Constraint Programming and Combinatorial Optimisation in Numberjack

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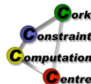

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Outline

1. Introduction
2. Basic Modeling
3. Advanced Modeling and Search
4. Controling Search
5. Behind the Scene

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Introduction

Numberjack cuts your exponential search tree into logs!

Numberjack

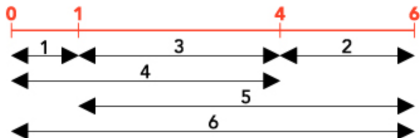
- A Python API for combinatorial optimization
- Why do we need yet another one?

Combinatorial Optimization

- A (broad) class of problems:
  - Make a number of decisions (Variables $\mapsto$ Domains)
  - Subject to a set of constraints (Constraints)
  - In order to optimize an objective
- A (broad) class of methods - Search and/or Inference
  - Constraint Programming (CP)
  - Boolean Satisfiability (SAT)
  - Mixed Integer Programming (MIP)
  - Local Search / Metaheuristics
- Scheduling, Timetabling, Configuration, Design, Diagnosis, Mathematics, Frequency assignment, Bioinformatics,...

Modeling - the constraint programming view

An Example: Golomb Ruler



- Place N marks on a ruler
- Not twice the same distance
- Minimise the length of the ruler

Modeling - the constraint programming view

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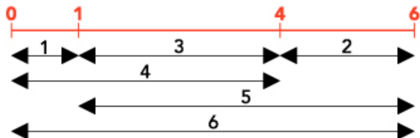

- Place N marks on a ruler
- Not twice the same distance
- Minimise the length of the ruler

Variables, Domains, Constraints and Objective

- Variables & Domains: The marks & their positions on the ruler: $m_1, m_2, m_3, m_4 \in [0, 6]$.
- Constraints: For all marks i, j, k, l: $(m_i - m_j) \neq (m_k - m_l)$.
- Objective: Minimise $last - first$.

Python

Python's Features

- High level & interpreted
- Duck typed
  - If it looks like a duck and quacks like a duck its a duck!
- Object oriented (or not!)
- Extensible
  - Modules in C/C++
- Large community

The basics

Data structures

- **list**

```
primes = [2, 3, 5, 7, 11]
cities = ["Cork", "Toulouse", "Ithaca"]
print cities[2]
>>> "Ithaca"

print cities[1:]
>>> ["Toulouse", "Ithaca"]
```

- **tuple**
- **dict**

The Basics

Loops

```
for item in <iterable>:
   do_something_with(item)
```

```
scores = [("Villa", 5),
          ("Sneijder", 5),
          ("Henry", 0)]

for player, goals in scores:
   print player, " scored ", goals, " goals"
```

The less basics

List comprehensions

One of Python's most useful feature for concise code is list comprehensions. The implementation is similar to that of Haskell.

```
print range(4)
>>> [0, 1, 2, 3]
print [x*x for x in range(4)]
>>> [0, 1, 4, 9]
```

List comprehensions generally take the following form.

```
[function(x) for x in <Iterable> (if <condition>)]
```

What is Numberjack?

A platform for combinatorial optimization

- Common language for diverse paradigms
- Back-end solvers are C/C++ libraries
  - CP (Mistral)
  - MIP (SCIP)
  - SAT (MiniSat, Walksat)
- API with a language "feel"
  - Readable, fast development time
  - Full access to the back-end solvers (SWIG)
- Open source (LGPL)
- Work in progress...
  - We need you to get involved.

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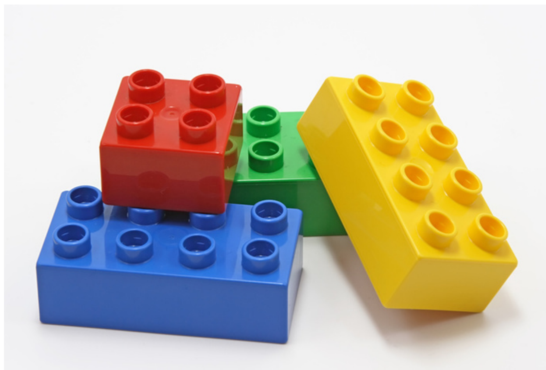
Basic Modeling and Solving



Building Blocks

- Basics of modeling in Numberjack
- Solving your problems

Expression Tree

Module

- Numberjack is a module
- Each solver is a module

```
from Numberjack import *
import MiniSat
```

Model

- Variables
- List of constraints
- Solver-abstracted

```
X = Variable( [2,4,5,8,10] )
model = Model( X != Y, ... )
model.add( Z == 2*x1 + x2 - 3*x3 )
```

