The multiagent project scheduling problem: complexity of finding a minimum-makespan Nash equilibrium

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1 Introduction

This paper considers cooperative projects that involve a set of agents (or organizations), each one being in charge of the execution of a part of the project. It is assumed that any agent is able to control the duration of its activities by leveling, at a given cost, the usage of extra resources. The outcome of agents not only depend on their decision strategy, but also on the strategies of their partners and on the satisfaction of the customer (Evaristo & van Fenema 1999), which depends on the project makespan: a fixed price is established for the project, together with rewards in case agents manage to finish the project earlier than expected (Estevez Fernandez 2008). Concerning the sharing of rewards among agents, we assume that it is balanced accordingly to some ratios that have been previously agreed upon the design phase of the actors' network (Cachon & Lariviere 2005).

The framework depicted here can be viewed as a particular non-cooperative game where players (*i.e.*, the agents and the customer) play together for performing a project, intending to maximize their objective (*i.e.*, profit maximization and makespan minimization, respectively).

This paper mainly discusses the complexity of finding a Nash equilibrium that minimizes the project makespan. It is structured as follows. Section 2 defines formally the multiagent project scheduling problem. Section 3 illustrates the notion of strategy efficiency and stability, which are at the heart of the problem considered in this abstract. Section 4 describes how Nash equilibria can be characterized using the notion of a residual cut. Section 5 show that the problem of minimizing the project makespan under the constraint that the found strategy is a Nash equilibrium is unary NP-hard.

2 Definitions and notation

More formally, a multi-agent project scheduling problem can be defined as a tuple $<\mathcal{G}, \mathcal{A}, \underline{P}, \overline{P}, C>$. $\mathcal{G} = (X, U)$ is an activity-on-arc graph that defines the project activity network. The set of nodes $X = \{0, 1, \ldots, n-1\}$ represents the project events, and the set of arcs U can be split into two subsets, U_R and U_D, corresponding to the real and dummy project activities, respectively. Classically, nodes 0 and $(n-1)$ represent the project beginning and project end, respectively. The activities of U_R are shared among a set $\mathcal{A} = \{A_1, \ldots, A_m\}$ of m agents. The set of m_u activities assigned to agent A_u is denoted $\mathcal{T}_u$. Basically, $\forall (A_u, A_v) \in \mathcal{A}^2$ with $u \neq v$, $\mathcal{T}_u \cap \mathcal{T}_v = \emptyset$. Moreover, $\cup_{u=1}^{m}(\mathcal{T}_u) = U_R$. $\underline{P}$ ($\overline{P}$)

is an application that associates each activity $(i,j) \in U$ with a minimal duration $\underline{p}_{i,j}$ (a normal duration $\overline{p}_{i,j}$, respectively), the processing time $p_{i,j}$ of each activity $(i,j) \in U_R$ being controllable, provided $p_{i,j} \in [\underline{p}_{i,j}, \overline{p}_{i,j}]$. For a dummy activity $(i,j) \in U_D$, we have $\underline{p}_{i,j} = \overline{p}_{i,j} = 0$. C is an application that associates a crashing unitary-cost $c_{i,j}$ to each activity $(i,j) \in U$ (provided that $c_{i,j} = 0, \forall (i,j) \in U_D$). The cost incurred by an agent A_u when shortening an activity $(i,j) \in \mathcal{T}_u$ from $\overline{p}_{i,j}$ to $p_{i,j}$ is $(\overline{p}_{i,j} - p_{i,j}) \times c_{i,j}$.

The agents' strategy, further referred as S, is defined as a vector that gathers the individual strategies, *i.e.*, $S = (P_1, \dots, P_m)$, where P_u is the strategy chosen by A_u, $\forall u = 1 \dots m$, *i.e.*, the duration vector associated with the activity set $\mathcal{T}_u$. The cost incurred by A_u for his strategy P_u is $\sum_{(i,j)\in\mathcal{T}_u} c_{i,j}(\overline{p}_{i,j} - p_{i,j})$. We denote S_{-u} the strategies played by the $(m-1)$ agents but A_u, (*i.e.*, $S_{-u} = (P_1, \dots, P_{u-1}, P_{u+1}, \dots P_m)$). Then, focusing on a particular agent A_u, S can be also written as a couple (S_{-u}, P_u). This notation will be helpful in the next section for defining some game concepts. We further refer to $\overline{S}$ ($\underline{S}$) as the particular strategy for which $p_{i,j} = \overline{p}_{i,j}$ ($p_{i,j} = \underline{p}_{i,j}$), for all $(i,j) \in U_R$, and note $\overline{D}$ ($\underline{D}$) the corresponding project makespan.

