Input-to-State Stability of Switched Systems under Dwell-Time Conditions

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References

Section 2:

M. Della Rossa and A. Tanwani (2022)
Instability of dwell-time constrained switched nonlinear systems.
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Section 3:

S. Liu, A. Tanwani and D. Liberzon (2021)
ISS and integral ISS of switched systems with nonlinear supply functions.
Mathematics of Controls, Signals, and Systems.
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Section 4:

G.-X. Zhang and A. Tanwani (2019)
ISS Lyapunov functions for cascade switched systems and sampled-data control.
Automatica, 105: 216–227, 2019.

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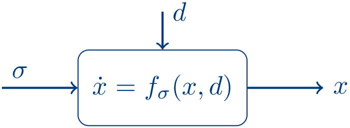
Switched Systems



Time-dependent switched systems:

- We have locally Lipschitz vector fields $\{f_p\}_{p \in \mathcal{P}}$, with some index set $\mathcal{P} \subset \mathbb{N}$.
- Switching signal $\sigma : [0, \infty) \to \mathcal{P}$ is piecewise constant, right-continuous, with finitely many discontinuities in a finite interval. We denote the set of such functions by Σ.
- The input $d : [0, \infty) \to \mathbb{R}^m$ is locally essentially bounded.
- State trajectory x is an absolutely continuous function.
- We can consider reset maps at switching times, but are avoided in this talk.

Stability Notions with Inputs

System class:

$$\dot{x} = f_\sigma(x, d)$$

Stability notions:

- The switched system is uniformly *input-to-state stable* (ISS) over Σ if there exists a class $\mathcal{KL}$ function β and a class $\mathcal{K}_\infty$ function γ such that

$$|x(t)| \leq \beta(|x_0|, t) + \gamma(\|d_{[0,t]}\|_\infty)$$

  for every $\sigma \in \Sigma$, every $x_0 \in \mathbb{R}^n$, and every input d.
- The switched system is uniformly *integral-ISS* over Σ if there exists a class $\mathcal{KL}$ function β and class $\mathcal{K}_\infty$ functions α, γ such that

$$\alpha(|x(t)|) \leq \beta(|x_0|, t) + \int_0^t \gamma(|d(s)|)\, \mathrm{d}s$$

  for every $\sigma \in \Sigma$, every $x_0 \in \mathbb{R}^n$, and every input d.
- In case of state resets with inputs, the definition of iISS includes an additional term due to value of input at impulse times.

(Average) Dwell-Time

We say that Σ_{τ_a} has average dwell-time (ADT) τ_a if, for each $\sigma \in \Sigma$, we have

$$N_\sigma(t,s) \leq N_0 + \frac{(t-s)}{\tau_a}$$

where $N_\sigma(t,s)$ is the number of switches on the interval $(s,t]$. We say that Σ_{τ_d} has dwell-time (DT) τ_d if, we have $N_0 = 1$.

- [Morse '96] formulated the notion of dwell-time
- [Hespanha/Morse '99] studied global asymptotic stability of switched system (0-GAS) using ADT notion
- Besides switched systems, the notion of dwell-time also appears in stability analysis of impulsive systems

ISS and (Average) Dwell-Time

Theorem (ISS for switched systems [Vu/Chatterjee/Liberzon '07])

Suppose that the switched system satisfies following two hypotheses:

- *Each subsystem* $\dot{x} = f_p(x,d)$ *is ISS, that is, there exist* $V_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ *such that, for every* $x \in \mathbb{R}^n$, $d \in \mathbb{R}^m$,

$$\underline{\alpha}_p(|x|) \leq V_p(x) \leq \overline{\alpha}_p(|x|)$$
$$\langle \nabla V_p(x), f_p(x,d) \rangle \leq -\lambda V_p(x) + \gamma_p(|d|)$$

for some $\underline{\alpha}_p, \overline{\alpha}_p, \gamma_p \in \mathcal{K}_\infty$, *and* $\lambda > 0$.