Solver

- Build from a model
- Accessors are wrapped

```
solver = MiniSat.Solver( model )
solver.solve()
```

Expression Tree

Variables - Leaves of an expression tree

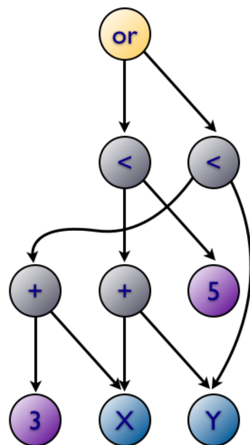
```
X = Variable(0,10)
X = Variable([1,3,5,7])
```

Predicates - Internal nodes of an expression tree

```
P = X+Y          # arythmetic value
Q = X+3 <= Y     # truth (logic) value
```

Constraints - Root node of an expression tree

```
C1 = (X+Y < 5) | (X+3 < Y)
C2 = AllDiff([x,y,z])
C3 = Sum([a,b,c,d]) >= e
```



Demo!

Golomb Ruler in Numberjack

```
m = 8
n = 2**m

marks = [Variable(n) for i in range(m)]

model = Model(

    Minimise( marks[-1] ),

    [marks[i-1] < marks[i] for i in range(1,m)],

    AllDiff( [ (first - second) for first, second in pair_of(marks) ] ),

    marks[0] == 0
)

solver = Mistral.Solver( model, marks )
solver.solve()
```

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Advanced Modeling

Advanced Features

- Advanced Numberjack constructs
- Controlling search

Language constructs

Variable containers: VarArray, Matrix

- Variable Indexing (Element constraint)
- Accessors (matrix.col, matrix.flat)
- Fancy printing

Adding new constraints and decompositions

- When a constraint name is not recognized by a solver, a decomposition is used, if it exists.
- Makes it easy to add in solver specific constraints

Domain specific modules

- Dedicated constructs for Scheduling
  - Tasks
  - Resources

Sudoku

Problem Definition

- 9 x 9 grid
- Place 1 to 9 in each column, row and square.
- Pain to model; array indexing is awful!

Sudoku

Create the variables

```
grid = Matrix(N*N, N*N, 1, N*N, 'cell_')
```

The Model

```
sudoku = Model( [AllDiff(row) for row in grid.row],
                [AllDiff(col) for col in grid.col],
                [AllDiff(grid[x:x+N, y:y+N])
                 for x in range(0,N*N,N)
                 for y in range(0,N*N,N)]
               )
```

- Every row/column/sub matrix must have different digits
  - **grid.col** returns the transpose of the matrix 'grid'
  - **grid[a:b, c:d]** returns a Matrix construct that contains rows a to b-1 and columns c to d-1.

Sudoku

Adding in a partially completed grid

```
model.add ( [(x == int(v)) for (x,v)
 in zip(grid.flat, "".join(open(clues)).split() )
 if v != '*'] )
```

Isn't Python cool?!

- clues is the path to a problem file
- Entry that is not * is a constraint
- Difficult in a modeling language

```
1 * 3
4 5 *
* * 9
```

(Water retaining) Magic Square

- N x N grid
- Contains numbers 1 to N^2
- Sum of rows, columns and diagonals equal.
- Known for at least 3000 years!
- Albrecht Dürer in 1514

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

Magic Square

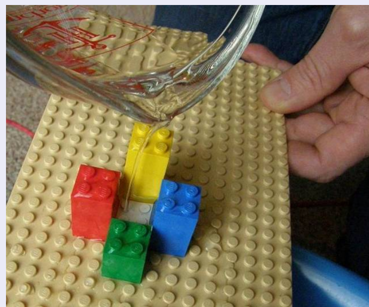
The Model

```
square = Matrix(N,N,1,N*N)
sum_val = N*(N*N+1)/2

model = Model(
    AllDiff( square ),
    [Sum(row) == sum_val for row in square.row],
    [Sum(col) == sum_val for col in square.col],
    Sum([square[a][a] for a in range(N)]) == sum_val,
    Sum([square[a][N-a-1] for a in range(N)]) == sum_val)
```

Water Retention Magic Squares

- Water retention (Craig Knecht 07)
- Magic square = 3D histogram
- Pour water on the histogram
- Maximize the amount of water



Water Retention Magic Squares

Dürer's Square

- One "lake"
- Water level = 9
- Water Depth: 3 and 2
- Water Retention = 5

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

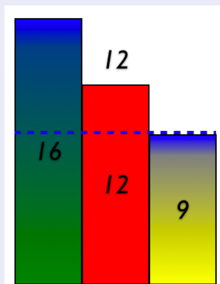
16	3	2	13
5	10	11	8
9	6+3	7+2	12
4	15	14	1

Water Retention Magic Square

How much water will it hold?

