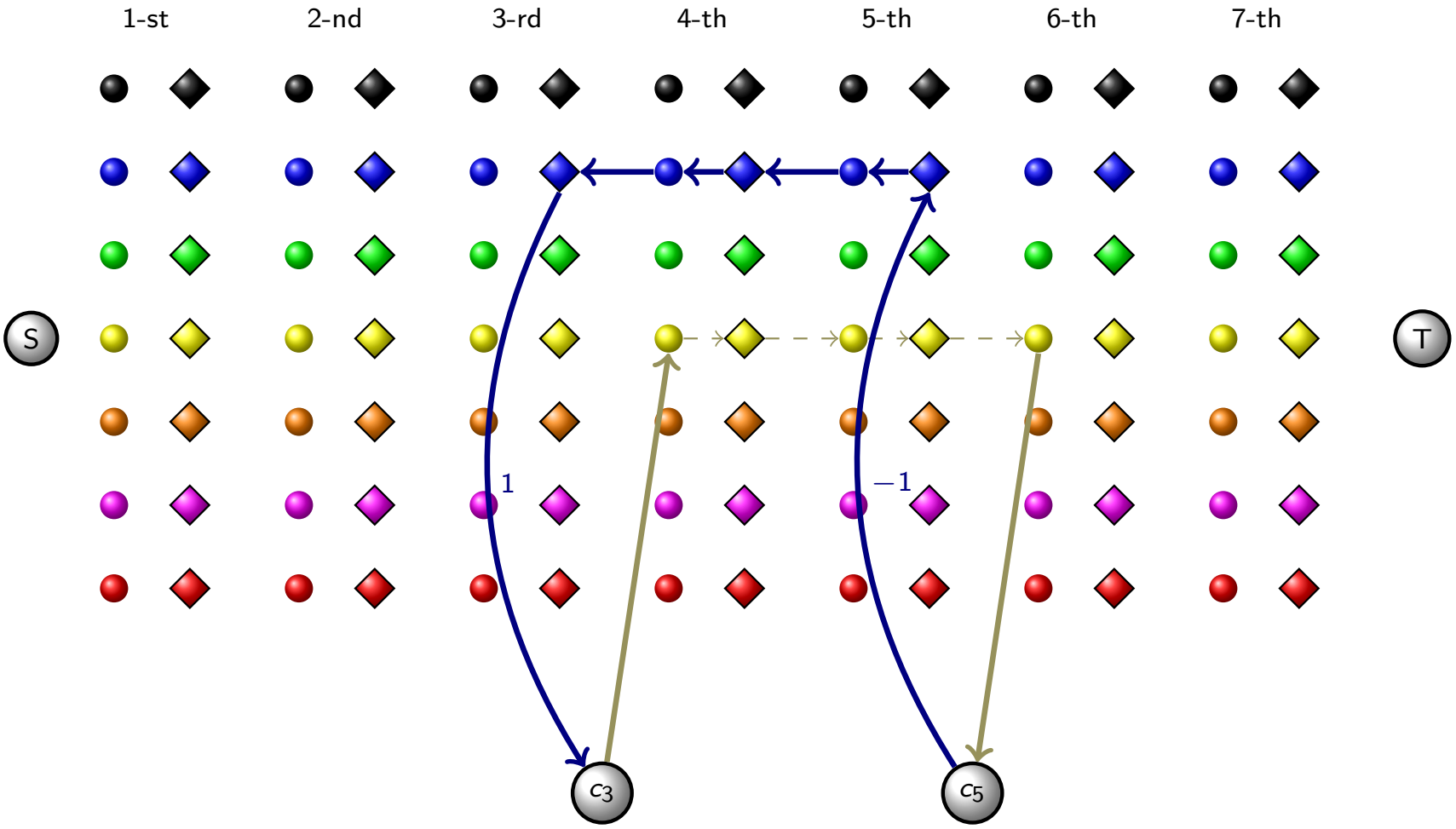

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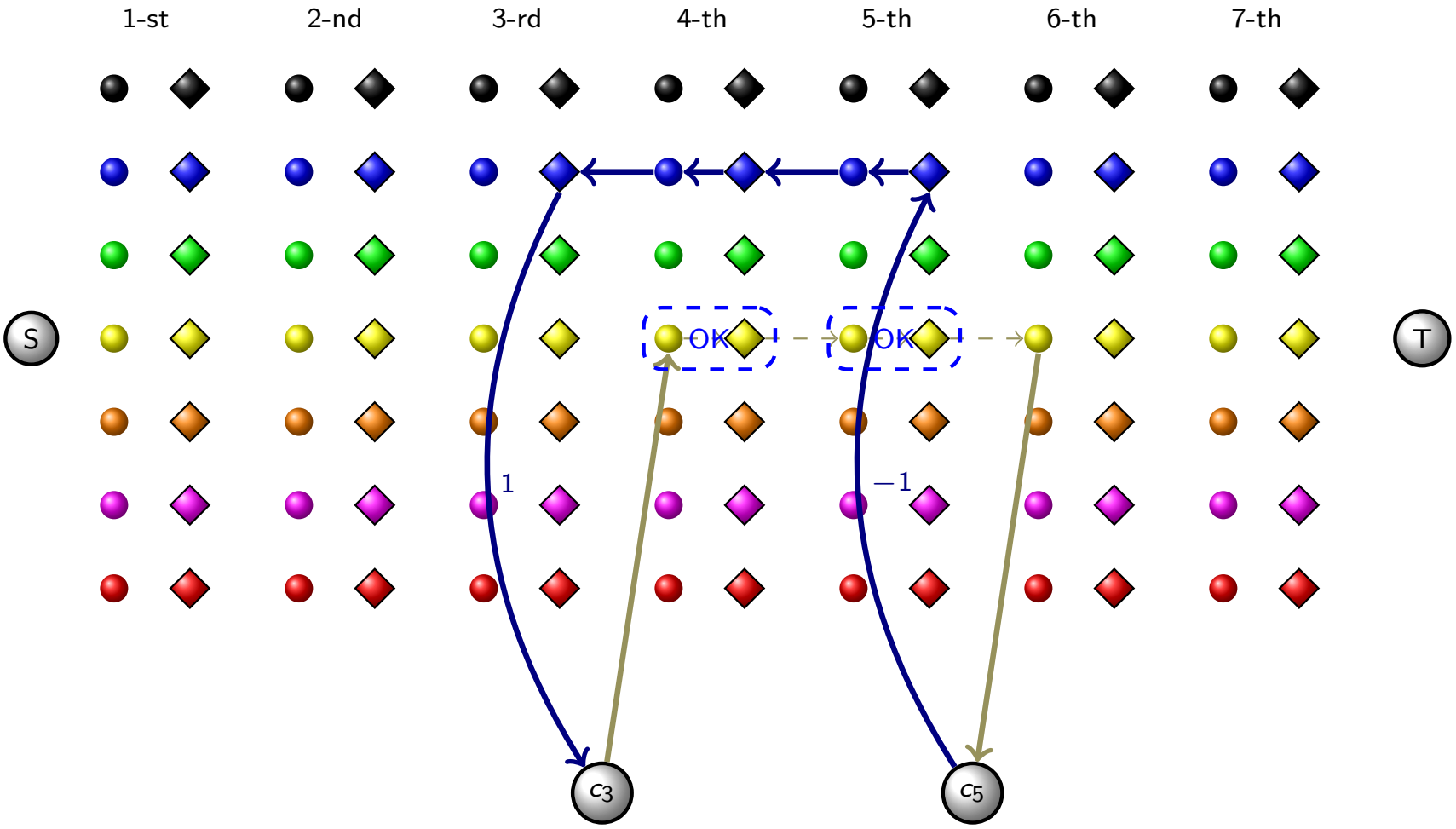

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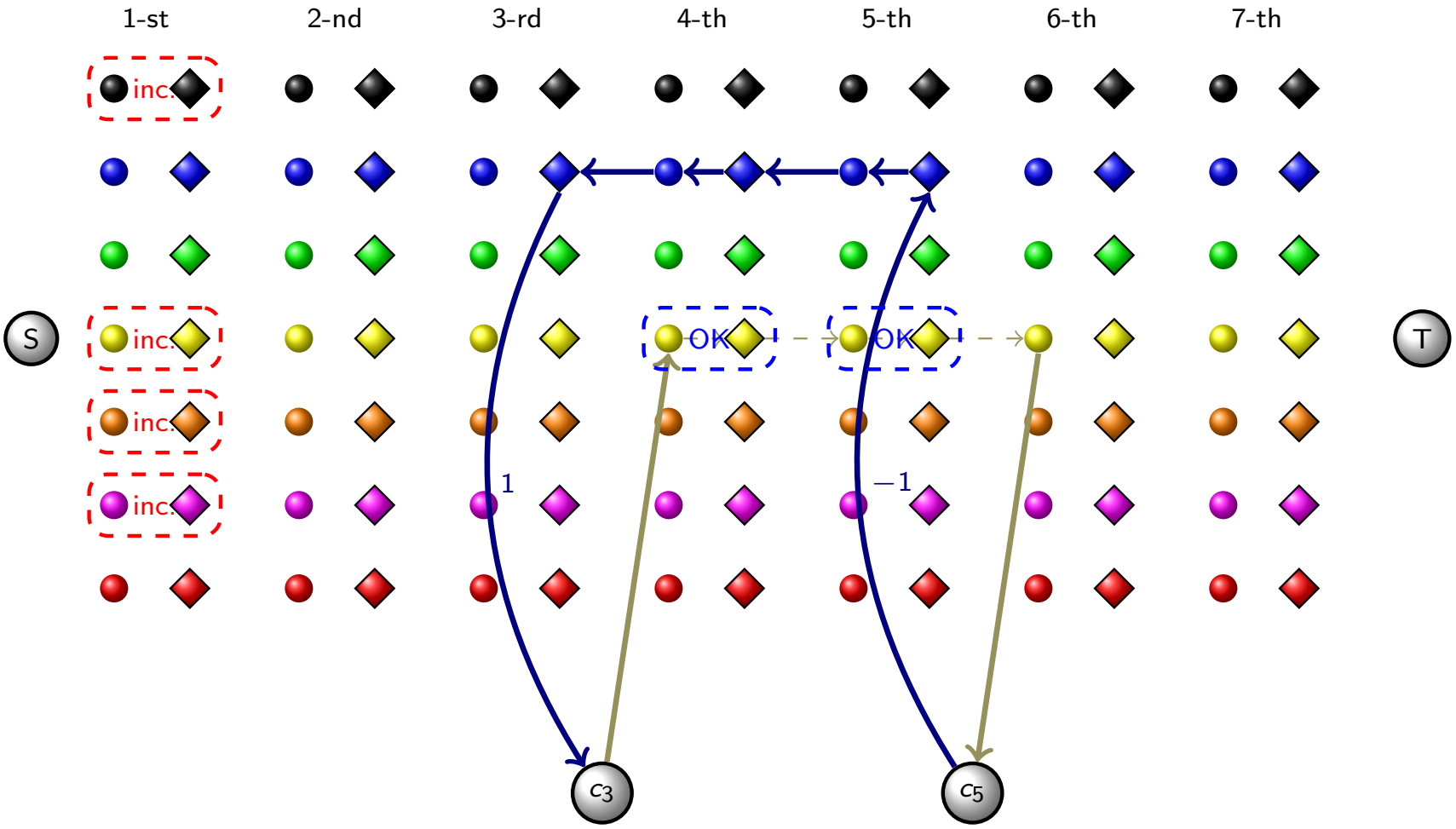

Propagation Algorithm (details at CP-AI-OR)

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Total time complexity: $O(n^2 d + nd^{1.5})$

experiments

Complexity of Propagating vs. Complexity of Solving

- Propagating is often harder
  - E.g. SWITCH: solving in $O(nd)$, propagating in $O(n^2d + nd^{1.5})$

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- Propagating is often harder
  - E.g. SWITCH: solving in $O(nd)$, propagating in $O(n^2d + nd^{1.5})$
- Sometimes, we can propagate "for free", i.e., with the same time complexity as for solving

Propagating for "Free": the AtMostSeqCard Constraint

The AtMostSeqCard Constraint

- A sequence of variables $[x_1, \ldots, x_n]$
  - $x_i = 1$ iff the bolt is of the type that Charlot cares about

Definition: $\textsc{AtMostSeqCard}(u, q, d, [x_1, \ldots, x_n]) \iff$

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Solving the AtMostSeqCard Constraint

- Finding a support (solving)

Solving the AtMostSeqCard Constraint

- Finding a support (solving)
  - We can be greedy and assign 1 whenever possible ⇒ `leftmost`

Example: $\overrightarrow{w}$ = `leftmost` $(u = 2,\ q = 4)$

$\mathcal{D}(x_i)$. 0 . 1 . . . 0 . 0 1 . . 1 1

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$\mathcal{D}(x_i)$	.	0	.	1	.	.	.	0	.	0	1	.	.	1	.	.	.	.	.	.	.	1
$\overrightarrow{w}[i]$																						

- Support: maximizing the cardinality while respecting capacities
  - Changing the value 1 to 0 has no impact on capacity constraints

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$\overrightarrow{w}[i]$		0		1				0		0	1			1								1

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$\overrightarrow{w}[i]$		0		1				0		0	1	0	0	1										1

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$\overrightarrow{w}[i]$	1	0		1				0		0	1	0	0	1									1

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$\overrightarrow{w}[i]$	1	0	0	1	1			0		0	1	0	0	1								1