- *There exists* $\mu > 1$ *such that, for all* $p, q \in \mathcal{P}$,

$$V_p(x) \leq \mu V_q(x), \qquad x \in \mathbb{R}^n.$$

Then, the system is uniformly ISS over Σ_{τ_a}, *with*

$$\tau_a \geq \frac{\ln \mu}{\lambda}.$$

Discussion

- By *compatible* Lyapunov functions, we mean that there exists $\mu > 1$ such that for each $p, q \in \mathcal{P}$, we have

$$V_p(x) \leq \mu V_q(x), \qquad \forall x \in \mathbb{R}^n.$$

- Basically, if each subsystem is ISS with *exponentially decaying compatible* Lyapunov function, then the switched system is uniformly ISS over Σ_{τ_a}, for τ_a large enough.
- Requiring each subsystem to be ISS seems natural, but what about *exponential decay* + *compatibility* condition?
- It turns out that *compatibility* is very restrictive and does not hold in general.
- What are the analogs of aforementioned results if *compatibility* among ISS Lyapunov functions of individual subsystems does not hold?

Other works: Different sorts of generalizations :

[Hespanha/Liberzon/Teel '08], [Muller/Liberzon '12], [Dashkovskiy/Mironchenko '13], [Kundu/Chatterjee '15], [Haimovich/Mancilla-Aguilar '19, '20], [Boarotto/Sigalotti '20], [Liu/Russo/Liberzon/Cavallo '21]

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Instability of Dwell-time Constrained Systems

Consider the autonomous system

$$\dot{x} = f_\sigma(x)$$

and the following implication

$$\{0\} \text{ is GAS for } \dot{x} = f_p(x),\ \forall\, p \in \mathcal{P} \overset{?}{\Longrightarrow} \exists\ \tau_d \text{ s.t. } \dot{x} = f_\sigma(x) \text{ is UGAS over } \Sigma_{\tau_d}$$

The answer is **NO.**

Counterexample: For $p \in \{1, 2\}$, consider

$$f_p^k(x) := |A_p x|^{k-1} A_p x$$

where each A_p is Hurwitz and is chosen as in [Liberzon '03, Page 19].

- $k = 1$ corresponds to the linear case. Earlier results apply.
- $k > 1$ corresponds to superlinear case. Slow decay near the origin.
- $k < 1$ corresponds to sublinear case. Slow decay far from the origin.

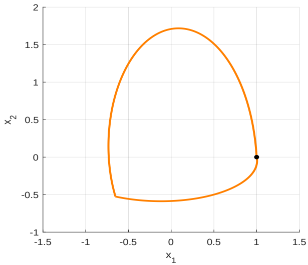
Instability but Ultimate Boundedness with $k > 1$

$$\dot{x} = f_\sigma(x) := |A_\sigma x| A_\sigma x, \qquad \sigma \in \{1, 2\}$$

- Pick $x_0 \in \mathbb{R}^n$, and find T_1, T_2 such that

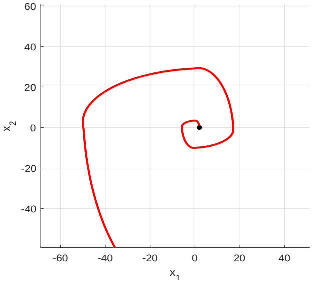
$$\phi_{f_1}(T_1, x_0) = \phi_{f_2}(-T_2, x_0)$$

- Consider a periodic (and hence dwell-time) switching signal with $\sigma = 1$ over $[0, T_1)$, and $\sigma = 2$ over $[T_1, T_1 + T_2)$, so that $\tau_{\mathrm{d}} = \min\{T_1, T_2\}$.
    - The state only evolves in a periodic orbit around the origin.
    - By scaling the initial condition, the dwell-time can be increased arbitrarily. The asymptotic behavior remains the same.
    - This time-scaling does not occur in linear vector fields.



Unbounded but locally converging with $k < 1$

- For each $\tau_d > 0$, there exist $x_0 \in \mathbb{R}^n$ with $|x_0|$ large enough, and $\sigma \in \Sigma_{\tau_d}$, such that, the corresponding solution diverges.



We can formalize such observations for two system classes:

- Each f_p is homogenous, that is, for every $\lambda > 0$, $f_p(\lambda x) = \lambda^k f_p(x)$, for some $k \in \mathbb{R}$.

Nonlinear Transformations

Consider $\{f_p\}_{p\in\mathcal{P}}$ homogenous of degree $k \neq 0$. Let

$$g_p(x) := \frac{|f_p(x)|^{\frac{1}{k}}}{|f_p(x)|} f_p(x), \;\; \forall\, x \in \mathbb{R}^n \setminus \{0\},$$

with $g_p(0) = 0$.