It is easy to see that, once a collective strategy S is chosen, the project completion time, denoted as $D(S)$, can be computed using a classical longest path algorithm on the graph $\mathcal{G}(S)$, the length of every arc $(i,j) \in U_R$ being set to the value $p_{i,j}$ chosen in strategy S. Parameter π denoting the daily reward given by the customer, the value $Z_u(S) = w_u \times \pi \times (\overline{D} - D(S)) - \sum_{(i,j)\in\mathcal{T}_u} c_{i,j}(\overline{p}_{i,j} - p_{i,j})$ corresponds to the profit of A_u for the strategy S, where $(\overline{D} - D(S))$ is the project-makespan reduction. Therefore, to any strategy S can be associated a profit vector $Z(S) = (Z_1(S), \dots, Z_m(S))$.

Every agent aiming at maximizing his profit, the multi-agent project scheduling problem can be seen as a multi-objective optimization problem:

$$
\begin{array}{lll}
\max \left\{ Z_u \right\} & , \forall A_u \in \mathcal{A} & (1) \\
\text{s.t} & & \\
t_j - t_i - p_{i,j} \geq 0 & , \forall (i,j) \in U & (2) \\
\underline{p}_{i,j} \leq p_{i,j} \leq \overline{p}_{i,j} & , \forall (i,j) \in U & (3) \\
t_i \in \mathbb{R} & , \forall i \in X & (4)
\end{array}
$$

Let us remark that, if $\mathcal{A}$ only contains a single agent, the problem turns into the classical project time/cost tradeoff problem, which is discussed in Phillips & Dessouky (1977).

3 Efficiency vs. stability

A strategy is said *efficient* if it corresponds to a Pareto-optimal solution. A strategy vector S is a Pareto-optimal solution in the strong (weak) sense if, for each agent A_u, it does not exist any alternate strategy vector S' such that $Z(S') > Z(S)$ ($Z(S') \geq Z(S)$, respectively), with at least one strict inequality. In other words, S is a Pareto strategy if no other strategy S' dominates S with respect to the agents' profit. In this abstract, we further refer to $\mathcal{S}^P$ as the set of weak Pareto strategies.

On the other hand, a strategy is *stable* if there is no incentive for any agent to modify it, so improving his profit. The stability of a strategy is important since it ensures that agents can trust each other. It is strongly connected to the notion of a Nash equilibrium. A strategy vector $S = (P_u, S_{-u})$ is a Nash equilibrium if for all agents A_u and any local alternate strategy P'_u, we have $Z_u(P_u, S_{-u}) \geq Z_u(P'_u, S_{-u})$. Let $\mathcal{S}^N$ be the set of Nash equilibria.

For illustration, let us consider the multiagent activity network of Figure 1 with two agents and five activities. Agent A_1 holds two activities: $\mathcal{T}_1 = \{a, c\}$ (represented with dotted arcs), while A_2 manages three activities: $\mathcal{T}_2 = \{b, d, e\}$ (drawn with plain arcs). We assume that the daily reward π equals 120 which is shared fairly: $w_1 = w_2 = 0.5$. When the duration of all the activities is set to their maximal value, *i.e.*, $S = (7, 9, 3, 8, 5)$, the project makespan is 15 and the reward is 0. With the strategy $S' = (7, 9, 2, 7, 5)$, where activities c and d have been decreased by one time unit, the project makespan becomes 14, giving the agents' profits $Z_1(S') = Z_2(S') = 120 \times 0.5 - 20 = 40$. The project makespan can be reduced more to 13 with the strategy $S'' = (6, 9, 3, 7, 4)$, leading the agents' profits $Z_1(S'') = Z_2(S'') = 50$.

Let us remark that it is impossible to get a project makespan lower than the one found for strategy S''. Moreover, S'' is efficient since it corresponds to a Pareto optimum that maximizes the profit of both agents. Nevertheless, S'' is not stable: agent A_2 can easily increase his profit, to the detriment of A_1, by simply increasing back the duration of his activities d and e, which gives the strategy $S''' = (6, 9, 3, 8, 5)$, with the corresponding makespan 14 and agents' profits $Z_2(S''') = 60$ and $Z_1(S''') = -10$. At the opposite, the strategy S', which is not a Pareto optimum (it is dominated by S''), is stable since no agent is able to improve his profit by himself.

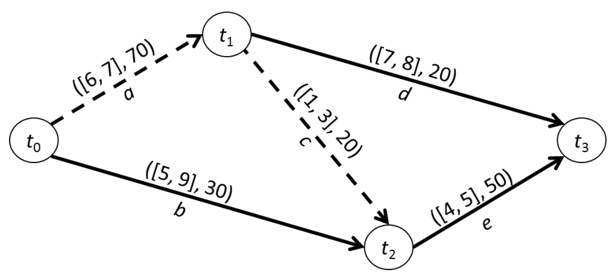

Fig. 1. A multiagent activity network with two agents and five activities

To sum up, the notion of a Pareto strategy seems to be important with respect to the agents profit, as represented by the objective function. The notion of a Nash equilibrium is complementary to the previous one and ensures organizational stability. Ideally, agents should choose a strategy being both a Pareto optimum and a Nash equilibrium, if such a strategy exists. On the other hand, from the customer viewpoint, the problem of finding a Nash equilibrium that minimizes the project makespan is also of interest. Indeed, its solution gives the lower makespan that can be reached for the customer, provided that the organization remains stable. It can be viewed as the problem of minimizing the price of anarchy for the customer viewpoint.