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$\overrightarrow{w}[i]$	1	0	0	1	1	0	0	0	1	0	1	0	0	1	1	0	0	1	1	0	0	1	

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$\overrightarrow{w}[i]$	1	0	0	1	1	0	0	0	1	0	1	0	0	1	1	0	0	1	1	0	0	1	

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$\mathcal{D}(x_i)$	.	0	.	1	.	.	.	0	.	0	1	.	.	1	.	.	.	.	.	.	.	.	1
$\overrightarrow{w}[i]$	1	0	0	1	1	0	0	0	1	0	1	0	0	1	1	0	0	1	1	0	0	1	

- Support: maximizing the cardinality while respecting capacities
  - Changing the value 1 to 0 has no impact on capacity constraints
  - Greedily inserting 1 works
- We can find a support in $O(nq)$
- We can achieve AC in $O(n^2q)$

Filtering the AtMostSeqCard Constraint

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- Can we propagate in the same time complexity?
  - There is nothing to propagate unless $|\overrightarrow{w}| = d$
  - Two runs of `leftmost` are enough

Example: $\overrightarrow{w}$ = `leftmost` $(u = 2,\ q = 4,\ d = 10)$

$\mathcal{D}(x_i)$	.	0	.	1	.	.	.	0	.	0	1	.	.	1	.	.	.	.	.	.	.	1	
$\overrightarrow{w}[i]$	1	0	0	1	1	0	0	0	1	0	1	0	0	1	1	0	0	1	1	0	0	1	

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$\mathcal{D}(x_i)$		.	0	.	1	.	.	.	0	.	0	1	.	.	1	.	.	.	.	.	.	.	1
$\overrightarrow{w}[i]$		1	0	0	1	1	0	0	0	1	0	1	0	0	1	1	0	0	1	1	0	0	1
$L[i]$	0	1	1	1	2	3	3	3	3	4	4	5	5	5	6	7	7	7	8	9	9	9	d

Example: $\overrightarrow{w}$ = `leftmost` $(u = 2,\ q = 4,\ d = 10)$

$\mathcal{D}(x_i)$		.	0	.	1	.	.	.	0	.	0	1	.	.	1	.	.	.	.	.	.	.	1
$\overrightarrow{w}[i]$		1	0	0	1	1	0	0	0	1	0	1	0	0	1	1	0	0	1	1	0	0	1
$\overleftarrow{w}[i]$		0	0	1	1	0	0	1	0	1	0	1	0	0	1	0	0	1	1	0	0	1	1
$L[i]$	0	1	1	1	2	3	3	3	3	4	4	5	5	5	6	7	7	7	8	9	9	9	d

Example: $\overrightarrow{w}$ = `leftmost` $(u = 2,\ q = 4,\ d = 10)$

$\mathcal{D}(x_i)$		.	0	.	1	.	.	.	0	.	0	1	.	.	1	.	.	.	.	.	.	.	1	
$\overrightarrow{w}[i]$		1	0	0	1	1	0	0	0	1	0	1	0	0	1	1	0	0	1	1	0	0	1	
$\overleftarrow{w}[i]$		0	0	1	1	0	0	1	0	1	0	1	0	0	1	0	0	1	1	0	0	1	1	
$L[i]$	0	1	1	1	2	3	3	3	3	4	4	5	5	5	6	7	7	7	8	9	9	9	d	
$R[i]$		d	d	d	9	8	8	8	7	7	6	6	5	5	5	4	4	4	3	2	2	2	1	0

Example: $\overrightarrow{w}$ = `leftmost` $(u = 2,\ q = 4,\ d = 10)$

$\mathcal{D}(x_i)$		.	0	.	1	.	.	.	0	.	0	1	.	.	1	.	.	.	.	.	.	.	1	
$\overrightarrow{w}[i]$		1	0	0	1	1	0	0	0	1	0	1	0	0	1	1	0	0	1	1	0	0	1	
$\overleftarrow{w}[i]$		0	0	1	1	0	0	1	0	1	0	1	0	0	1	0	0	1	1	0	0	1	1	
$L[i]$	0	1	1	1	2	3	3	3	3	4	4	5	5	5	6	7	7	7	8	9	9	9	d	
$R[i]$		d	d	d	9	8	8	8	7	7	6	6	5	5	5	4	4	4	3	2	2	2	1	0

Example: $\overrightarrow{w}$ = `leftmost` $(u = 2,\ q = 4,\ d = 10)$

$\mathcal{D}(x_i)$		.	0	.	1	.	.	.	0	.	0	1	.	.	1	.	.	.	.	.	.	.	1	
$\overrightarrow{w}[i]$		1	0	0	1	1	0	0	0	1	0	1	0	0	1	1	0	0	1	1	0	0	1	
$\overleftarrow{w}[i]$		0	0	1	1	0	0	1	0	1	0	1	0	0	1	0	0	1	1	0	0	1	1	
$L[i]$	0	1	1	1	2	3	3	3	3	4	4	5	5	5	6	7	7	7	8	9	9	9	d	
$R[i]$		d	d	d	9	8	8	8	7	7	6	6	5	5	5	4	4	4	3	2	2	2	1	0

Example: $\overrightarrow{w}$ = `leftmost` $(u = 2,\ q = 4,\ d = 10)$

$\mathcal{D}(x_i)$		.	0	.	1	.	.	.	0	1	0	1	.	.	1	.	.	.	.	.	.	.	1	
$\overrightarrow{w}[i]$		1	0	0	1	1	0	0	0	1	0	1	0	0	1	1	0	0	1	1	0	0	1	
$\overleftarrow{w}[i]$		0	0	1	1	0	0	1	0	1	0	1	0	0	1	0	0	1	1	0	0	1	1	
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- We can achieve AC in $O(nq)$, that is, $O(n^2)$
- We can do better: `leftmost` can run in $O(n)$ details
- Total complexity of $O(n)$, optimal!

NP-hard Constraints

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NP-hard Constraints: the SoftAllEqual Constraints

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- Two versions
  - Maximise the number of equalities
  - Minimise the number of value skip

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At least k pairs of equal variables in the set $\{x_1, \ldots, x_n\}$

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Alice	✓			✓
Bob		✓		
Charlie	✓		✓	✓
Dave		✓	✓	
Eve		✓	✓	
Frank				✓
Guy	✓		✓	✓
Hugo	✓	✓		
Ian			✓	✓
Joe	✓	✓	✓	

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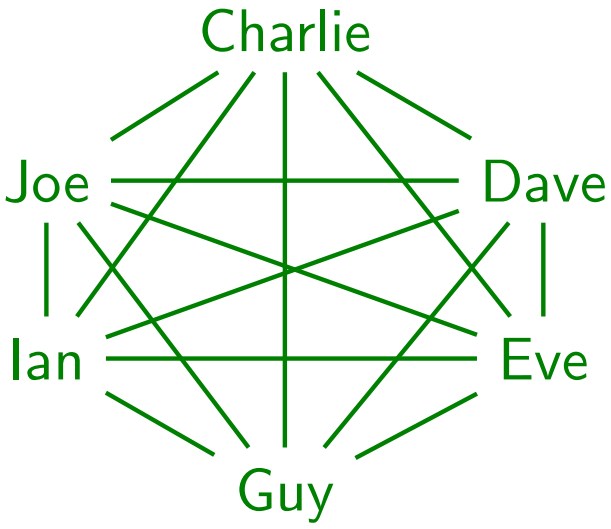
Alice — Hugo

Bob



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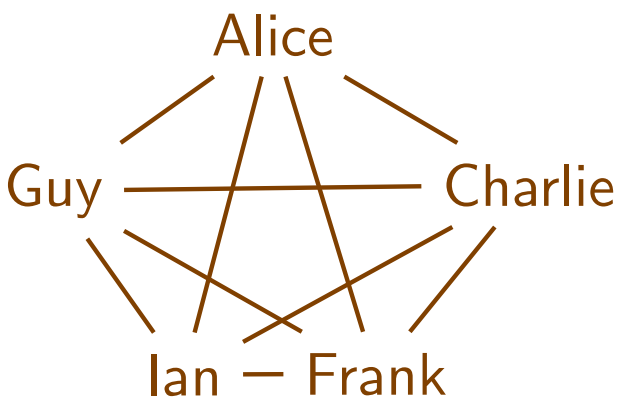

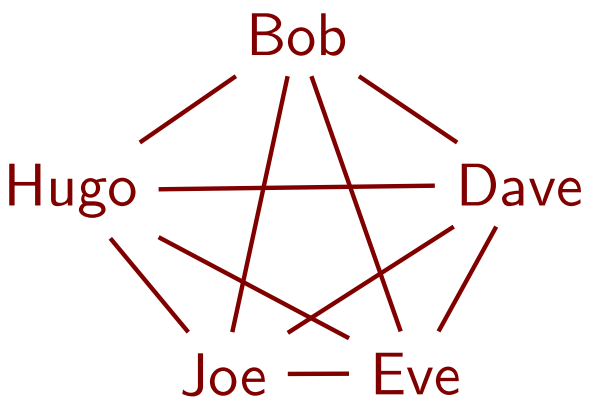

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- There is an algorithm exponential only in the number of values (V)
- Exponential in something that is potentially smaller than the data nd
  - Fixed Parameter Tractable

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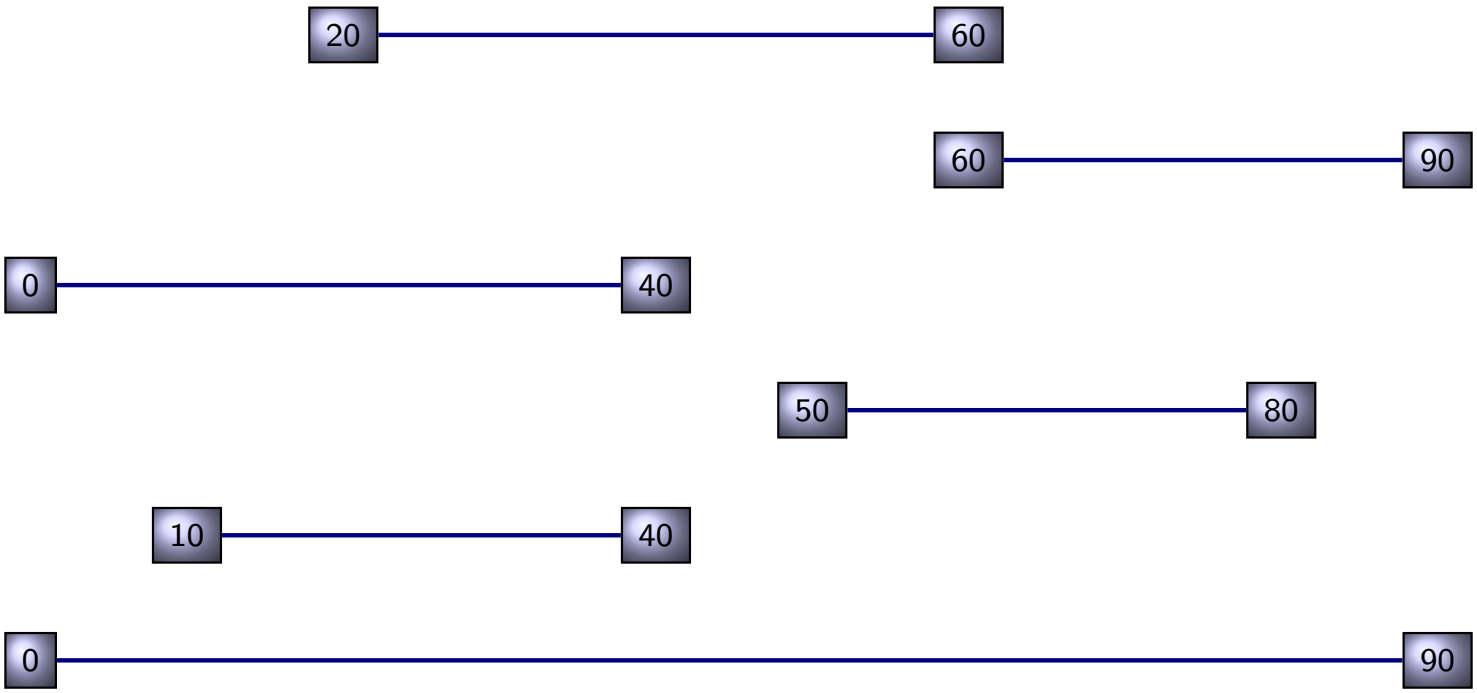
- Achieving AC is NP-hard
- What about BC?