- For each $p \in \mathcal{P}$, the vector field g_p is homogenous of degree 1.

Applying the definition to our example:

- For the example treated earlier, with $f_p(x) := |A_p x|^{k-1} A_p x$, we have

$$g_p(x) = A_p x$$

- There exists matrices $A_p \in \mathbb{R}^{n\times n}$, such that the solutions of

$$\dot{x} = A_{\sigma^*} x$$

diverge to $+\infty$ for some switching signal σ^* (with positive dwell-time)

Instability Statements [Della Rossa/ AT '21]

Consider $\{f_p\}_{p\in\mathcal{P}}$ homogenous of degree $k \neq 0$. Assume that

- there exists a dwell-time switching signal σ^* such that the solutions of

$$\dot{x} = g_{\sigma^*}(x)$$

  diverge to $+\infty$ as $t \to \infty$.

Then, the following statements hold:

- $(k > 1)$ For every $\tau_d > 0$, the origin $\{0\}$ is not Lyapunov stable over Σ_{τ_d}.
- $(k < 1)$ For every $\tau_d > 0$, there exist $x(0)$ arbitrarily large, and a $\sigma \in \Sigma_{\tau_d}$, such that the resulting solution diverges to $+\infty$.

Additional remark:

- We can get some positive statements in the form of ultimate boundedness for $k > 1$, and local asymptotic stability for $k < 1$.

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ISS with Multiple Lyapunov Functions

Let us return to original question:

$$\dot{x} = f_\sigma(x, d)$$

and consider the implication

$$\left\{ \begin{array}{c} \dot{x} = f_p(x,d) \text{ is ISS }, \ \forall\, p \in \mathcal{P} \\ + \\ \text{auxilliary conditions} \end{array} \right\} \implies \begin{array}{l} \exists\ \tau_a \text{ s.t. } \dot{x} = f_\sigma(x,d) \\ \quad \text{is uniformly ISS over } \Sigma_{\tau_a} \end{array}$$

- Conditions based on ISS Lyapunov functions for each mode
- We consider nonlinear supply functions in the dissipation inequalities, without compatibility conditions
- More general average dwell-time bounds which guarantee ISS
- The approach can be generalized to allow some subsystems to be non-ISS

Assumptions on System Data

Each subsystem is ISS, that is, there are $\mathcal{C}^1$ functions $V_p : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, $p \in \mathcal{P}$, satisfying :

(L1) There exist $\underline{\alpha}, \overline{\alpha}, \alpha, \gamma \in \mathcal{K}_\infty$ such that

$$\underline{\alpha}(|x|) \leq V_p(x) \leq \overline{\alpha}(|x|), \quad \forall x \in \mathbb{R}^n, p \in \mathcal{P},$$
$$\left\langle \frac{\partial}{\partial x} V_p(x), f_p(x,d) \right\rangle \leq -\alpha(V_p(x)) + \gamma(|d|) \quad \forall x \in \mathbb{R}^n, d \in \mathbb{R}^m, p \in \mathcal{P},$$

(L2) There exist $\chi \in \mathcal{K}_\infty$ such that

$$V_q(x) \leq \chi(V_p(x)) \quad \forall x \in \mathbb{R}^n, p, q \in \mathcal{P}.$$

Modification for integral ISS:

- If we take α to be positive definite, then each subsystem is integral ISS, and we can develop results for integral ISS of switched systems.

Construction of (i)ISS Lyapunov Function

For some constant $c > 0$, consider the function $\psi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ as

$$\psi(t) := \min_{r \in [0,t]} \{\alpha(r) + c(t - r)\}$$

Key assumption: It holds that

$$\boxed{\zeta^* := \sup_{s>0} \int_s^{\chi(s)} \frac{1}{\psi(r)} dr < \infty}$$

- Finiteness of ζ^* essentially captures a trade-off between continuous and discrete dynamics.
- If α is linear, and χ is linear, we recover the earlier results

ADT Bounds and (i)ISS Lyapunov Function

Hybrid system description [Goebel et. al. '12]:

$$(x,p,\tau) \in \mathcal{C} : \begin{cases} \dot{x} = f_p(x) \\ \dot{p} = 0 \\ \dot{\tau} \in [0, \frac{1}{\tau_a}] \end{cases} \qquad (x,p,\tau) \in \mathcal{D} \begin{cases} x^+ = x \\ p^+ \in \mathcal{P} \\ \tau^+ = \tau - 1 \end{cases}$$

where $\mathcal{C} := \mathbb{R}^n \times \mathcal{P} \times [0, N_0]$, and $\mathcal{D} := \mathbb{R}^n \times \mathcal{P} \times [1, N_0]$.