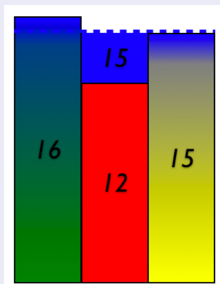
- Keep track of the total height: **sea level**

1st case: landmasses



- The "sea level" around is lower than its own height

2nd case: sea level



- The "sea level" around is greater than its own height

Water Retention Magic Square

Modeling the sea level

- Variables to represent the total level

```
water = Matrix(N,N,1,N*N)
```

Water on the edges

- The edges can't hold any water
  - Matrices allow expression of equality among variable vectors

```
water.row[0] == square.row[0]
water.row[N-1] == square.row[N-1]
water.col[0] == square.col[0]
water.col[N-1] == square.col[N-1]
```

Water Retention Magic Square

Water in the middle

- Sea level at a cell: max of its own height or the surrounding level

```
[water[a][b] ==
 Max((square[a][b],        # own height
      Min((water[a-1][b], water[a][b-1],     # sea level
           water[a+1][b], water[a][b+1])))) # on the cells
                                             # nearby
 for a in range(1,N-1)
 for b in range(1,N-1)]
```

Objective

- Goal is to maximise the water collected

```
    Maximise( Sum( water ) ) # - (N*N*(N*N-1)/2)
```

Water Retention Model

```
square = Matrix(N,N,1,N*N,'s')
water = Matrix(N,N,1,N*N,'w')
model = Model(
    Maximise( Sum( water ) ),   # objective

    # regular magic-square constraints
    AllDiff(square),
    [Sum(row) == N*(N*N+1)/2 for row in square.row],
    [Sum(col) == N*(N*N+1)/2 for col in square.col]
    Sum([square[i][i] for i in range(N)]) == N*(N*N+1)/2,
    Sum([square[i][N-i-1] for i in range(N)]) == N*(N*N+1)/2,

    # channeling with sea level
    water.row[0] == square.row[0],
    water.row[N-1] == square.row[N-1],
    water.col[0] == square.col[0],
    water.col[N-1] == square.col[N-1],
    [water[a][b] ==
     Max((square[a][b], Min((water[a-1][b], water[a][b-1],
                             water[a+1][b], water[a][b+1]))))
     for a in range(1,N-1) for b in range(1,N-1)]
    )
```

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Controling Search

Programming search & strategies

- Prototyping, search/heuristics/solvers
- Programming your own search method

Problem Solving

Simple is good

- Getting a solution quickly (Development time + Solving time)
- Prototyping, testing ideas

Parameter tuning

- Selecting
  - Variable Ordering / Branching
  - Restarts and randomization

Example (Mistral)

Problem Solving

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Example (Mistral)

- `solver.setHeuristic('MinDomain', 'RandomSplit')`
  Select the variable with 'min domain' first, branch by splitting the domain around a random pivot

Problem Solving

Simple is good

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Example (Mistral)

- `solver.setHeuristic('MinDomain', 'RandomSplit')`
  Select the variable with 'min domain' first, branch by splitting the domain around a random pivot
- `solver.solveAndRestart(LUBY, 300)` Set the restart policy to the Luby sequence with a base of 300

Programming Search

Search Primitives

- `solver.propagate()`: Reach a fixed point for some inference method.
- `solver.save()`: Save the current state.
- `solver.post( decision )` (`decision` is a unary constraint):
  - `x == v`, `x > v`, ...
- `solver.deduce()`: Post the complement of the decision taken at this level:
  - resp. `x != v`, `x <= v`, ... ("right" branch)
- `solver.undo()`: Restore the last saved state.

Water Retention Magic Squares

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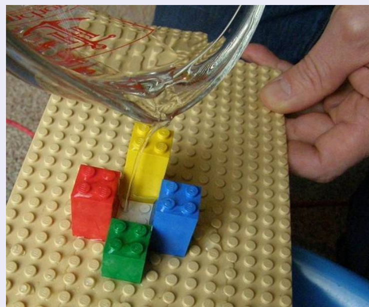

Solving the Water Retention Magic Square Problem

Challenges

- Getting good upper bounds for the objective function is hard
  - Any breach, and all the water is gone!
  - i.e., getting a good lower bound is difficult

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Solution

- There is a common sense strategy to find Magic Square that retains a large amount of water

Solving the Water Retention Magic Square Problem

Challenges

- Getting good upper bounds for the objective function is hard
  - Any breach, and all the water is gone!
  - i.e., getting a good lower bound is difficult

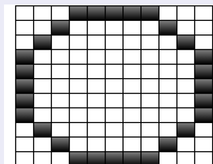
Solution

- There is a common sense strategy to find Magic Square that retains a large amount of water
  - Building a wall, i.e., high values toward the edges