4 Characterization of a Nash equilibrium

A Nash equilibrium can be characterized using the notion of residual cut in critical activity networks. As discussed in Hadjiat & Maurras (1997), a residual cut characterizes a way to increase or decrease the tension in a graph at a given cost. Given a cut $\omega = (\Omega, \overline{\Omega})$ (with $0 \in \Omega$ and $n - 1 \in \overline{\Omega}$), let us refer to ω^+ (to ω^-) as the set of critical activities (i, j) such that $i \in \Omega$ and $j \in \overline{\Omega}$ (such that $i \in \overline{\Omega}$ and $j \in \Omega$, respectively). A cut ω is said residual with respect to a strategy S if there exists $\delta \in \mathbb{R}$ such that $\forall (i, j) \in \omega^+$, $p_{i,j} + \delta \in [\underline{p}_{i,j}, \overline{p}_{i,j}]$ and $\forall (i, j) \in \omega^-$, $p_{i,j} - \delta \in [\underline{p}_{i,j}, \overline{p}_{i,j}]$. A cut identifies a way to increase ($\delta > 0$) or decrease ($\delta < 0$) the project makespan by δ with the unitary cost

$\text{cost}(\omega) = \pm(\sum_{(i,j)\in\omega^+} c_{i,j} - \sum_{(i,j)\in\omega^-} c_{i,j})$, negative if $\delta > 0$, positive otherwise ($\delta < 0$). We will refer to $\text{cost}_u(\omega)$ as the unitary cost given by ω projected on agent A_u.

We also need to introduce the notion of a *poor* strategy. A strategy S with project duration $D(S)$ is said poor if there exists an alternative strategy $S' = (P'_u, S_{-u})$ with the same makespan $D(S') = D(S)$, only differing from S by the strategy taken by A_u, such that $Z_u(S) < Z_u(S')$ and $Z_v(S) \leq Z_v(S')$, for all $A_v \neq A_u$. Obviously, any poor strategy cannot be a Pareto or a Nash one.

Now, we are able to state the following proposition that aims at characterizing a Nash equilibrium. This proposition is not proven in the paper because of the length limitation.

Proposition 1. *A non-poor strategy S is a Nash equilibrium if and only if there is no residual cut $\omega(\Omega, \overline{\Omega})$ satisfying one of the two following conditions: (1) $\delta > 0$ and $\exists A_u \in \mathcal{A}$ with $-cost_u(\omega) \geq w_u\pi$ or (2) $\delta < 0$ and $\forall (i,j) \in \omega^+$, $(i,j) \in \mathcal{T}_u$ and $cost_u(\omega) < w_u\pi$.*

5 Complexity of finding $S \in \mathcal{S}^N$ that minimizes $D(S)$

In this section, we take an interest in minimizing the project makespan with the constraint that the corresponding strategy is a Nash equilibrium. Such an optimization problem is of interest for the customer who aims at minimizing the project makespan. It is also interested for the set of agents since the lower the makespan, the greater the rewards. Constraining the strategy to be a Nash equilibrium ensures that no agent will be tempted to increase back the makespan (stability), to the detriment of the others.

We state the following complexity result (which is not proven in the paper because of the length limitation). Simply note that it is based upon a reduction from problem 3-partition, which is known to be NP-complete in the strong sense (Garey & Johnson 1979). The detailed of the proof will be presented during the conference.

Proposition 2. *Determining whether it exists a Nash strategy S such that $D(S) < \lambda$ is strongly NP-complete.*

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References

Cachon, G. P. & Lariviere, M. A. (2005), 'Supply chain coordination with revenue-sharing contracts: Strengths and limitations', *Management Science* **51**(1), pp. 30–44.

Estevez Fernandez, M. (2008), A game theoretical approach to sharing penalties and rewards in projects, Discussion Paper 2008-84, Tilburg University, Center for Economic Research.

Evaristo, R. & van Fenema, P. C. (1999), 'A typology of project management: emergence and evolution of new forms', *International Journal of Project Management* **17**(5), 275–281.

Garey, M. R. & Johnson, D. S. (1979), *Computers and Intractability: A Guide to the Theory of NP-Completeness*, W. H. Freeman & Co., New York, NY, USA.

Hadjiat, M. & Maurras, J. F. (1997), 'A strongly polynomial algorithm for the minimum cost tension problem', *Discrete Mathematics* **165-166**, 377 – 394. Graphs and Combinatorics.

Phillips, Steve, J. & Dessouky, M. I. (1977), 'Solving the project time/cost tradeoff problem using the minimal cut concept', *Management Science* **24**(4), pp. 393–400.