Theorem ([Liu/AT/Liberzon MCSS'21])

Suppose (L1), (L2) hold, and $\zeta^* < \infty$. Then, the switched system is uniformly (i)ISS over Σ_{τ_a}, for each $\tau_a > \zeta^*$. The corresponding (i)ISS hybrid Lyapunov function is

$$V(x,p,\tau) := \varphi^{-1}\Big(e^{2c\zeta\tau}\varphi\big(V_p(x)\big)\Big)$$

for $\zeta > \zeta^*$, and $\varphi : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is given by $\varphi(s) = \begin{cases} \exp\left(\int_1^s \frac{2c}{\psi(r)}\, dr\right), & s > 0 \\ 0, & s = 0. \end{cases}$

Discussions and Extensions

Extensions covered in [Liu/AT/Liberzon MCSS'21]:

- We can consider the case where some subsystems are not (i)ISS. With such subsystems, we associate a positive definite function α_u and the inequality

$$\left\langle \frac{\partial}{\partial x} V_p(x), f_p(x,d) \right\rangle \leq \alpha_u(V_p(x)) + \gamma(|d|) \quad \forall x \in \mathbb{R}^n, d \in \mathbb{R}^m,$$

  To guarantee (i)ISS, we must limit the average activation time of such subsystems:

$$(1+\kappa)\eta + \frac{\zeta^*}{\tau_a} < 1$$

  where $\eta \in [0,1]$ is the fraction of activation times of unstable subsystems, and $\kappa := \sup_{r>0} \frac{\alpha_u(r)}{\psi(r)}$.
- We can allow for reset maps at switching times, $x^+ = g_{p,q}(x,d)$. The growth estimate at switching times changes and involves an additional gain function.
- Several interesting examples: iISS of switched bilinear systems; and stability with slow-fast switching.

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Lyapunov Function for Cascade Interconnections

$$\xrightarrow{d} \boxed{\dot e = f_{o,\sigma}(e,d)} \longrightarrow \boxed{\dot x = f_{c,\sigma}(x,e)}$$

$$\underline{\alpha}_o(|e|) \le V_{o,p}(e) \le \overline{\alpha}_o(|e|) \qquad \underline{\alpha}_c(|x|) \le V_{c,p}(x) \le \overline{\alpha}_c(|x|)$$

$$\dot V_{o,p}(e) \le -\alpha_{o,p}(V_{o,p}(e)) + \gamma_{o,p}(|d|) \qquad \dot V_{c,p}(x) \le -\alpha_{c,p}(V_{c,p}(x)) + \gamma_{c,p}(V_{o,p}(e))$$

Lyapunov function for the cascade: [Sontag/Teel '95] and [AT et. al. CDC'15]

Proposition (ISS of each subsystem)

For each $p \in \mathcal{P}$, and a suitably chosen function $\ell_p : \mathbb{R}_{\ge 0} \to \mathbb{R}_{\ge 0}$, let

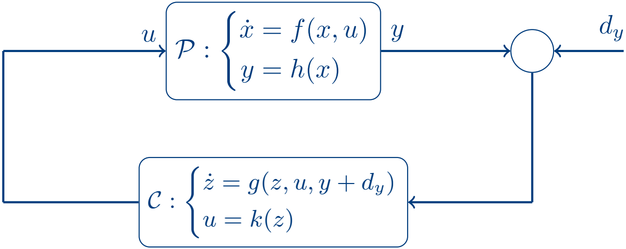
$$V_p(e,x) := \ell_p(V_{o,p}(e)) + V_{c,p}(x).$$

Then, there exist $\underline{\alpha}_p, \overline{\alpha}_p, \alpha_p \in \mathcal{K}_\infty$, and $\gamma_p \in \mathcal{K}$, such that

$$\underline{\alpha}_p(|(x,e)|) \le V_p(x,e) \le \overline{\alpha}_p(|(x,e)|)$$

$$\left\langle \nabla V_p(x,e), \begin{pmatrix} f_{c,p}(x,e) \\ f_{o,p}(e,d) \end{pmatrix} \right\rangle \le -\alpha_p(V_p(x,e)) + \gamma_p(|d|)$$

Where do we see Cascade Systems?