A bc Algorithm for SoftAllEqual



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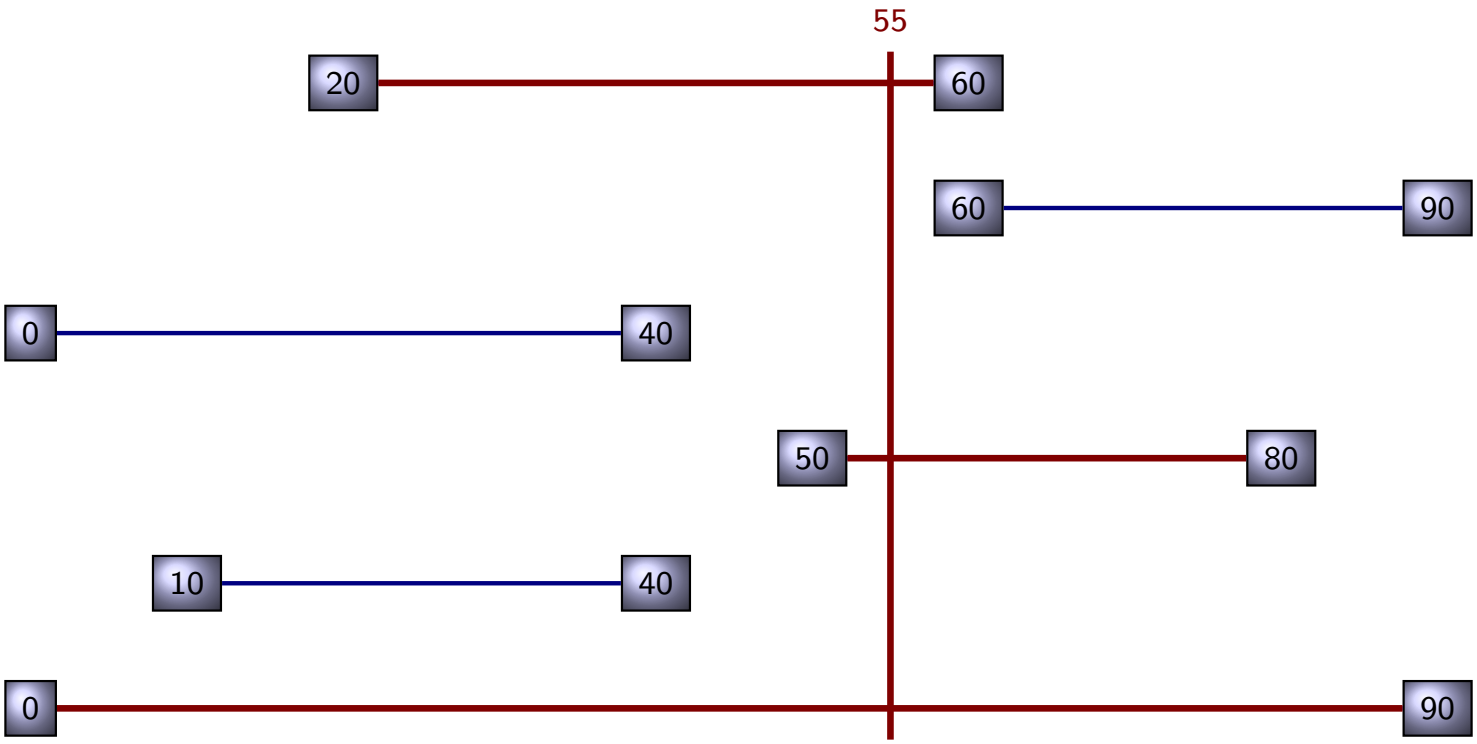

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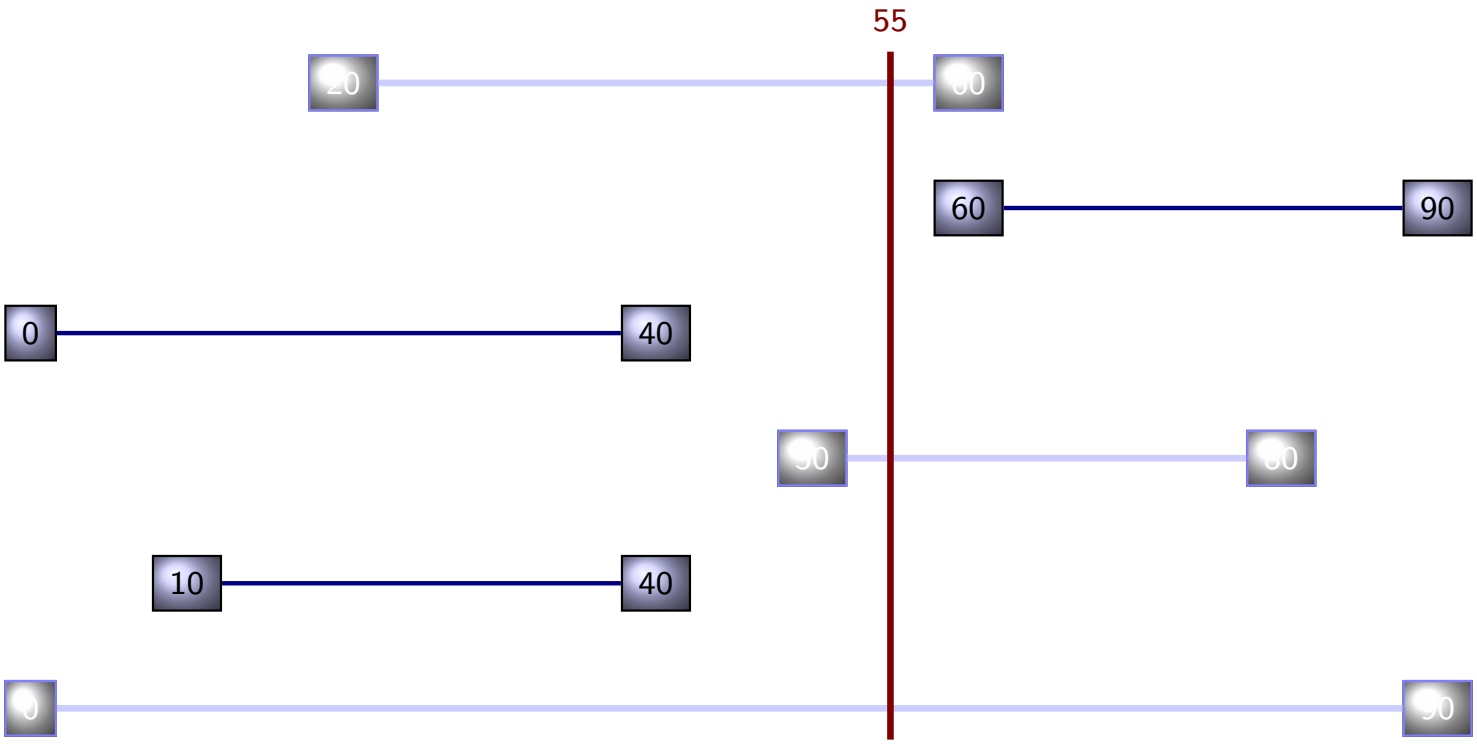

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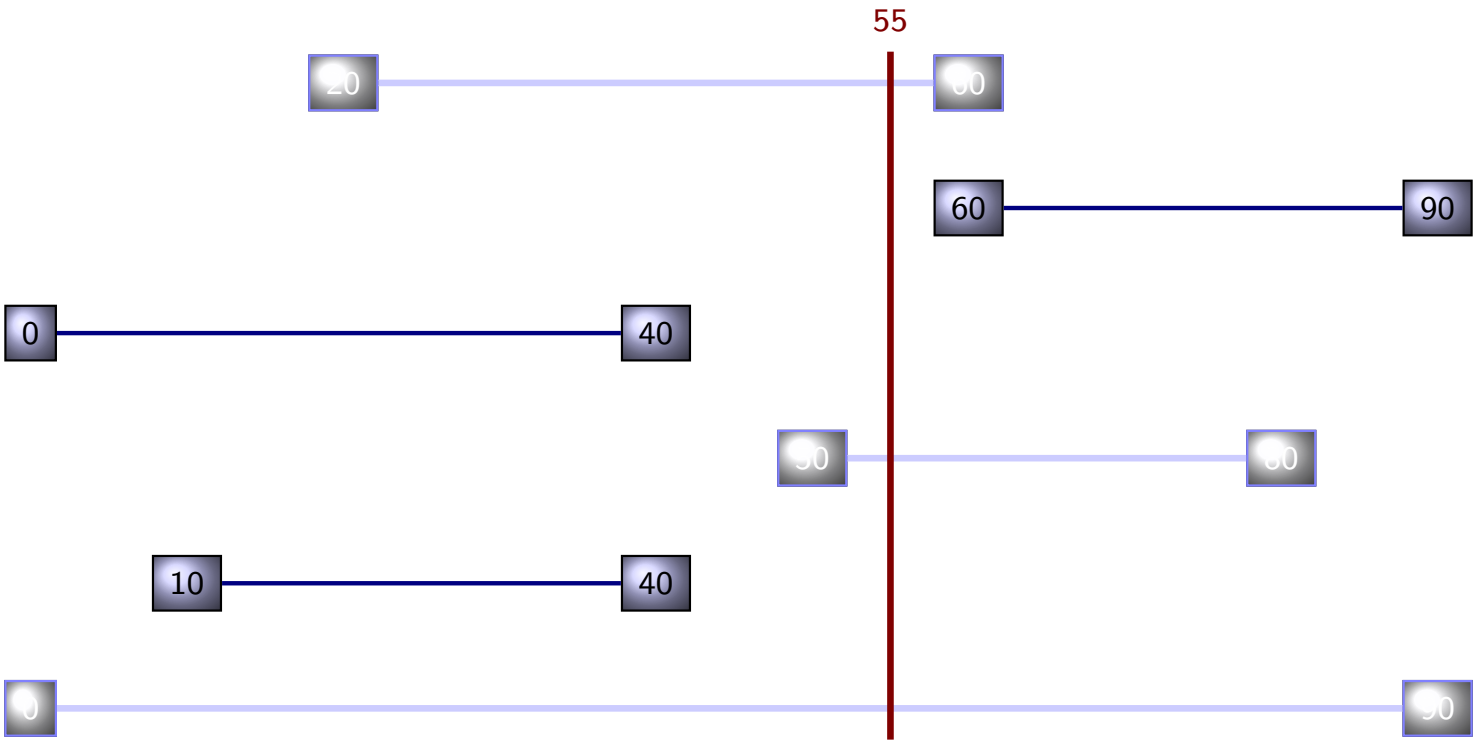

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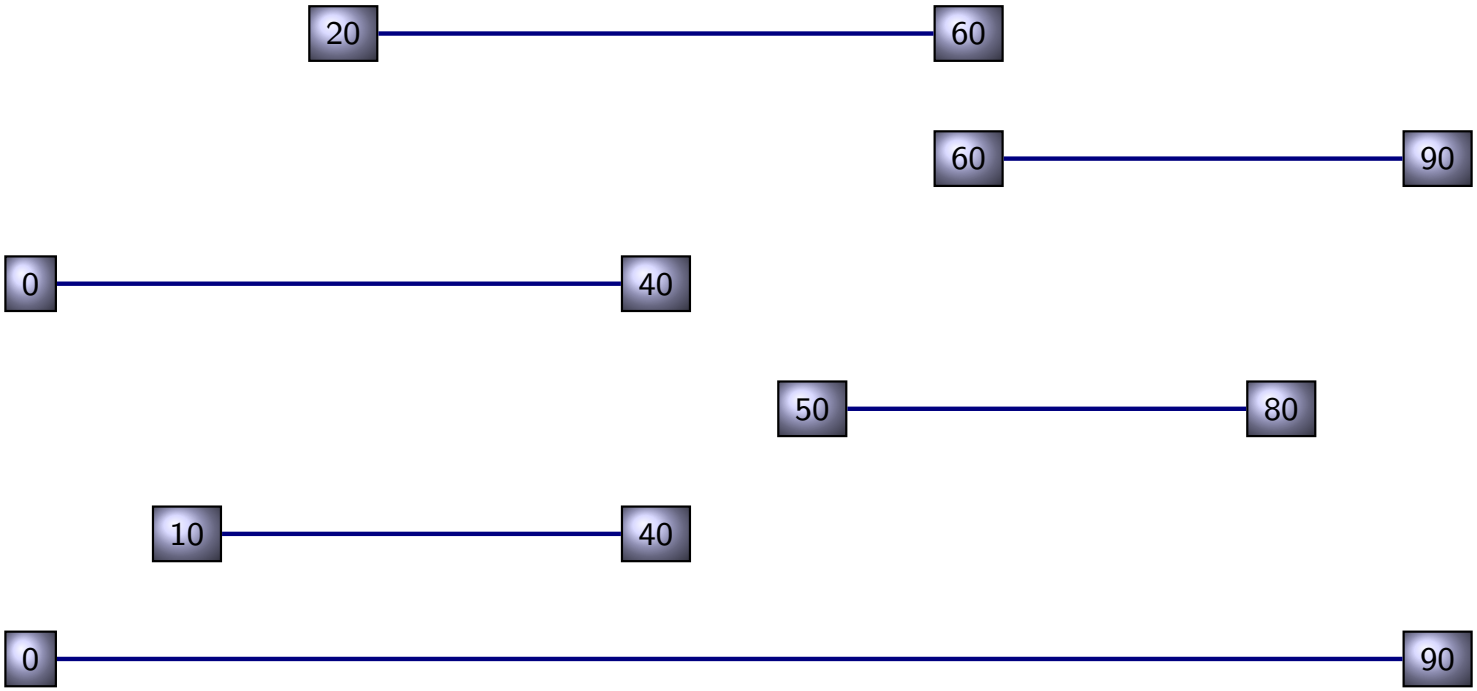
  - Polynomial size search tree: Dynamic Programming