Solving the Water Retention Magic Square Problem

Building a wall

- The wall should not span over a whole row/column
- We want it as high as possible
  - It makes it difficult to find a Magic Square



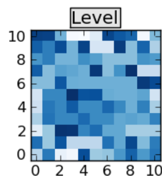
Strategy

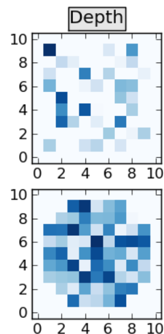
```
wall = ... # assumed to be a set of variables (cells)
for height in range(N*N-len(wall), N*N-(len(wall)+K), -1):
    solver.save()
    for brick in wall: solver.post(brick > height)
    solver.solveAndRestart()
    if solver.is_sat(): break;
    solver.undo()
```

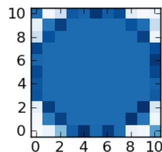
Demo!

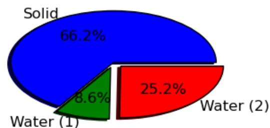
Water Retention Test

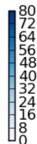
- Default search (1):
- Wall search (2):
- Water Retention:











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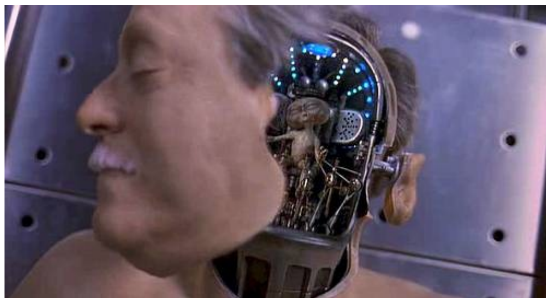
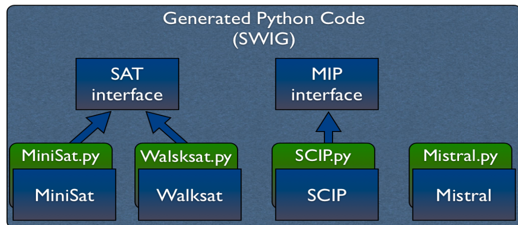
Behind the Scene



The machinery

- Wrapping a new solver
- SAT and MIP encodings

Everything can be wrapped in Python



Architecture

- C++ standardized interfaces
  - CP: Adding expression trees (direct wrapping)
  - MIP: Adding linear equations (encoder)
  - SAT: Adding clauses (encoder)
- Modules are automatically generated (SWIG)

SAT Encoding

Variables - direct/order encoding

- Two atoms for each value: $x = v$ and $x \leq v$
- A linear-sized clause: $\bigvee_{v \in D(x)} x = v$
- A linear number of 2-clauses:
  - $x \leq v \rightarrow x \leq v + 1$
  - $x = v \rightarrow x \leq v$
  - $x = v \rightarrow \neg(x \leq v - 1)$

Constraints

- Example: disequality $(x \neq y)$
  - $\forall v \in D(x) \cap D(y) \;\; \neg(x = v) \vee \neg(y = v)$
- Example: precedence $(x \leq y)$
  - $\forall v \in D(x) \setminus [-\infty, max(y)] \;\; x \leq v$
  - $\forall v \in D(y) \setminus [min(x), \infty] \;\; \neg(y \leq v)$
  - $\forall v \in D(x) \cap D(y) \;\; y \leq v \rightarrow x \leq v$

MIP Encoding

Variables

- A linear integer variable with the same bounds
- Channelled with 0/1 variables standing for $x = v$
  - *Only if required by a constraint*
  - $\sum_{v \in D(x)} (x = v) = 1$
  - $\sum_{v \in D(x)} (x = v) * v = x$

Constraints

- Example: disequality $(x \neq y)$
  - $\forall v \in D(x) \cap D(y) \quad (x = v) + (y = v) \leq 1$
- Example: Alldiff$(x_1, \ldots, x_n)$
  - $\forall v \in \bigcup_{1 \leq i \leq n} D(x_i) \quad \sum_{1 \leq i \leq n} (x_i = v) \leq 1$
  - With "domain" equations: network flow

Conclusion

Yet another platform?

- Yes, but not a new language
  - API, hence deeper control the back-end solvers
  - Python is an established programming language
  - It is easy to plug into other applications

Future work

- More back-end solvers
  - CPlex
  - CP Optimizer
  - OR tools (Google CP solver)
  - ToolBar!
  - ...
- Encodings
  - Need to be more exhaustive
  - Also need research work
    - Good thesis topic!

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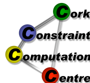

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