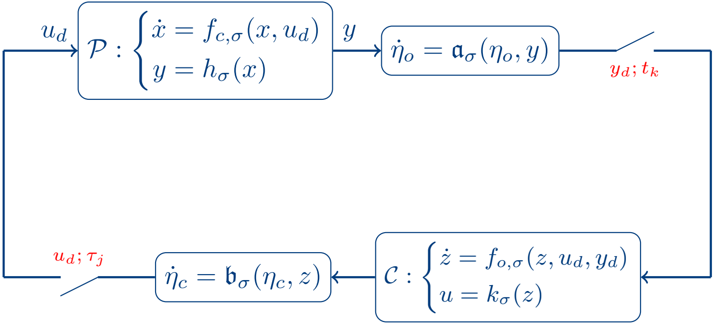


- **Dynamic output feedback:** Stabilize nonlinear plant with partial state output measurements
- **Cascade principle:** Design an estimator, and a static full state feedback separately
- **Robustness:** ISS guarantees performance in the presence of measurement noise

$$\xrightarrow{d_y} \boxed{\dot e = f_o(e, d_y)} \xrightarrow{e} \boxed{\dot x = f_c(x, e)} \xrightarrow{x}$$

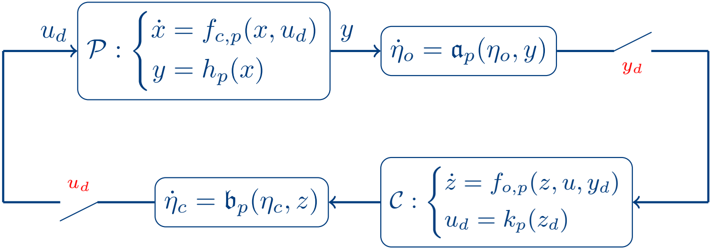
Sampled-Data Feedback Stabilization of Switched Systems

Switched System with Dynamic Output Feedback Control:



- **Asynchronous Sampling:** The sampling times for outputs and inputs are determined based on some algorithm to preserve asymptotic stability.
- **Dynamic Event-Based Sampling:** Design dynamic filters that determine sampling times for output feedback controllers.

Sampling Algorithms



Sampling algorithm:

$$\dot{\eta}_o := -\beta_{o,p}(\eta_o) + \rho_{o,p}(|y|) + \gamma_{o,p}(|y - y_d|)$$

$$\dot{\eta}_c := -\beta_{c,p}(\eta_c) + \rho_{c,p}\left(\frac{|z|}{2}\right) + \gamma_{c,p}(|z - z_d|)$$

Event-triggered sampling strategy:

$$\begin{cases} \dot{y}_d = 0 \\ \dot{z}_d = 0, \end{cases} \quad \text{and} \quad \begin{cases} y_d^+ = y, & \text{if } |y - y_d| \geq \sigma_{o,p}(\eta_o) \\ z_d^+ = z, & \text{if } |z - z_d| \geq \sigma_{c,p}(\eta_o) \end{cases}$$

Stability Result

Theorem ([Zhang/AT '19])

Consider the closed-loop system under appropriate assumptions on Lyapunov functions associated with nominal system. If the functions in sampling algorithms are appropriately designed, and the average dwell-time τ_a satisfies:

$$\tau_a > \zeta^* := \sup_{s \geq 0} \int_s^{(1+\varepsilon)\chi(s)} \frac{dr}{\psi(r)}$$

where ψ and χ are computed from individual Lyapunov functions, then a desired set $\mathcal{A}$ is globally asymptotically stable for the closed-loop hybrid system.