Dynamic Programming



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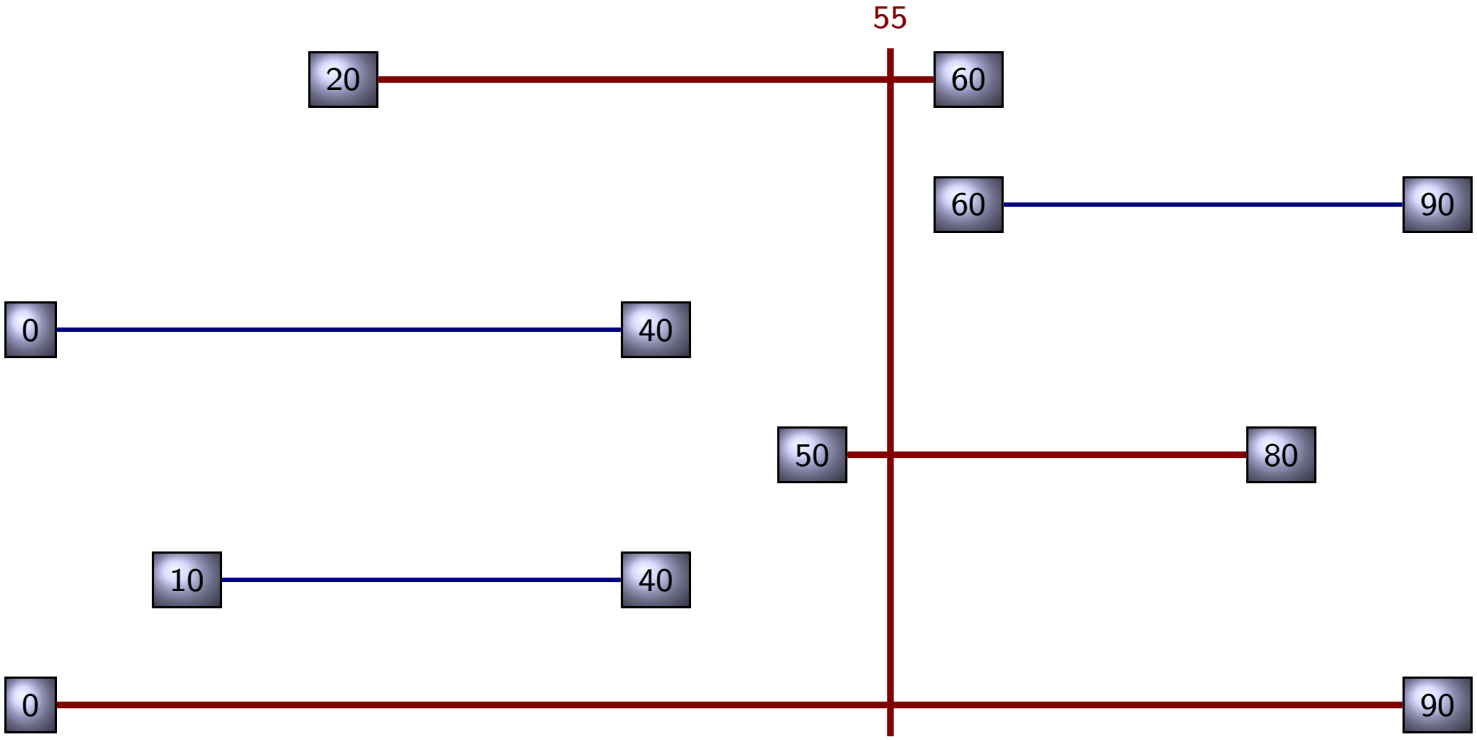

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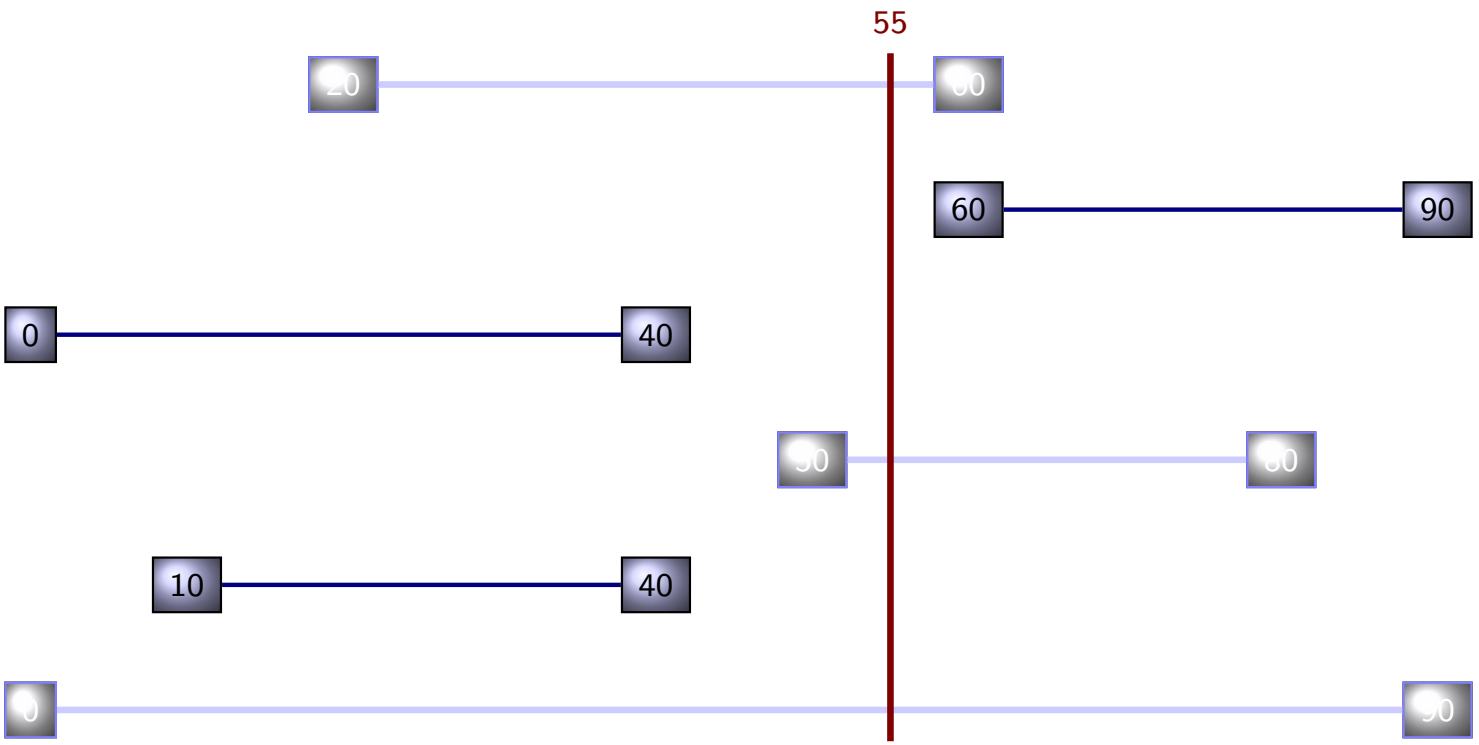

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  - table content on the splitted parts: $T_{a,v-1} + T_{v+1,b}$

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  - Table of size V^2, for each cell, go through the V values and count occurrences ($O(n)$)
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- The best tradeoff might not be obvious
  - Theoretical comparison
  - Empirical comparison

Assessing the Tradeoff: The AtMostNValue Constraint

Definition: $\textsc{AtMostNValue}([x_1, \ldots, x_n], N) \iff$

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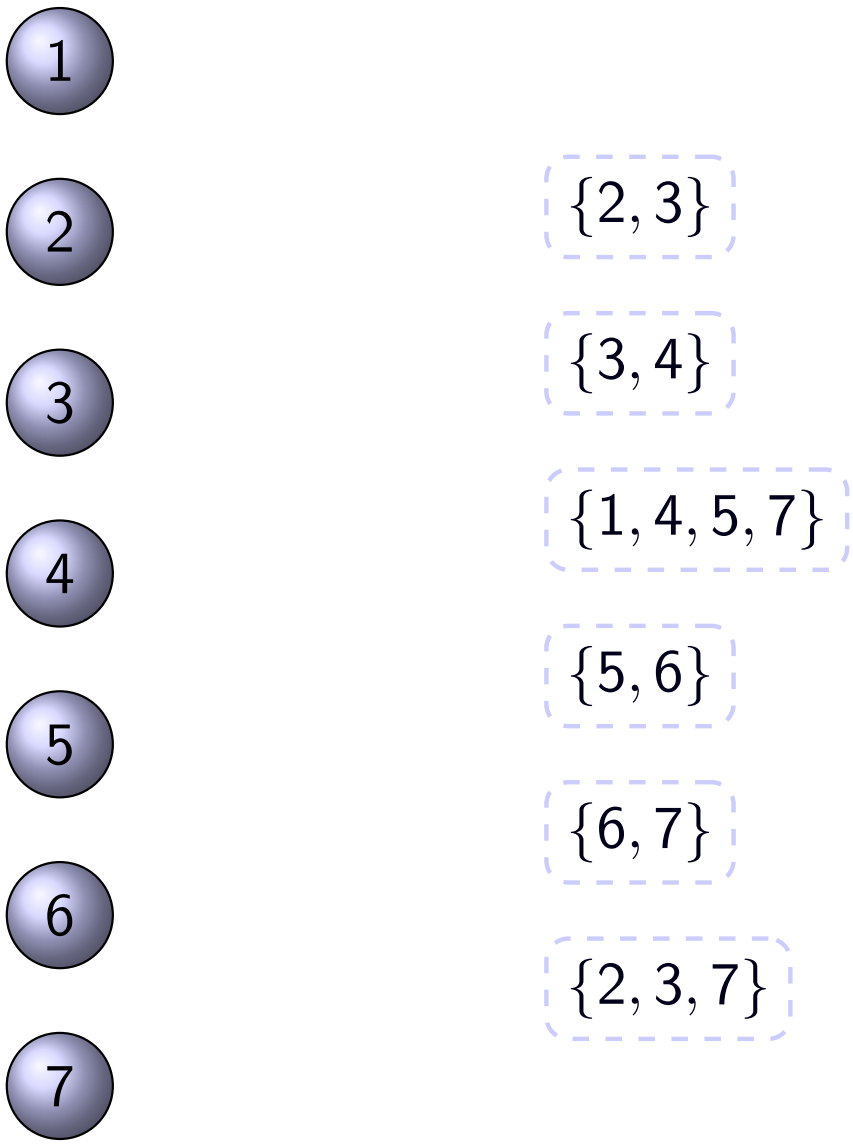

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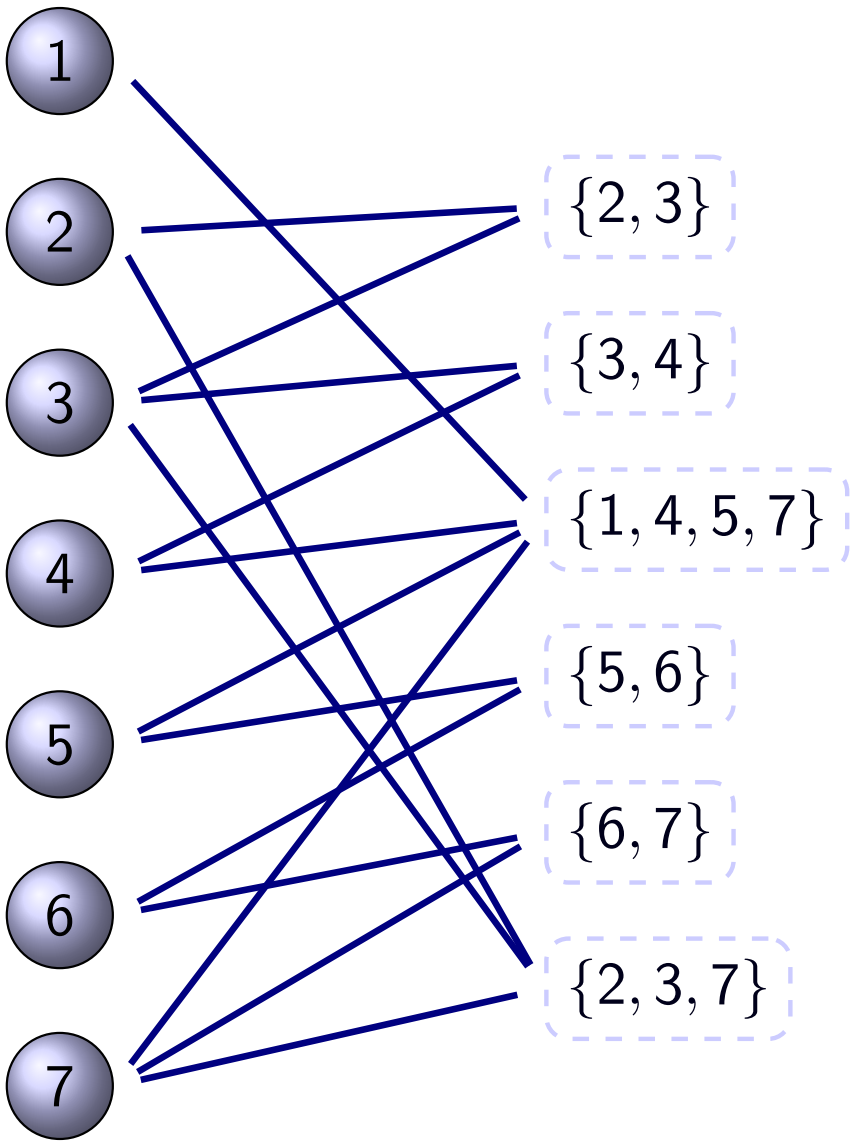

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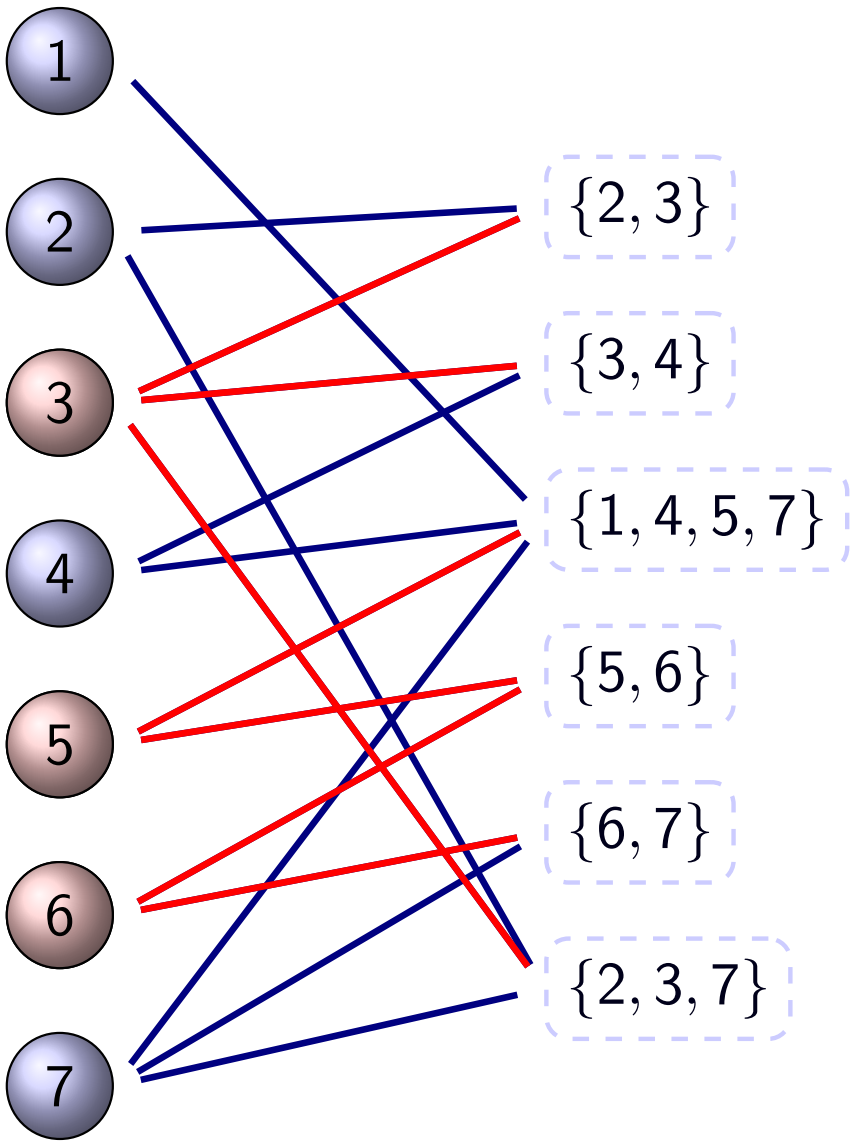

Hitting Set, Clique Cover

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Hitting Set, Clique Cover

- Can we approximate Minimum Hitting Set to filter AtMostNValue?
  - We need a lower bound; approximating MHS gives us an upper bound
  - Another approach: Clique Cover of the Intersection Graph

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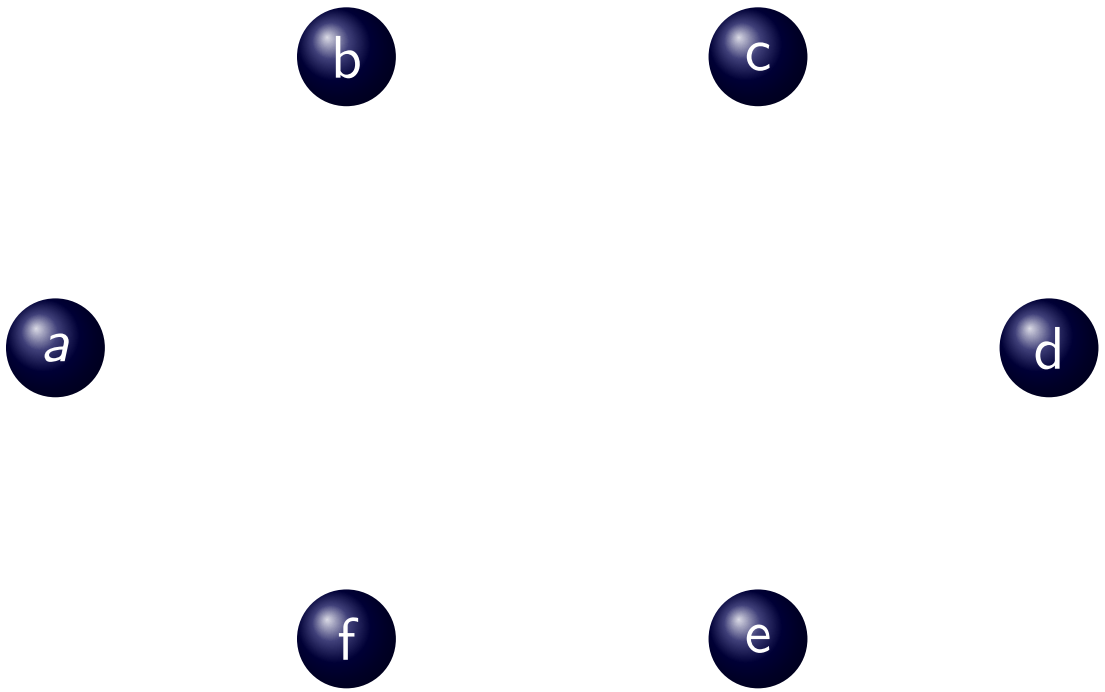

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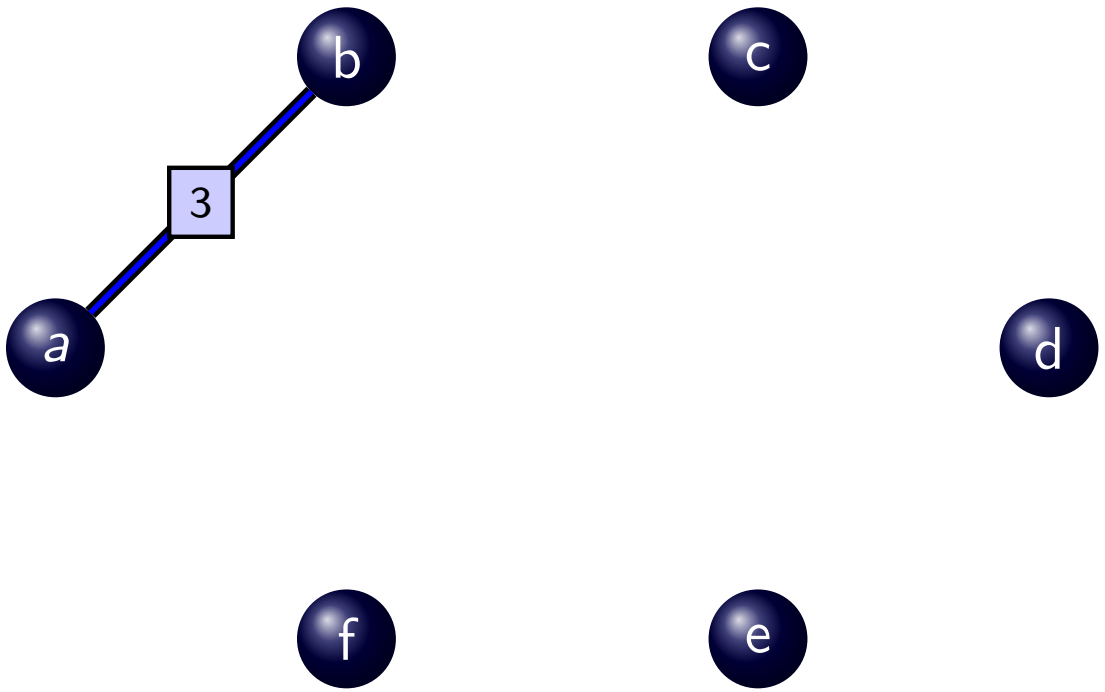

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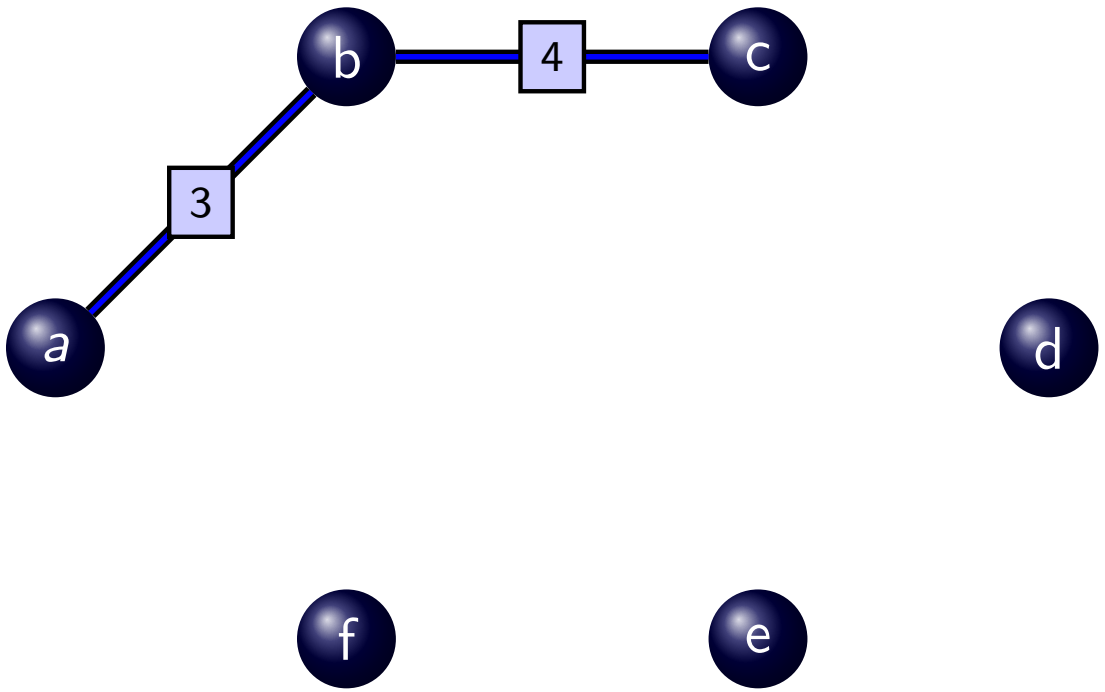

Intersection Graph: Discrete domains

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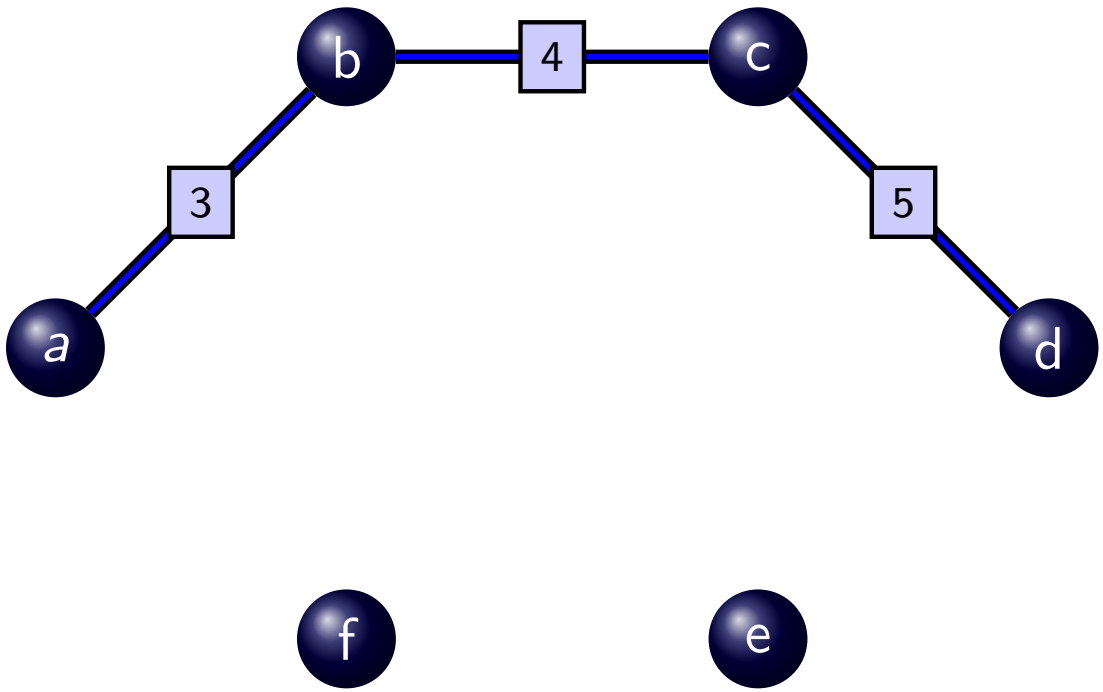