Proof outline

- Construct function $W(\xi) := \exp(\zeta\tau)\varphi(V_p(x,e) + \eta_o + \eta_c)$, where $\zeta \in (\zeta^*, \tau_a)$.
- The function V_p is defined as $V_p(x,e) := \ell_p(V_{o,p}(e) + V_{c,p}(x))$.
- Analyze the flow equations and the jump relations
- Use the invariance principle based arguments to conclude convergence to desired set

Example and Simulation Result

Consider a feedback switched system as follows:

The first linear subsystem is

$$p = 1 : \begin{cases} \dot{x} = A_1 x + B_1 u_d \\ y = C_1 x \end{cases}$$

The feedback controller associated is:

$$p = 1 : \begin{cases} \dot{z} = A_1 z + B_1 u_d + L_1(y - C_1 z) \\ u_d = -K_1 z, \end{cases}$$

where we choose $A_1 = \begin{bmatrix} 0.5 & -1 \\ 0 & 0.5 \end{bmatrix}$, $B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $C_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}$, $L_1 = \begin{bmatrix} 3.5 \\ -3 \end{bmatrix}$, $K_1 = \begin{bmatrix} -1.5 & 2.5 \end{bmatrix}$.

The second nonlinear subsystem is

$$p = 2 : \begin{cases} \dot{x}_1 = x_2 + 0.25\,|x_1| \\ \dot{x}_2 = \mathsf{sat}(x_1) + u_d \\ y = x_1 \end{cases}$$

The feedback controller associated is:

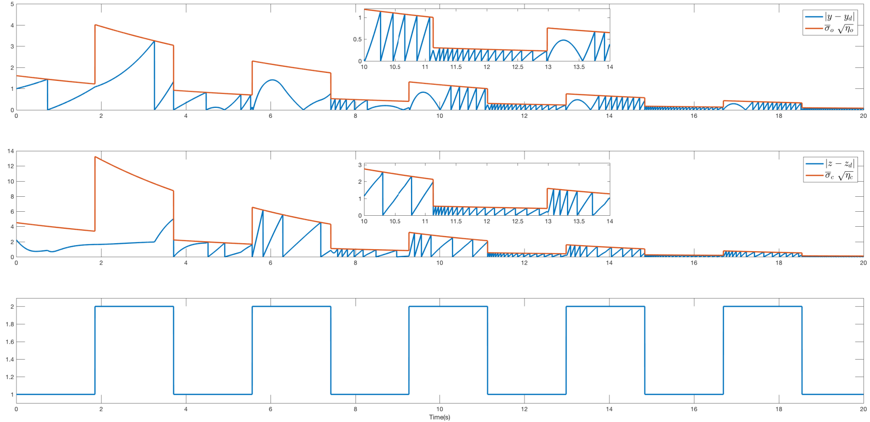
$$p = 2 : \begin{cases} \dot{z}_1 = z_2 + 0.25\,|y| + l_1(y - z_1) \\ \dot{z}_2 = \mathsf{sat}(y) + u_d + l_2(y - z_1) \\ u_d = \mathsf{sat}(z_1) + K_2 z \end{cases},$$

where we choose $K_2 = \begin{bmatrix} -2 & -2 \end{bmatrix}$ and $L_2 = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

Take $V_{o,p}(e) = e^\top P_{o,p} e$ and $V_{c,p} = x^\top P_{c,p} x$, then for $p \in \{1, 2\}$, we have:

$$\dot{V}_{o,p} \leq -a_{o,p} V_{o,p}(e) + \overline{\gamma}_{o,p}\,|y - y_d|^2$$
$$\dot{V}_{c,p} \leq -a_{c,p} V_{c,p}(x) + \overline{\gamma}_{c,p} V_{o,p}(e) + \overline{\gamma}_{c,p}\,|z - z_d|^2\,,$$

Example and Simulation Result



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Summary

- Exponentially decaying *compatible* Lyapunov functions were used earlier for computing lower bounds on ADT
- In general, if all subsystems are 0-GAS (resp. ISS), then 0-GAS (resp. ISS) may not hold for switched nonlinear systems even for arbitrarily large ADT
- In addition to each subsystem being ISS, we need some additional condition for ISS of switched systems with finite lower bound on ADT
- The additional condition essentially boils down to the finiteness of

  $$\sup_{s>0} \int_s^{\chi(s)} \frac{1}{\psi(r)} \,\mathrm{d}r$$

  Recall that ψ is related to decay functions during continuous flows, and χ is related to growth at switching times.
- The construction of Lyapunov functions finds utility in design of sampled-data systems