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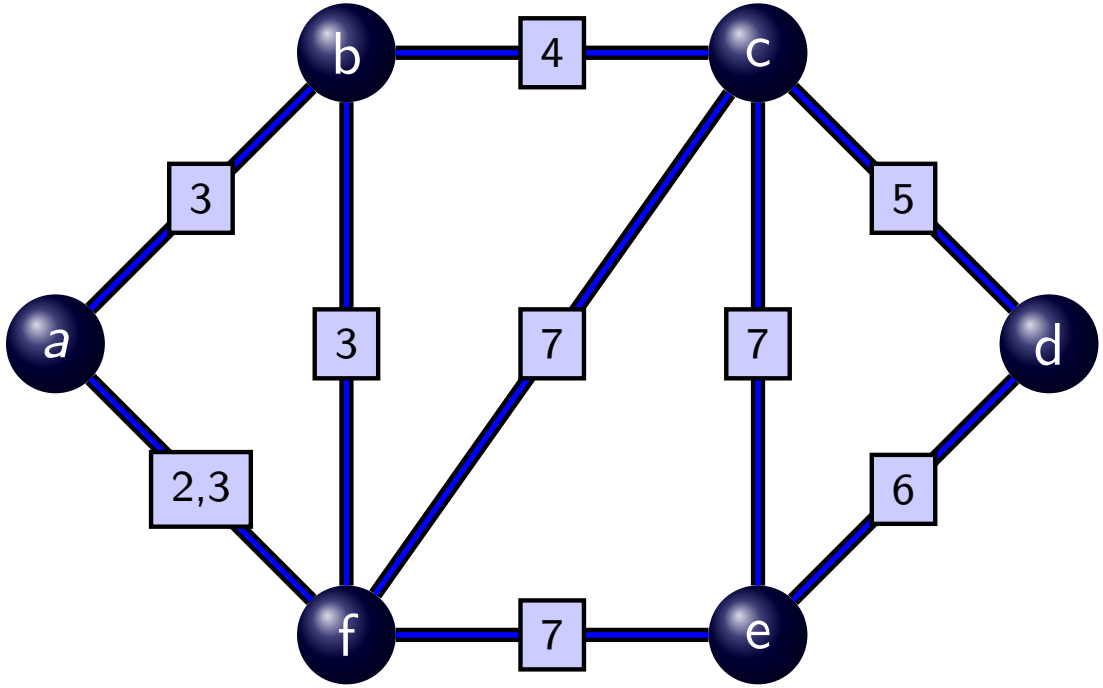

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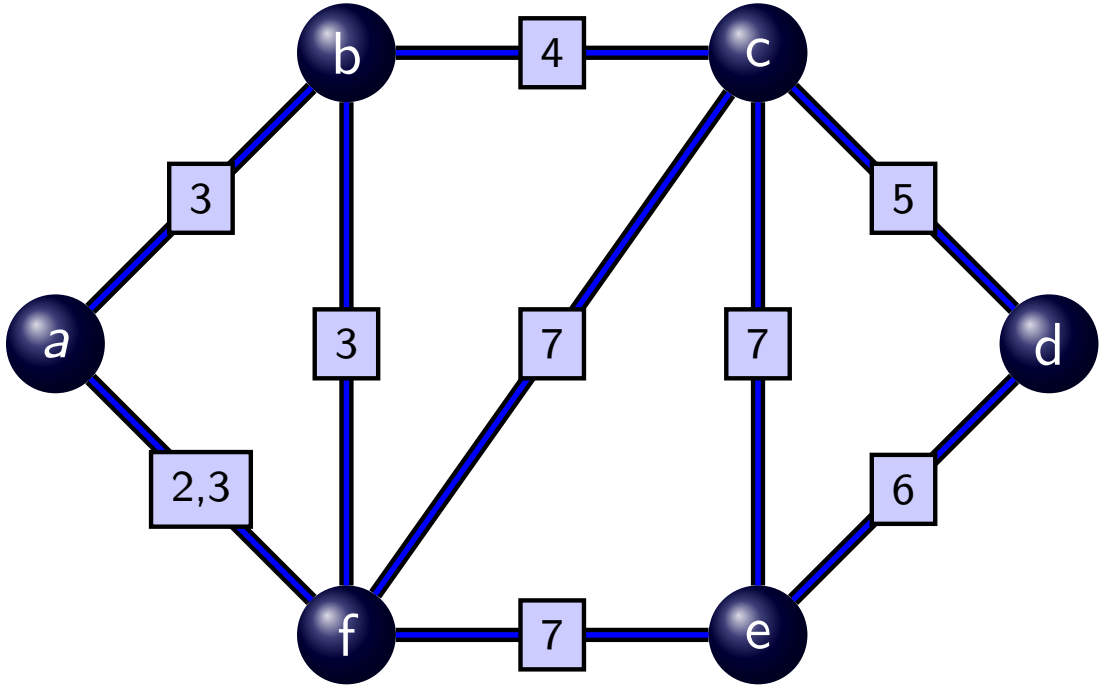

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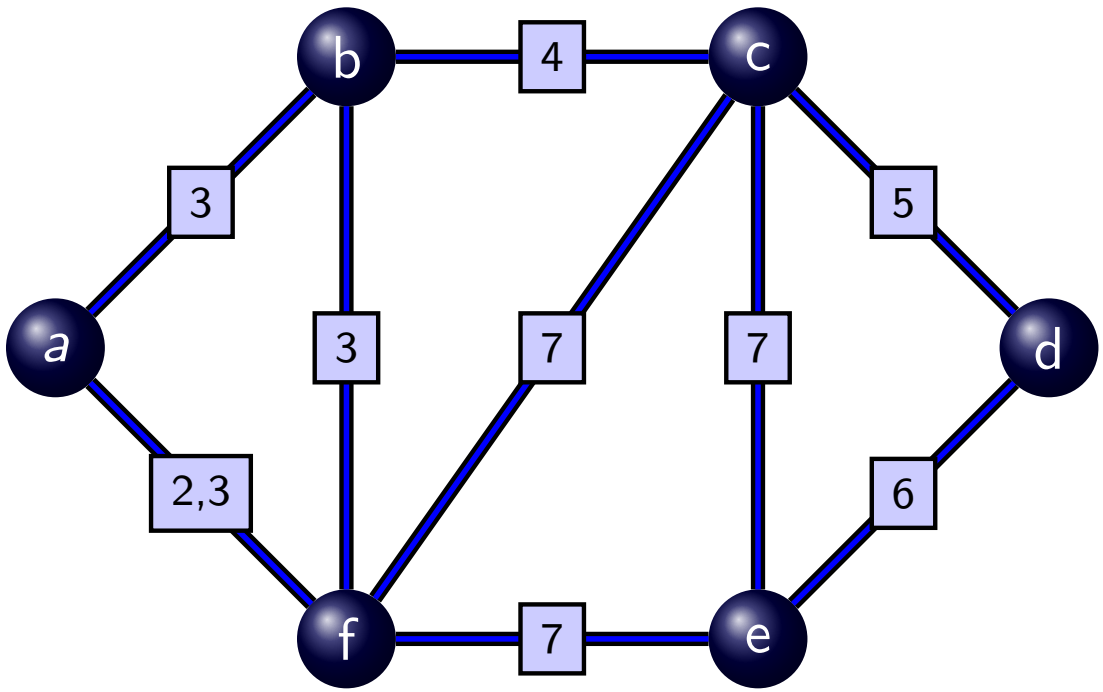

- A Minimum Clique Cover is a lower bound
- However, finding a Minimum Clique Cover is NP-hard

Intersection Graph: Interval domains

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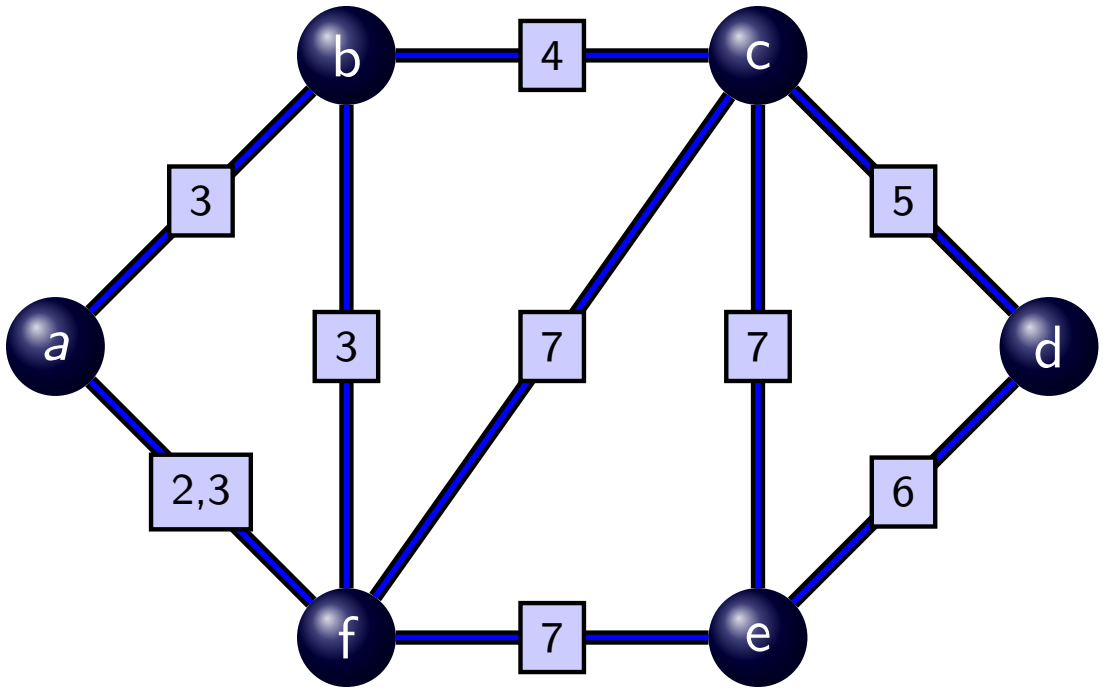

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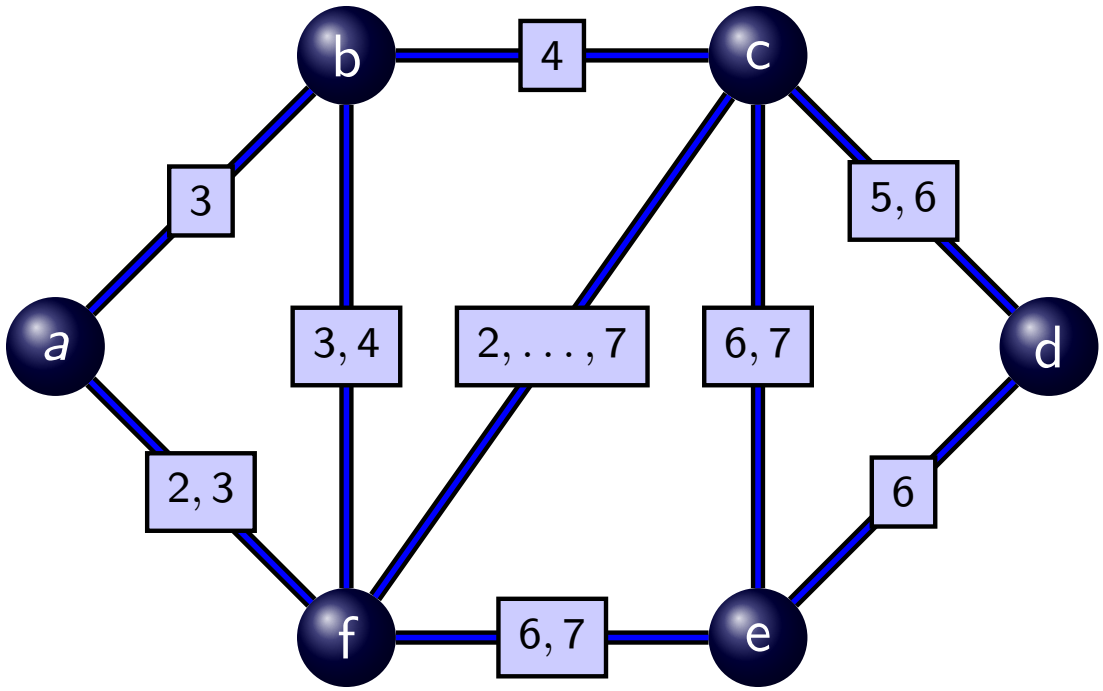

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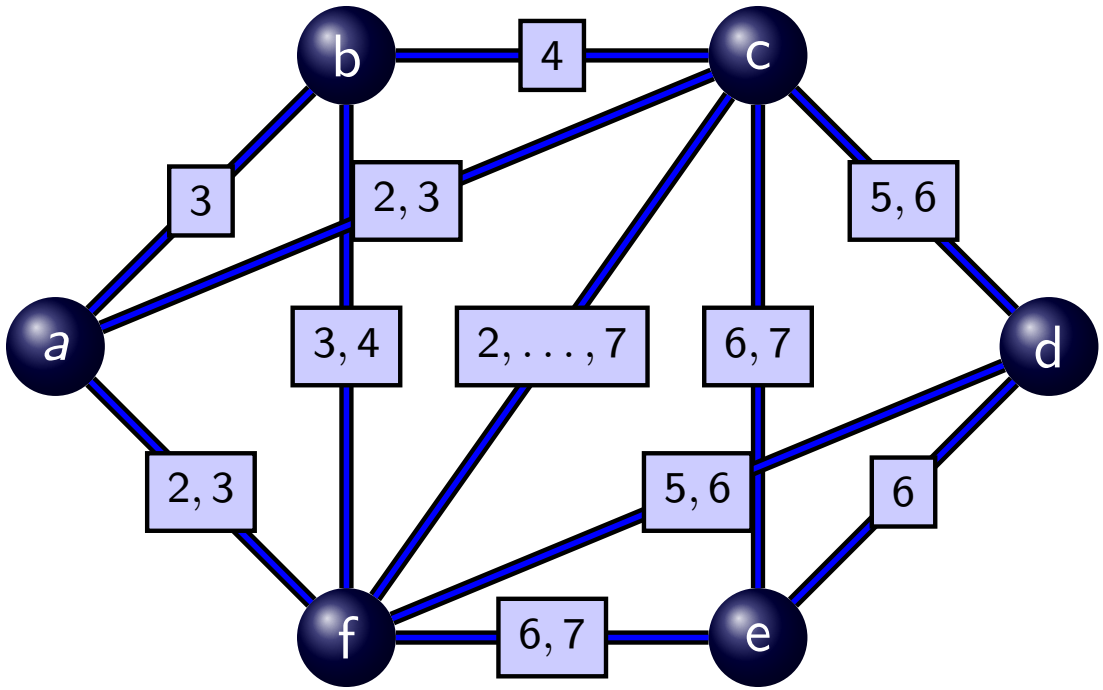

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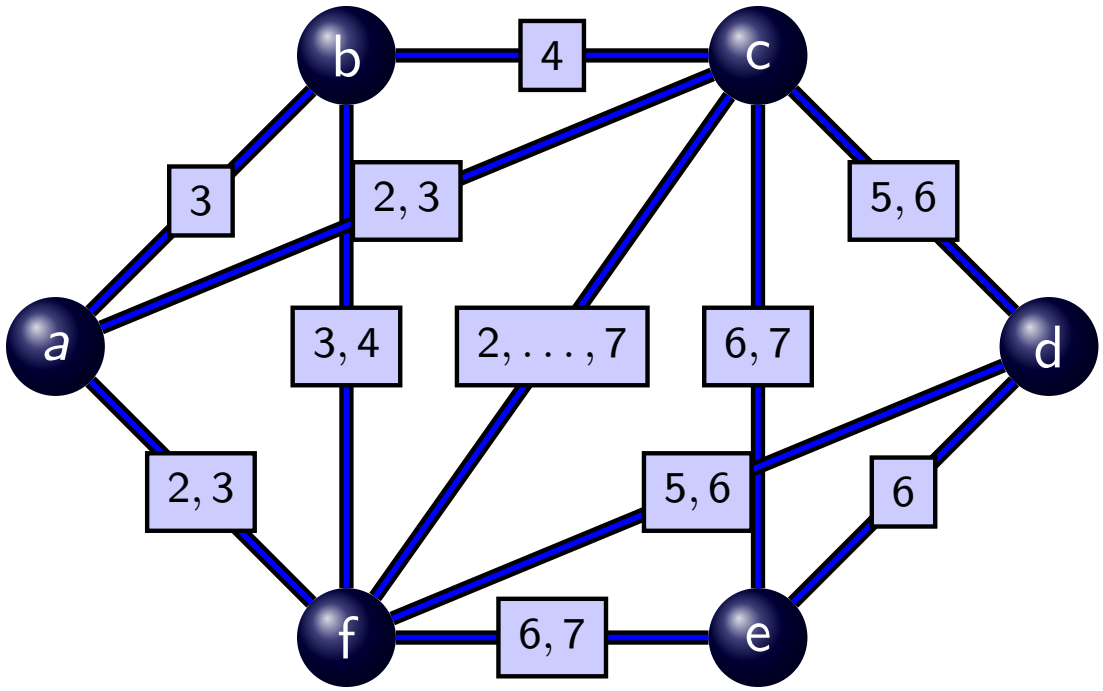

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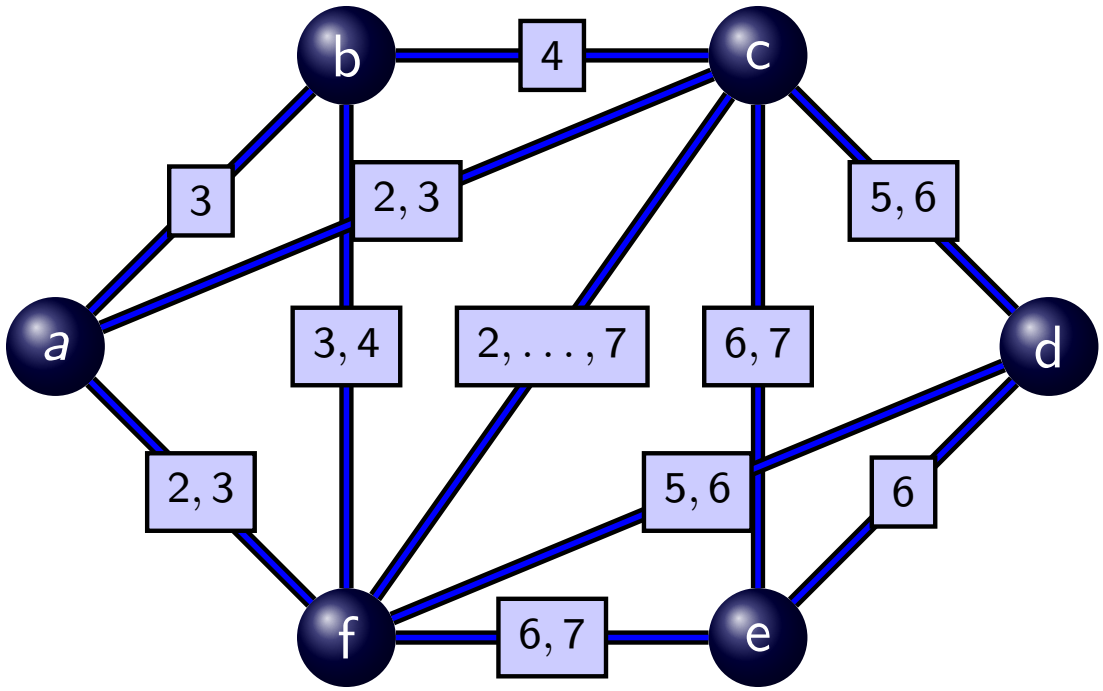

- Finding a Minimum Clique Cover is polynomial ($O(n \log n)$)
- A Minimum Clique Cover is an exact lower bound [Beldiceanu 2001]

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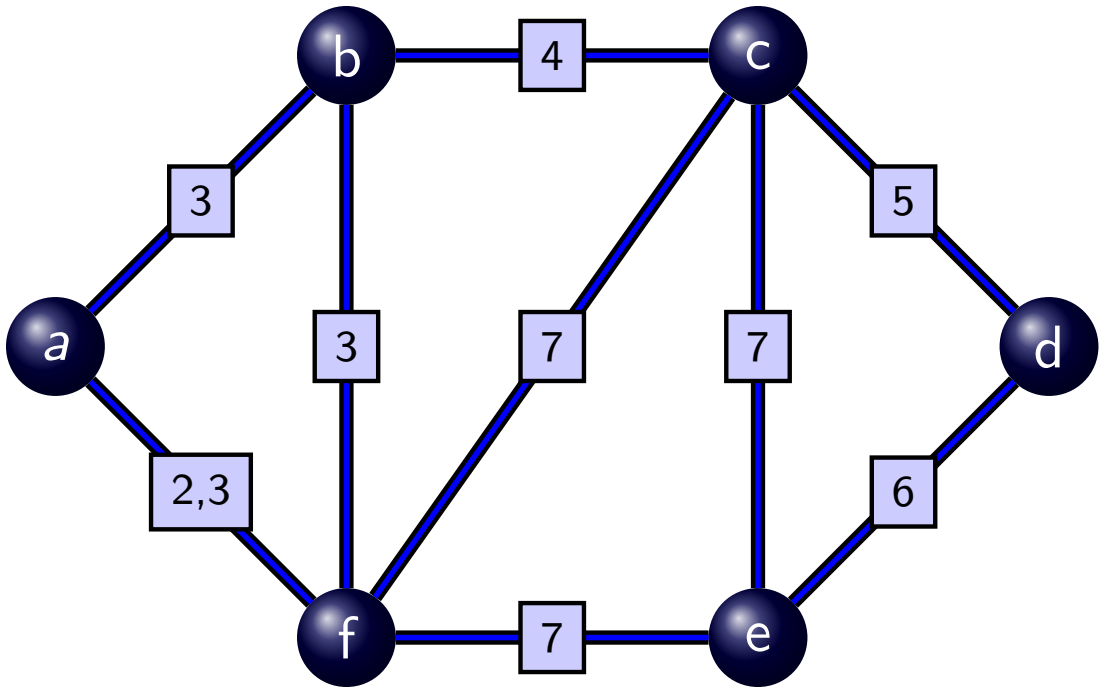

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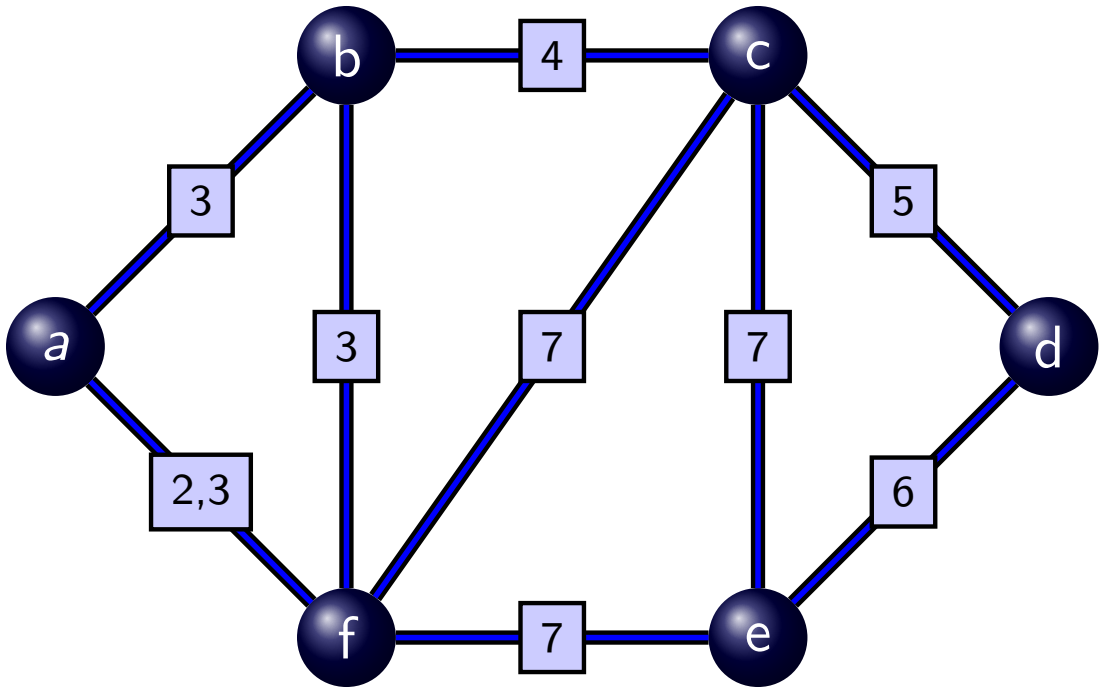

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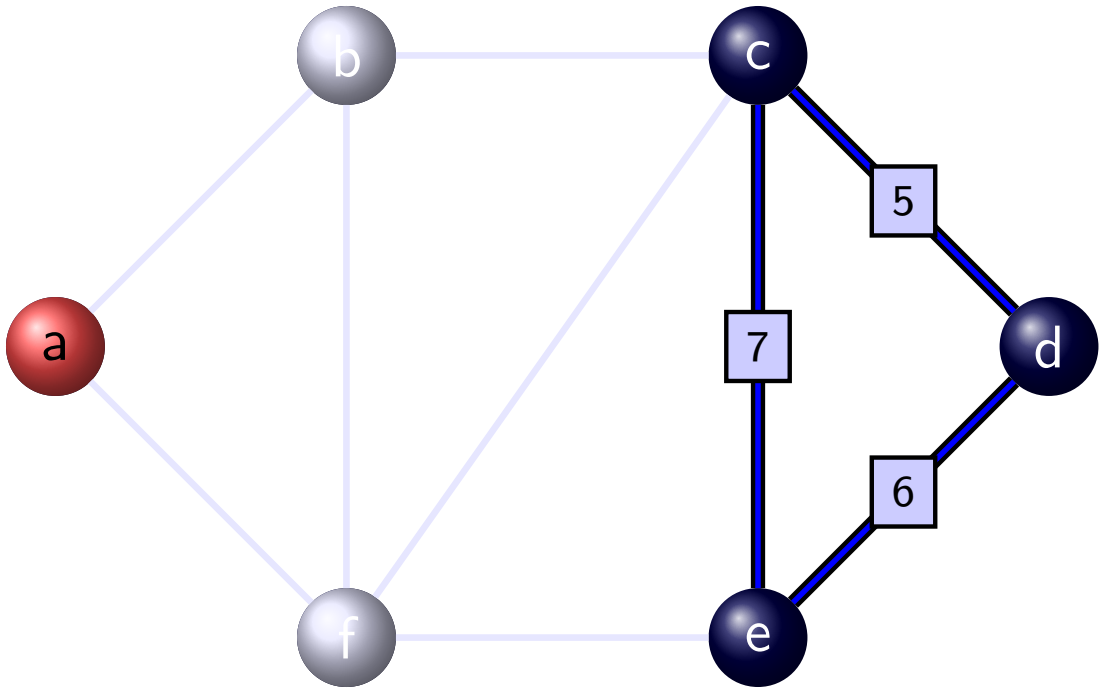

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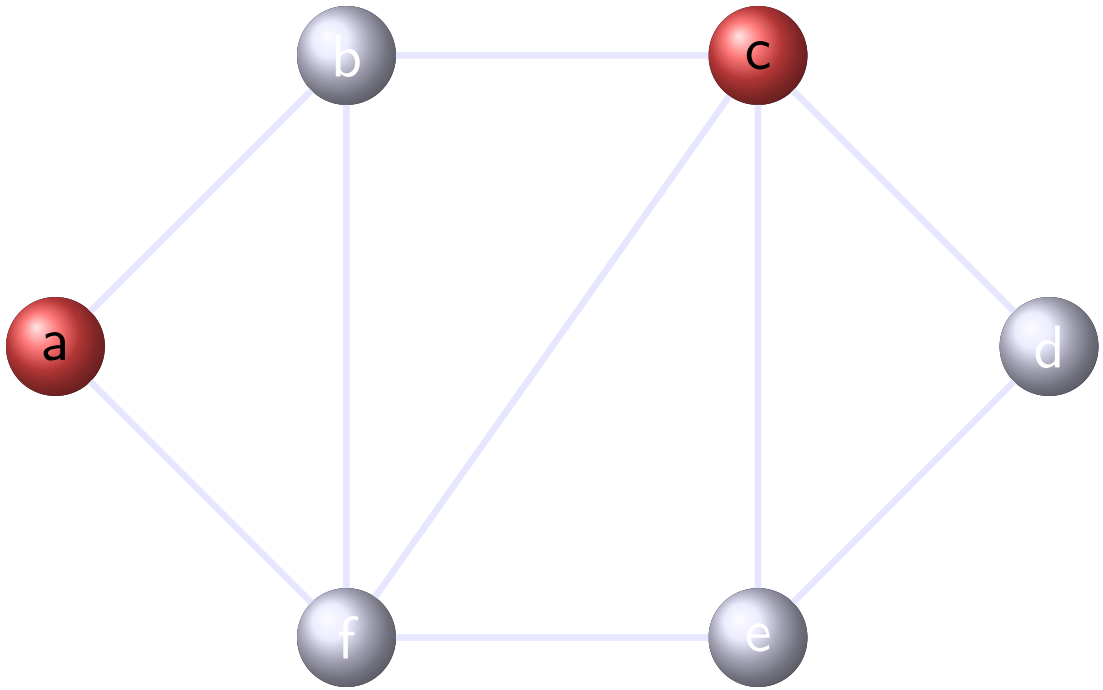

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Turán's Lower Bound

- There are even simpler lower bounds
  - [Turán 1941]

$$\alpha(G) \leq \left\lceil \frac{n^2}{2m+n} \right\rceil$$

Turán's Lower Bound

- There are even simpler lower bounds
  - [Turán 1941]
  - [Favaron et al. 1993]

$$\alpha(G) \leq \left\lceil \frac{n^2}{2m+n} \right\rceil \leq \left\lceil \frac{2n - \frac{2m}{\lceil \frac{2m}{n} \rceil}}{\lceil \frac{2m}{n} \rceil + 1} \right\rceil$$

Linear Program

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$$\begin{aligned} \text{minimize} \quad & \sum_{v \in \mathcal{D}} y_v \\ \text{subject to} \quad & \sum_{v \in x_i} y_v \geq 1, \; x_i \in \mathcal{X} \\ & y_v \geq 0 \qquad , \; v \in \mathcal{D} \end{aligned}$$

number of values

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- An integral solution is a Hitting Set
- The linear relaxation gives a lower bound

Tradeoff: Theory

- Which solution is best?

experiments

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- Which solution is best?
  - Better time complexity vs. better propagation

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  - $\phi(C, \mathcal{D})$ the domain representation / set of solutions after applying ϕ
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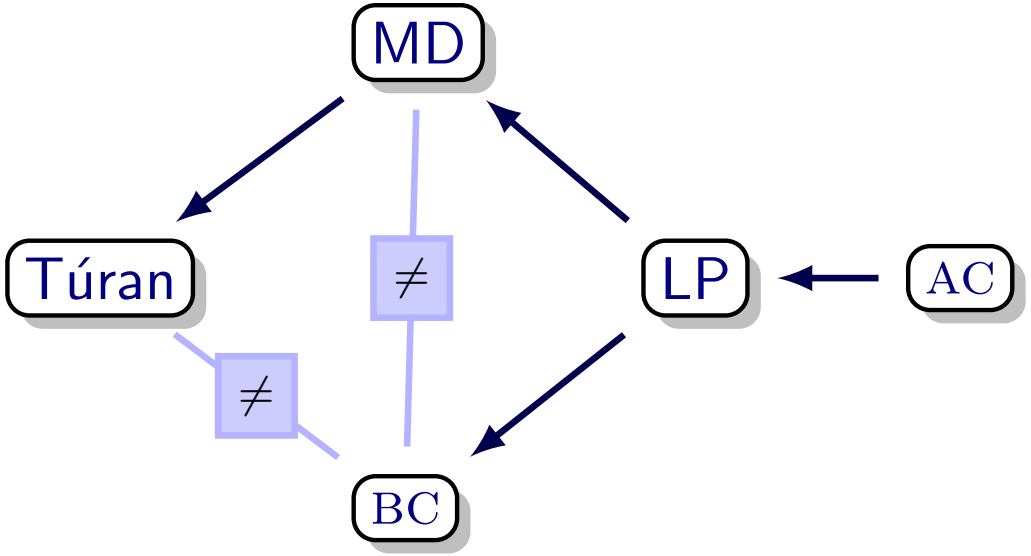

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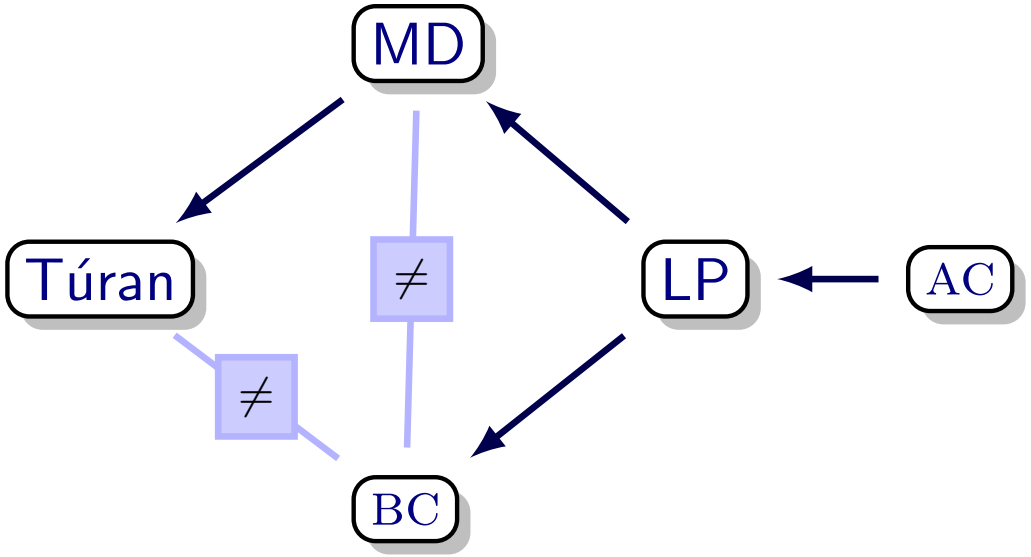

Túran	amortised
BC	$O(n \log n)$
MD	$O(n^2 d)$
LP	$O(nd^4)$
AC	NP-hard

experiments

Conclusion

- When solving a problem: find the algorithm with best complexity

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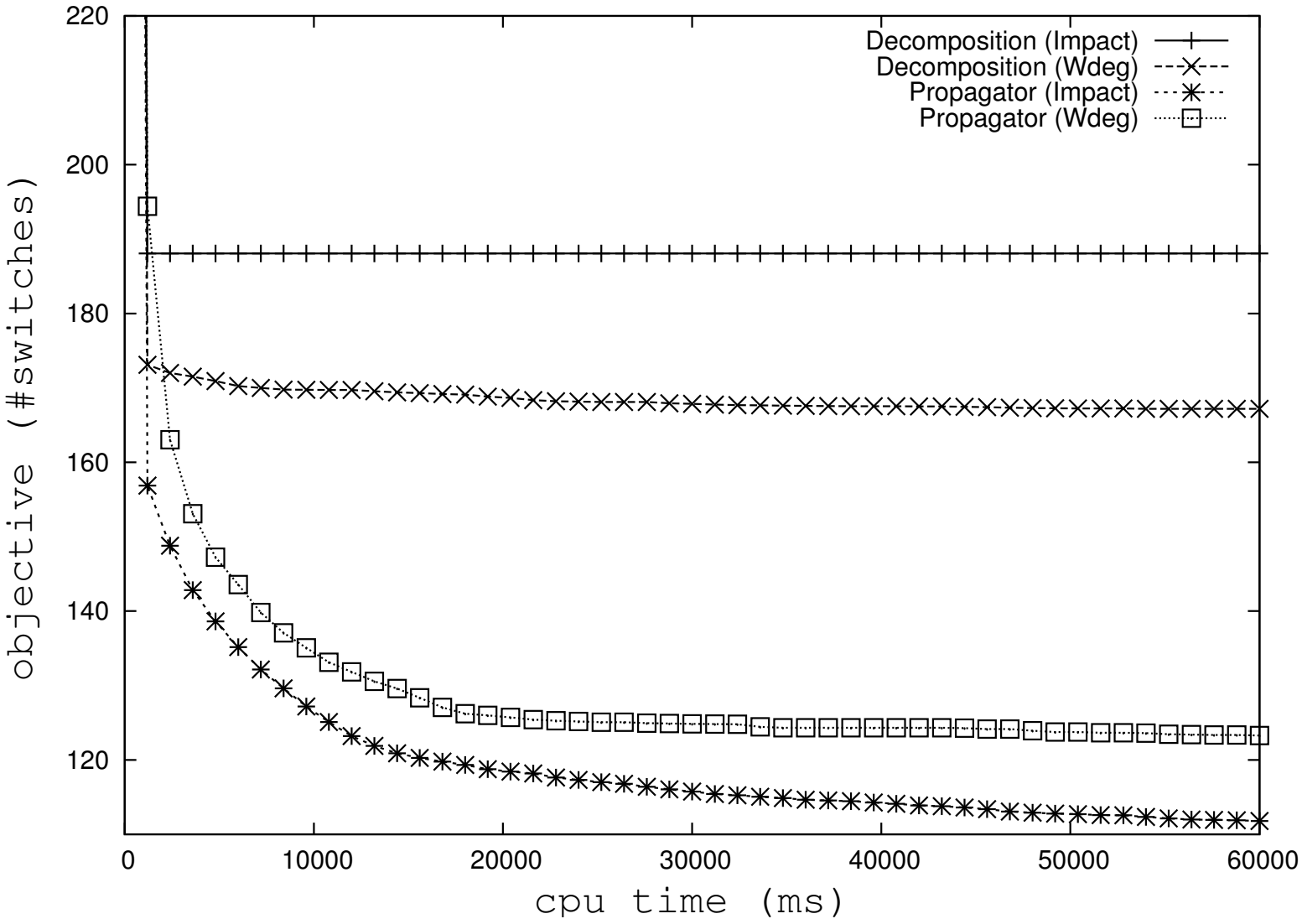
Conclusion

- When solving a problem: find the algorithm with best complexity
  - Here we also want to achieve the highest possible consistency
    - ★ AC? BC? MDD-C? (SAC, MaxRPC, ...)
    - ★ Even something undefined
- When achieving a given consistency is NP-hard
  - Consider a lower consistency
  - Consider approximations
  - Consider parameterized complexity
- Other related questions:
  - Is it decomposable?
  - What consistency/decomposition is the strongest?
- Experimental tradeoff!

Questions?

Experimental Evaluation (120 garments, 3 machines, 7 colors)



back

Experimental results

[label=expeamsc]

Table: Car-sequencing

Models	G1 ($70 \times 34 \times 5$) 11900		G2 ($4 \times 34 \times 5$) 680		G3 ($5 \times 34 \times 5$) 850		G4 ($7 \times 34 \times 5$) 1190	
	#sol	time	#sol	time	#sol	time	#sol	time
sum	8480	13.93	95	76.60	0	> 1200	64	43.81
gsc	11218	3.60	325	110.99	31	276.06	140	56.61
amsc	10702	4.43	**360**	**72.00**	16	8.62	**153**	**33.56**
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- SUM: simple decomposition
- GSC (Global Sequencing Constraint)
  - same as ATMOSTSEQCARD for an option + demand for each type of car requiring this option
  - NP-hard, approximation

Experimental results

Table: Crew-Rostering

Heuristic	Lexicographic					
Model	satifisable (1140)			unsatisfiable (385)		
	#sol	time	avg bts	#sol	time	avg bts
sum	0	-	-	170	0.05	86
gsc	25	308.93	74074	175	2.56	262
amsc	**125**	**164.36**	**1828347**	**213**	**1.76**	**22621**
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- MultiAtMostSeqCard: several capacity constraints together

back

Linear time algorithm

- We need to do each of the following in $O(1)$:
  - access the cardinality of the jth subsequence containing x_i: $card(i, j)$

1 0 0 1 1 0 . 0 . 0 1 . . 1 1

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- $card(7, 0) = 2$
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- $card(i, j) = \sum_{k=1}^{i} + c[i + j \bmod q]$

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- $card(i, j) = \sum_{k=1}^{i} + c[i + j \bmod q]$

back

Linear time algorithm

- We need to do each of the following in $O(1)$:
  - check whether there exists a subsequence of cardinality u: $\#sub(i, k)$
    - ★ jump from x_i to x_{i+1}
    - ★ increment all subsequences containing x_i

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						0	1	2	3

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